Palestine Journal of Mathematics

Vol. 8(1)(2019) , 235–241

A NOTE ON DIRECT-INJECTIVE MODULES

Sanjeev Kumar Maurya and A. J. Gupta

Communicated by Robert Wisbauer

MSC 2010 Classifications: Primary 16D40, 16D10, 16S50.

Keywords and phrases: Direct-injective modules; endoregular modules; SIP and SSP modules.

We would like to thanks the referee(s) for valuable comments and suggestions on the manuscript.

Abstract. In this paper, we study some more properties on direct-injective modules in the context of endoregular, SSP and SIP modules. We find the equivalent condition for a direct-injective module to be divisible. We also show that the endomorphism ring of an R-module M is a division ring if and only if M is a direct-injective module with (*) property. Finally, we study dc-rings and find their connections with hereditary rings and SSI-rings.

1 Introduction

Throughout this paper, all rings are associative rings with unity and all modules are unitary right R-modules. For a right R-module M, $S = End_R(M)$ denotes the endomorphism ring of M. For $\phi \in S$, $Ker(\phi)$ and $Im(\phi)$ stand for kernel and image of ϕ respectively. The notations $N \leq M$, $N \leq^{ess} M$ and $N \leq^{\oplus} M$ means that N is a submodule, an essential submodule and a direct summand of M respectively. $Mat_n(R)$ denotes the $n \times n$ matrix ring over R and $r_M(I) = \{m \in M | Im = 0\}$.

The notion of direct-injective modules was introduced by W. K. Nicholson [10] in 1976. This notion is the generalization of quasi-injective module. A right R-module M is said to be direct-injective if given a direct summand N of M with inclusion $i_N : N \to M$ and any monomorphism $g : N \to M$ there exist $f \in End_R(M)$ such that $fog = i_N$. Recall that a module M is called a $C2$-module if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. Nicholson and Yousif [11, Theorem 7.13] showed that the class of direct-injective modules is equivalent to the class of $C2$-modules.

According to Rizvi et al. [8], a right R-module M is said to be an endoregular module if $End_R(M)$ is a von Neumann regular ring. For any right R-module M if $End_R(M)$ is a von Neumann regular ring then M is a direct-injective module. Thus, every endoregular module is a direct-injective module but the converse need not be true. We give an example of a direct-injective module that is not an endoregular module.

In Section 2 of this paper, we discuss the conditions under which every direct-injective module is an endoregular module. We also show that a projective module M is endoregular if and only if M is direct-injective and $Im(s)$ is projective for all $s \in S$. According to Wilson [16], a right R-module M is said to have summand sum property (summand intersection property) called SSP-module (SIP-module) if sum (intersection) of two direct summands of M is a direct summand of M. In this Section, we also characterize direct-injective modules in terms of SSP and SIP module.

According to Sharpe and Vamos [12], an element e of E is said to be 'divisible' if for every r of R which is not a right zero-divisor, there exist $e^{'} \in E$ such that $e = re^{'}$. If every element of E is divisible, then E is said to be a divisible module. In [6], Han and Choi proved that every direct-injective module is divisible; however we can show that their result is incorrect, since $\mathbb{Z}_4$ as $\mathbb{Z}$-module is a direct-injective module but not divisible. In this Section, we find the condition for a direct-injective module to be divisible. According to Tiwari and Pandeya [13], a right R-module M is said to satisfy (*) property if every non-zero endomorphism of M is a monomorphism and any module with (*) property is indecomposable. In this Section, we show that the endomorphism ring of an R-module M is a division ring if and only if M is a direct-injective module with (*) property.

In Section 3, we investigate some other properties of direct-injective modules. In this Section, we give the condition under which a direct-injective module satisfies finite exchange property. We also find the condition for a submodule of a direct-injective module to be direct-injective. We show that the class of co-Hopfian, weakly co-Hopfian and Dedekind finite modules are equivalent for a class of direct-injective module. We also show that every singular module in $\sigma[M]$ is direct-injective if and only if it is injective in $\sigma[M]$, where $\sigma[M]$ is the full subcategory of Mod-R whose objects are all R-modules subgenerated by M [17]. At the end of this section, we also study about dc- rings. A ring R is said to be dc- ring if every cyclic R-module is direct-injective. A ring R is said to be an SSI ring if every semisimple R-module is injective. Finally, we show that a commutative SSI ring is a dc-ring and every self-injective hereditary ring is a dc-ring.

2 Characterization of Direct-injective Modules in terms of Endoregular, SSP and SIP Modules

Nicholson and Yousif [11, Theorem 7.13] showed that the class of direct-injective modules are equivalent to the class of $C2$-modules. Throughout the paper, we consider direct-injective modules as a $C2$-modules. We need the following lemmas for clarity.

Lemma 2.1. *[1, Theorem 16] Let M be a right R-module then, $S = End_R(M)$ is a von Neumann regular ring if and only if $Ker(s)$ and $Im(s)$ are direct summands of M for all $s \in S$.*

Lemma 2.2. *[5, Lemma 2.1] Let M be a module and $S = End(M)$. Then the following conditions are equivalent:*

(i) M is a $C2$-module (or direct-injective).

(ii) For any $s \in S$, $Im(s)$ is a direct summand of M if $Ker(s)$ is a direct summand of M.

According to Rizvi et al. [8], a right R-module M is an endoregular module if $End_R(M)$ is a von Neumann regular ring. By Lemma 2.1, every endoregular module is direct-injective but the converse need not be true. Here, we give a counterexample which shows that a direct-injective module need not be an endoregular module.

Example 2.3. Let $M = \mathbb{Z}_4$ as $\mathbb{Z}$-module. Then M is a direct-injective module because it is a quasi-injective module but $End_{\mathbb{Z}}(\mathbb{Z}_4)$ is not a von Neumann regular ring hence it is not an endoregular module.

Now we discuss the conditions under which direct-injective modules are endoregular. According to G. Lee et al. [9], a right R-module M is a Rickart module if $Ker(\phi)$ is a direct summand of M for all $\phi \in End_R(M)$.

Theorem 2.4. *The following conditions are equivalent for a module M and $S = End_R(M)$*

(i) M is an endoregular module;

(ii) M is a direct-injective module and $M \bigoplus M$ is an SIP module;

(iii) M is a direct-injective module and a Rickart module.

Proof. (1) $\Rightarrow$ (2). It is easy to see that M is an endoregular module implies that M is a direct-injective module. Now, we show that $M \bigoplus M$ is an SIP module. Set $S^{'} = End_R(M \bigoplus M) \cong Mat_2(S)$, $Mat_2(S)$ is von Neumann regular ring as S is von Neumann regular ring and has SSP due to the fact that every von Neumann regular ring has the SSP. So $S^{'}$ has SSP. Then by [3, Lemma 2.1], for the any pair of idempotents $\alpha, \beta \in S^{'}$ there exist idempotents $e, e^{'} \in S^{'}$ such that $\alpha\beta S^{'} = eS^{'}$ and $S^{'}\alpha\beta = S^{'}e^{'}$. Since, $Ker(\alpha\beta) = r_M(S^{'}\alpha\beta) = r_M(S^{'}e^{'}) = (1 - e^{'})M \bigoplus M$. So $Ker(\alpha\beta) \leq^{\oplus} (M \bigoplus M)$ which shows that $M \bigoplus M$ is an SIP modules.

(2) $\Rightarrow$ (3). Since $M \bigoplus M$ is an SIP module therefore $Ker(s)$ is a direct summand of M for all $s \in S$. Hence, M is a Rickart module.

(3) $\Rightarrow$ (1). Since M is a Rickart module so $Ker(s)$ is a direct summand of M for all $s \in S$. Also given that M is direct-injective then, by Lemma 2.2 $Im(s)$ is a direct summand of M for all $s \in S$. Hence, by Lemma 2.1 M is an endoregular module.

Corollary 2.5. *A module M is an endoregular module if M is a Rickart module and $M \bigoplus M$ is a $C3$ module.*

We have observed that a projective module need not be an endoregular module. For example, $\mathbb{Z}$-module $\mathbb{Z}^n$ is not an endoregular module for any $n \in N$. In the next proposition, we give an equivalent condition for a projective module to be an endoregular module.

Proposition 2.6. *A projective module M is an endoregular module if and only if M is direct-injective and $Im(s)$ is projective for all $s \in S = End_R(M)$.*

Proof. Suppose a projective module M is an endoregular module then M is a direct-injective module because $Ker(s)$ and $Im(s)$ are direct summands of M for all $s \in S$. Since M is projective there exist a projective submodule K of M such that $M = Ker(s) \bigoplus K$ and $Im(s) \cong \frac{M}{Ker(s)} \cong K$. Therefore, $Im(s)$ is projective.

Conversely, suppose that $Im(s)$ is projective for all $s \in S$. Then $Ker(s)$ is a direct summand of M but M is direct-injective implies that $Im(s)$ is a direct summand of M, so by Lemma 2.1, M is an endoregular module.

Theorem 2.7. *Let M is a direct-injective module then the following assertions hold.*

(i) $S = End_R(M)$ is a right SSP ring if M is a Rickart module.

(ii) $Mat_2(S)$ is a right SSP ring if M is a Rickart module.

Proof.

(i) Let M be a Rickart module then $Ker(s)$ is a direct summand of M $\forall s \in S = End(M)$. Since M is a direct-injective module, hence $Im(s)$ is a direct summand of M $\forall s \in S = End(M)$. Then by [1, Theorem 16], S is a von Neumann regular ring. Since every von Neumann regular ring is a right SSP ring, so S is a right SSP ring.

(ii) Since S is a von Neumann regular ring, therefore $Mat_2(S)$ is a von Neumann regular ring. Hence, $Mat_2(S)$ is a right SSP ring.

Remark 2.8. Since every right SSP ring is also a right SIP ring, therefore S and $Mat_2(S)$ are also right SIP ring if M is a Rickart module and a direct-injective module.

An element m of a module M over a ring R is said to be torsion element there exist a regular element $r \in R$ such that $rm = 0$. A module M over a ring R is called a torsion module if all its elements are torsion element and M is called torsion-free if zero is the only torsion element of M. Every torsion-free module need not be direct injective. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is a torsion-free module but not a direct-injective module. A module M over a ring R is said to be divisible if $rM = M$ for all regular element $r \in R$.

Proposition 2.9. *Let R be a commutative domain and M be a torsion-free module. Then M is a direct-injective module if and only if M is a divisible module.*

Proof. Suppose M is a direct-injective module and let r be a non-zero element of R. Since R is a commutative domain, so r is a regular element of R. Let us define $f : M \to M$ by $f(m) = rm$, $m \in M$. Then f is clearly a monomorphism. As M is a direct-injective module, $f(M) \leq^{\oplus} M$. Then there exist a submodule K of M such that $M = f(M) \bigoplus K = rM \bigoplus K$. Then $rK = 0$ implies that $K = 0$. Thus, $rM = M$ for all regular $r \in R$. Hence, M is a divisible module.

Conversely, let M is a divisible R-module. Since M is also a torsion-free module over a commutative domain, therefore by [12, Proposition 2.7], M is an injective module and so M is a direct-injective module.

Remark 2.10. It is observed that every direct-injective module need not be divisible. For example, $\mathbb{Z}$-module $\mathbb{Z}_4$ is a direct-injective module but not a divisible module. This shows that [6, Theorem 2.1] is incorrect.

An R-module M is said to satisfy (*) property if each non-zero endomorphism of M is a monomorphism [13]. With the help of this property, we find the condition under which the endomorphism ring of a direct-injective module is a division ring. Recall that a module M is called co-Hopfian [14], if every injective endomorphism $f : M \to M$ is an automorphism. In the next result, we generalize Schur's, Lemma.

Proposition 2.11. *Let M be a right R-module and $S = End_R(M)$. Then the following conditions are equivalent:*

(i) S is a division ring;

(ii) M is a direct-injective module with (*) property;

(iii) M is a co-Hopfian module with (*) property;

(iv) S is a von-Neumann regular ring and M is an indecomposable module.

Proof. (1) $\Rightarrow$ (2) Since every division ring is von Neumann regular ring, therefore, S is a von Neumann regular ring and hence, M is a direct-injective module. Also S is a division ring, so every non-zero endomorphism $f \in S$ is an automorphism, hence a monomorphism. Therefore, M satisfy (*) property.

(2) $\Rightarrow$ (3) Let $f \in S$ is an injective endomorphism. Then $f(M) \cong M \leq^{\oplus} M$, so $f(M) \leq^{\oplus} M$, as M is direct-injective. Since every module with (*) property is indecomposable, therefore $f(M) = M$. Thus, f is an automorphism implies M is co-Hopfian.

(3) $\Rightarrow$ (4) Let f is a non-zero endomorphism in S. Since M has (*)property, therefore f is a monomorphism. Since, M is a co-Hopfian module, therefore f becomes an automorphism. Hence, $Ker(f) = 0$ and $Im(f) = M$. This implies that $Ker(f)$ and $Im(f)$ are direct summands of M. Therefore, S is a von Neumann regular ring. It is easy to see that every module with (*) property is indecomposable.

(4) $\Rightarrow$ (1) Since S is a von Neumann regular ring, so $Ker(f)$ and $Im(f)$ are direct summands of M for each $f \in S$. To show that S is a division ring we have to show that each non-zero endomorphism $f \in S$ is an automorphism. Since, M is an indecomposable module, $Ker(f) = 0$ and $Im(f) = M$. Hence, f is an automorphism as desired.

Corollary 2.12. *(i) Let M be a cyclic torsion-free direct-injective module, then $S = End_R(M)$ is a division ring.*

(ii) Let M be an uniform torsion-free direct-injective module, then $S = End_R(M)$ is a division ring.

Proof. Since every cyclic torsion-free and uniform torsion-free modules satisfy (*) property. Therefore, the proof follows from Proposition (2.11).

A ring is said to be an *abelian* ring if all its idempotents are central. A module M is said to be an abelian module if its endomorphism ring is an abelian ring.

Proposition 2.13. *Let M be an abelian endoregular module with (*) property. Then $End_R(M)$ is a division ring.*

Proof. Since M has the (*) property, each non-zero endomorphisms are monomorphisms. Since M is an abelian endoregular module, $M = Ker(s) \bigoplus Im(s)$ for all $s \in End_R(M)$ [8]. So each injective endomorphism becomes an automorphism. Thus, each non-zero endomorphism is invertible so $End_R(M)$ is a division ring.

3 Some Properties of Direct-injective Modules

In this section, we give the condition under which every direct-injective module satisfies the finite exchange property. We also find the condition under which a submodule of a direct-injective module is a direct-injective module. We also study about the ring for which every cyclic R-module is a direct-injective module.

A right R-module M is said to satisfy the exchange property if for every right R-module A and any two direct sum decompositions $A = M' \bigoplus N = \bigoplus_{i \in \mathcal{I}} A_i$ with $M' \cong M$, there exist submodules B_i of A_i such that $A = M' \bigoplus (\bigoplus_{i \in \mathcal{I}} B_i)$. M is said to satisfy finite exchange property if this hold only for any finite index set $\mathcal{I}$. A ring R is said to be an exchange ring if the module R_R satisfy the exchange property. Warfield [15] proved that a module M has the finite exchange property if and only if $End_R(M)$ is an exchange ring.

Proposition 3.1. *Let M be a direct-injective module such that $Ker(s)$ lies under a direct summand of M for any $s \in End_R(M)$. Then M satisfies the finite exchange property.*

Proof. Let M be a direct-injective module and $S = End_R(M)$. Since $Ker(s)$ lies under a direct summand of M for every $s \in End_R(M)$, therefore, S is a semiregular ring [10]. Since every semiregular ring is an exchange ring, hence S is an exchange ring. This proves that M has the finite exchange property.

Corollary 3.2. *Every endoregular module has the finite exchange property.*

Proposition 3.3. *Let M be a direct-injective module and N is a submodule of M. Then N is a direct-injective module if $\frac{M}{N}$ is a free module.*

Proof. Since $\frac{M}{N}$ is free module then the short exact sequence $0 \to N \to M \to \frac{M}{N} \to 0$ splits. So N is a direct summand of M. Hence, N is a direct injective module.

Corollary 3.4. *Let M be a finitely generated direct-injective module over a principle ideal domain, then the torsion submodule of M is a direct-injective module.*

Haghani and Vedadi [4] called an R-module M weakly co-Hopfian if for any injective endomorphism f of M, $f(M) \leq^{ess} M$. An R-module M is called Dedekind-finite if $M \cong M \bigoplus N$ for some module N, then $N = 0$. Co-Hopfian modules are weakly co-Hopfian and weakly co-Hopfian modules are Dedekind-finite [4]. But these classes of modules are equivalent for the class of direct-injective modules.

Proposition 3.5. *Let M be a direct-injective module. Then the following are equivalent:*

(i) M is co-Hopfian;

(ii) M is weakly co-Hopfian;

(iii) M is Dedekind-finite.

Proof. (1) ⇒ (2) ⇒ (3). They are clear.

(3) ⇒ (1) Let $f : M \to M$ be an injective endomorphism. Since M is direct-injective, so $f(M) \leq^{\oplus} M$. Let $M = f(M) \bigoplus N$ for some $N \leq M$. We define a homomorphism $g : M \bigoplus N \to M$ by $g(m, n) = f(m) + n$. Then $M \bigoplus N \cong M$ and M is Dedekind-finite, $N = 0$. Hence, $f(M) = M$, so f is an automorphism as desired.

According to Wisbauer[17], for a module M, $\sigma[M]$ denotes the full subcategory of Mod-R whose objects are all R-modules subgenerated by M and $E_M(N)$ denotes the M-injective hull of a module N which is the trace of M in the injective hull $E(N)$ of N, i.e. $E_M(N) = \sum\{f(M) : f \in Hom(M, E(N))\}$. According to Dung et al.[2], an R-module N is called singular in $\sigma[M]$ or M-singular if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq^{ess} L$. Every module $N \in \sigma[M]$ contains a largest M-singular submodule which we denote by $Z_M(N)$. If $Z_M(N) = 0$, then N is called non-singular in $\sigma[M]$.

Theorem 3.6. *Assume $Z_M(M) = 0$. Then the following are equivalent:*

(i) Every singular module in $\sigma[M]$ is injective;

(ii) Every singular module in $\sigma[M]$ is direct-injective.

Proof. (1) $\Rightarrow$ (2). This is clear.

(2) $\Rightarrow$ (1). Let N be a singular module in $\sigma[M]$. By [2, Proposition 4.1], $E_M(N)$ the M-injective hull of N is also singular in $\sigma[M]$. Then, $N \bigoplus E_M(N)$ is also singular in $\sigma[M]$, so $N \bigoplus E_M(N)$ is direct-injective by hypothesis. Thus, the inclusion map $i : N \to E_M(N)$ splits, so $N \leq^{\oplus} E_M(N)$. As N is essential in $E_M(N)$, $N = E_M(N)$. So, N is M-injective. Hence, every singular module in $\sigma[M]$ is injective.

Corollary 3.7. *The following conditions are equivalent for a right non-singular ring R:*

(i) Every singular right R-module is direct-injective;

(ii) Every singular right R-module is injective;

(iii) Every cyclic singular right R-module is injective;

(iv) Every singular right R-module is semisimple.

Proof. (1) $\Rightarrow$ (2). Let M be a singular right R-module. Then, $M \bigoplus E(M)$ is also singular right R-module, so $M \bigoplus E(M)$ is direct-injective by hypothesis. Hence, $M = E(M)$ which implies that M is injective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). They are clear by [2, Corollary 7.1].

(4) $\Rightarrow$ (1). This is clear.

A ring R is said to be a dc-ring if every cyclic R-module is a direct-injective module. A ring R is said to be a qc-ring if every cyclic R-module is a quasi-injective module. Since every quasi-injective module is a direct-injective module, therefore every qc-ring is a dc-ring. Semisimple artinian rings are obviously dc-rings. In this section, we also find the connections of dc-rings with SSI rings and hereditary rings.

Proposition 3.8. *A commutative SSI-ring is a dc-ring.*

Proof. A ring R is SSI ring if and only if R is a Noetherian V-ring. Since R is a commutative SSI ring implies that R is a commutative Noetherian V-ring. Since commutative V-ring is regular. So R is a Noetherian regular ring, hence R is a semisimple artinian ring. Therefore, R is a dc-ring.

Proposition 3.9. *Every self injective hereditary ring is a dc-ring.*

Proof. Since R is a hereditary ring, a quotient of an injective module is direct-injective [18, Theorem 4]. Also, R is self-injective implies that every cyclic R-module is isomorphic to the quotient of an injective R-module, so every cyclic R-module is a direct-injective module. Therefore, R is a dc-ring.

References

[1] G. Azumaya, On generalized semi-primary rings and Krull-Remak-Schmidt's theorem. *Japan J. Math.* **19**, 525–547 (1948).

[2] N. V. Dung, D. V. Huynh, P. F. Smith, R. Wisbauer: *Extending Modules*, Harlow, New York, Longman (1994).

[3] J. L. Garcia, Properties of direct summands of modules, *Comm. Algebra* **17**, 73–92 (1989).

[4] A. Haghany, M. R. Vedadi, Modules whose injective endomorphisms are essential, *J. Algebra* **243**, 765–779 (2001).

[5] P. T. Hai, On modules and rings satisfy condition (C), *Asian European Journal of Mathematics* **9**, 1650045 (14 pages) (2016).

[6] C. W. Han, S. J. Choi, Generalizations of the quasi-injective module, *Comm. Korean Math. Soc.* **10** no. 4, 811–813 (1995).

[7] Y. Ibrahim, X. H. Nguyen, M. F. Yousif, Y. Zhou, Rings whose cyclics are C3 modules, *Journal of Algebra and Its Applications* **15**, No. 8, (2016) 1650152 (18 pages).

[8] G. Lee, S. T. Rizvi, C. S. Roman, Modules whose endomorphism rings are von Neumann regular, *Comm. Algebra* **41**, 4066–4088 (2013).

[9] G. Lee, S. T. Rizvi, C. S. Roman, Rickart Modules, *Comm. Algebra* **38** (11), 4005–4027 (2010).

[10] W. K. Nicholson, Semiregular modules and rings, *Canad. J. Math.* **28**, No(5), 1105–1120 (1976).

[11] W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius Rings*, Cambridge University Press (Tract in Mathematics No 158), Cambridge (2003).

[12] D. W. Sharpe, P. Vamos, *Injective Modules*, Cambridge University Press, (1972).

[13] A. K. Tiwari, B. M. Pandeya, Modules whose non zero endomorphisms are monomorphism, *in: Proc. Int. Symp. Algebra and its Application*, New Delhi, 1981, in: Lect. Notes Pure Appl. Math. 91 Marcel Dekker, Inc. New York, (1984).

[14] K. Varadarajan, Hopfian and co-Hopfian objects, *Publ. Mat* **36**, 293–317 (1992).

[15] R. B. Warfield Jr., Exchange rings and Decomposition of modules, *Math. Ann.* **199**, 31–36 (1972).

[16] G. V. Wilson, Modules with the direct summand intersection property, *Commun. Algebra* **14**, 21–38 (1986).

[17] R. Wisbauer, *Foundations of Module and Ring Theory*, Philadelphia: Gordon and Breach, (1991).

[18] W. Xue, Characterization of rings using direct-projective modules and direct-injective modules, *J. Pure Appl. Algebra* **87**, 99–104 (1993).

Author information

Sanjeev Kumar Maurya and A. J. Gupta, Department of Mathematical Sciences, IIT(BHU) Varanasi, Varanasi, India 221005, INDIA.
E-mail: sanjeevm50@gmail.com, agupta.apm@itbhu.ac.in

Received: October 29, 2017.
Accepted: December 28, 2017