Palestine Journal of Mathematics

Vol. 8(1)(2019) , 23–34

FUZZY SESQUILINEAR FORM AND ITS PROPERTIES

S. Ghosal and T. Bag

Communicated by Faud Kittaneh

2010 AMS Classification: 46S40, 03E72.

Keywords and phrases: Fuzzy real number, fuzzy inner product, fuzzy sesquilinear form.

The authors are grateful to the referees for their valuable suggestions in rewriting the paper in the present form. The authors are also thankful to the Editor-in-Chief of the journal (PJM) for their valuable comments which helped us to revise the paper.

The present work is partially supported by Special Assistance Programme (SAP) of UGC, New Delhi, India [Grant No. F. 510/4/DRS/2009 (SAP-I)].

Abstract. In this paper, an idea of fuzzy sesquilinear form is introduced. Riesz representation theorem for sesquilinear form is established in fuzzy setting.

1 Introduction

Metric, norm and inner product structures are the main tools of functional analysis. So to develop fuzzy functional analysis, fuzzy metric, fuzzy norm and fuzzy inner product play the important role. Several authors studied fuzzy metric space as well as fuzzy normed linear space and a large number of papers have been published. We refer some of them which are related to our work [see 1-14]. Study on fuzzy inner product spaces are relatively recent. Idea of real probabilistic inner product space is introduced by Sklar [1]. Following his concept, Biswas [3], EI-Abyed and Hamouly [4], Kohli and Kumar [9], Majumder and Samanta [10], Hasankhani, Nazari and Saheli [7], Goudarzi and Vaezpour [6], Mukherjee and Bag [13] introduced the concept of fuzzy inner product space in different approaches and developed many results in such spaces.
In this paper, following the definition of fuzzy inner product given by Hasankhani et.al [7], an idea of fuzzy sesquilinear form is introduced as a fuzzy real number. Definition of bounded fuzzy sesquilinear form is given and concept of fuzzy norm of sesquilinear form is introduced. Riesz representation theorem for sesquilinear form has been established in fuzzy setting.
It is to be noted that Hasankhani et.al [9] considered the fuzzy real number in the sense of Kaleva et. al [10] to define fuzzy inner product whose induced fuzzy norm is Felbin's type [6] fuzzy norm. In this paper we consider Xiao and Zhu [14] type fuzzy real number and the induced fuzzy norm is Bag and Samanta [3] type fuzzy norm. In [3], it is shown that all the result which are valid in Felbin's fuzzy norm [6] are also valid in Bag and Samanta [3] type fuzzy norm.
The organization of the paper is as follows:
Section 2 comprises some preliminary results which are used in this paper.Riesz theorem for fuzzy bounded linear operator is modified in section 3. Definition of fuzzy sesquilinear form is given in Section 4. Riesz representation theorem is established in fuzzy setting in Section 5.

2 Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.
According to Mizumoto & Tanaka [11], a fuzzy real number is a mapping
$x: R \to [0, 1]$ over the set R of all reals.
x is called convex if $x(t) \geq min(x(s), x(r))$ where $s \leq t \leq r$.
If there exists $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semicontinuous convex normal fuzzy set of R (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then

$[a_\alpha\ ,\ b_\alpha]$ means the interval $(-\infty\ ,\ b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
x is called non-negative if $x(t)\ = 0,\ \forall t\ <\ 0$.
For any real number r, $\bar{r}$ is defined by $\bar{r}(t)\ = 1$ if $t =\ r$ and $\bar{r}(t)\ = 0$ if $t \neq\ r$.
Kaleva & Seikkala [8] (Felbin [5]) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$. A partial ordering " $\preceq$ " in E is defined by $\eta\ \preceq\ \delta$ if and only if $a^1_\alpha \le a^2_\alpha$ and $b^1_\alpha \le\ b^2_\alpha$ for all $\alpha\ \in\ (0\ ,\ 1]$ where $[\eta]_\alpha\ =\ [a^1_\alpha\ ,\ b^1_\alpha]$ and $[\delta]_\alpha\ =\ [a^2_\alpha\ ,\ b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha\ \in (0\ ,\ 1]$.

According to Mizumoto and Tanaka [11], the arithmetic operations $\oplus$, $\ominus$, $\odot$, $\oslash$ on $E \times E$ are defined by
$(x \oplus y)(t)\ =\ Sup_{s\in R}\ min\ \{x(s)\ ,\ y(t-s)\},\ \ t \in R,$
$(x \ominus y)(t)\ =\ Sup_{s\in R}\ min\ \{x(s)\ ,\ y(s-t)\},\ \ t \in R,$
$(x \odot y)(t)\ =\ Sup_{s\in R, s\neq 0}\ min\ \{x(s)\ ,\ y(\frac{t}{s})\},\ \ t \in R.$
$(\eta \oslash \delta)(t)\ =\ Sup_{s\in R}\ min\ \{\eta(st)\ ,\ \delta(s)\},\ \ t \in R.$

Definition 2.1. [5] The absolute value $|\eta|$ of $\eta\ \in F(R)$ is defined by

$$|\eta|(t) = \begin{cases} max(\eta(\text{t}), \eta(\text{-t})) & \text{if } t\ \ge 0 \\ 0 & \text{if } t\ < 0 \end{cases}$$

Lemma 2.2. *[8] Let $\eta, \gamma\ \in F(R)$ and $[\eta]_\alpha\ = [\eta^-_\alpha,\ \eta^+_\alpha],\ [\gamma]_\alpha\ = [\gamma^-_\alpha,\ \gamma^+_\alpha]\ \forall\ \alpha \in (\ 0,\ 1]$.*
Then (i)$[\eta \oplus \gamma]_\alpha\ = [\eta^-_\alpha + \gamma^-_\alpha,\ \eta^+_\alpha + \gamma^+_\alpha]$
(ii)$[\eta \ominus \gamma]_\alpha\ = [\eta^-_\alpha - \gamma^+_\alpha,\ \eta^+_\alpha - \gamma^-_\alpha]$
(iii)$[\eta \odot \gamma]_\alpha\ = [\eta^-_\alpha\gamma^-_\alpha,\ \eta^+_\alpha\gamma^+_\alpha]$ for $\eta, \gamma\ \in F^+(R)$
(iv)$[\bar{1} \oslash \eta]_\alpha\ =\ [\frac{1}{\eta^+_\alpha},\ \frac{1}{\eta^-_\alpha}]$ if $\eta^-_\alpha > 0$
(v)$[|\eta|]_\alpha\ = [max(\ 0,\ \eta^1_\alpha,\ \ -\eta^2_\alpha), max(\ |\eta^1_\alpha|,\ \ |\eta^2_\alpha|)]$

Definition 2.3. [5] Let X be a vector space over R.
Let $||\ ||\ : X \to R^*(I)$ and the mappings
$L, U\ : [0\ ,\ 1] \times [0\ ,\ 1] \to [0\ ,\ 1]$ be symmetric, nondecreasing in both arguments and satisfying $L(0\ ,\ 0) =\ 0$ and $U(1\ ,\ 1) =\ 1$.
Write $[||x||]_\alpha\ =\ [||x||^1_\alpha\ ,\ ||x||^2_\alpha]$ for $x \in X,\ 0\ < \alpha\ \le 1$ and suppose for all $x \in X$, $x \neq \underline{0}$, there exists $\alpha_0\ \in (0\ ,\ 1]$ independent of x such that for all $\alpha\ \le\ \alpha_0$,
(A) $||x||^2_\alpha\ <\ \infty$,
(B) $\inf||x||^1_\alpha\ >\ 0$.
The quadruple $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is called a fuzzy normed linear space and $||\ ||$ is a fuzzy norm if
(i) $||x||\ =\ \bar{0}$ if and only if $x\ =\ \underline{0}$ (the null vector),
(ii)$||rx||\ =\ |r|||x||,\ x \in X,\ r \in R$,
(iii) for all $x, y\ \in X$,
(a) whenever $s\ \le\ ||x||^1_1,\ t \le ||y||^1_1$ and $s + t\ \le ||x + y||^1_1$,
$||x + y||(s + t)\ \ge\ L(||x||(s)\ ,\ ||y||(t))$.
(b) whenever $s\ \ge\ ||x||^1_1,\ t \ge ||y||^1_1$ and $s + t\ \ge ||x + y||^1_1$,
$||x + y||(s + t)\ \le\ U(||x||(s)\ ,\ ||y||(t))$.

Remark 2.4. [5] For the case when $U =\ \bigvee(max)$ and $L =\ \bigwedge(min)$, then the condition (iii) is equivalent to
$||x + y||\ \preceq ||x|| \oplus ||y||$ and $||\ ||^i_\alpha\ :\ i = 1, 2$ are crisp norms on X and $(X\ ,\ ||\ ||,\ L\ ,\ U)$ is simply denoted as $(X\ ,\ ||\ ||)$.

Definition 2.5. [14] A mapping $\eta : R \to [0,\ 1]$ is called a fuzzy real number, whose α level set is denoted by
$[\eta]_\alpha = \{t\ :\ \eta(t) \ge \alpha\},\ 0 < \alpha \le 1$, if it satisfies two axioms:
(N1) There exists $t_0 \in R$ such that $\eta(t_0) = 1$.
(N2) each $\alpha \in (0,\ 1]$; $[\eta]_\alpha = [\eta^-_\alpha,\ \eta^+_\alpha]$,
where $-\infty,\ \eta_\alpha\ \le\ \ +\infty$.
The set of all fuzzy real numbers is denoted by F.

Since to each $r \in R$, one can consider $r \in F$ defined by $r(t) = 1$ if $t = r$ and $r(t) = 0$ if $t \neq r$, R can be embedded in F.

Lemma 2.6. *[14] $\eta \in F$ if and only if $\eta : R \to [0,\ 1]$ satisfies :*
1. η normal, convex and upper semicontinuous.
2. $lim_{t\to\infty}\eta(t) = 0$.

Definition 2.7. [14] Let $\eta \in F$. Then η is called a positive fuzzy real number if $\eta(t) = 0\ \forall t < 0$. The set of all positive fuzzy real numbers is denoted by F^+.

Definition 2.8. [7]Let X be a vector space over R. A fuzzy inner product on X is a mapping $< ., . >: X \times X \to F(R)$ (set of fuzzy real numbers) such that for all vectors $x,\ y,\ z \in X$ and all $r \in R$,
$(IP1)\langle x + y,\ z\rangle = \langle x,\ z\rangle \oplus \langle y,\ z\rangle;$
$(IP2)\langle rx,\ y\rangle = \bar{r} \odot \langle x,\ y\rangle;$
$(IP3)\langle x,\ y\rangle = \langle y,\ x\rangle;$
$(IP4)\langle x,\ x\rangle \succeq \bar{0};$
$(IP5)\ \inf_{\alpha\in(0\ ,\ 1]} \langle x,\ x\rangle^-_\alpha > 0$ if $x \neq \underline{0}$;
$(IP6)\langle x,\ x\rangle = \bar{0}$ if and only if $x = \underline{0}$.
The vector space X equipped with a fuzzy inner product is called a fuzzy inner product space.
A fuzzy inner product on X defines a fuzzy number
$||x|| = \sqrt{\langle x,\ x\rangle}, \qquad \forall x \in X.$
This is a well defined fuzzy norm.
A fuzzy Hilbert space is a complete fuzzy inner product space.

Definition 2.9. [2] Let $(X, ||\ ||)$ and $(Y, ||\ ||^*)$ be two fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. T is said to be strongly fuzzy bounded if there exists a real number $k > 0$ such that $||Tx||^* \oslash ||x|| \ \preceq\ \bar{k}\ \forall x(\neq \underline{0}) \in\ X$.

Proposition 2.10. *[2] Let $T : (X, ||\ ||_1) \ \to (Y, ||\ ||_2)$ be a strongly fuzzy bounded linear operator and $\{[||T||^{*1}_\alpha, ||T||^{*2}_\alpha]; \alpha \in (0,\ 1]\}$ be a family of nested bounded closed intervals of real numbers. Define a function $||T||^* : R \to [0,\ 1]$ by*
$||T||^*(t) = \vee\{\alpha\ \in\ (0,\ 1]\ :\ t \in [||T||^{*1}_\alpha, ||T||^{*2}_\alpha]\}$.
Then $||T||^*$ is a fuzzy real number (fuzzy interval) and it is the fuzzy norm of T.

Theorem 2.11. *[7] Let Y be any subspace of a fuzzy inner product space X such that the normed spaces $(Y,\ ||\ ||^-_\alpha)$ are complete, for all $\alpha \in (0,\ 1]$. Then $X = Y \oplus Z$ where $Z = Y^\perp$.*

Lemma 2.12. *[13] Let $(X,\ ||\ ||)$ be a fuzzy normed linear space. If f is a strongly fuzzy continuous mapping on X, then $N(f)$ is a fuzzy closed subspace of X.*

Theorem 2.13. *[13][Riesz]*
Let H be a fuzzy Hilbert space and H^* be its first strong fuzzy dual space. Then for any $f \in H^*$ satisfying $N(f) = \{x \in H : f(x) = 0\}$ is complete w.r.t. $||\ ||^-_\alpha$, f can be represented as $||f(x)|| =< x, y >\ \forall x \in H$, where y is unique and depends on f such that $||f||^+_\alpha = ||y||^2_\alpha\ \forall\ \alpha \in (\ 0,\ 1]$, where $[||f||]_\alpha\ = [\ ||f||^-_\alpha,\ ||f||^+_\alpha]$, and $[||y||]_\alpha\ = [\ ||y||^1_\alpha,\ ||y||^2_\alpha], \forall\ \alpha \in (\ 0,\ 1]$.

3 α-Inner products and Modified Riesz theorem

In this section, we show that α-level sets of fuzzy inner products are crisp inner products and Riesz Theorem is introduced by Mukherjee and Bag[13] has been modified.

Theorem 3.1. *Let X be a vector space over R and $< .,\ . >: X \times X \to F(R)$ be a fuzzy inner product (Hasankhani type). Let $[< x,\ y >]_\alpha = [< x,\ y >^1_\alpha, < x,\ y >^2_\alpha] \forall\ \alpha \in (\ 0,\ 1]$. Then $\{< .,\ . >^1_\alpha\!:\ \alpha \in (\ 0,\ 1]\}$ and $\{< .,\ . >^2_\alpha\!:\ \alpha \in (\ 0,\ 1]\}$ are families of crisp inner products from $X \times X \to R$.*

Proof. We have $||x|| = \sqrt{< x,\ x >}$ and $[< x,\ y >]_\alpha = [< x,\ y >^1_\alpha, < x,\ y >^2_\alpha] \forall\ \alpha \in (\ 0,\ 1]$.
Now $< x,\ y > = < y,\ x >$
$\Rightarrow < x,\ y >^1_\alpha = < y,\ x >^1_\alpha,\ \ < x,\ y >^2_\alpha = < y,\ x >^2_\alpha$
Again $< x + y,\ z > = < x,\ z > \oplus < y,\ z >$
$\Rightarrow < x + y,\ z >^1_\alpha = < x,\ z >^1_\alpha + < y,\ z >^1_\alpha$ and $< x + y,\ z >^2_\alpha = < x,\ z >^2_\alpha + < y,\ z >^2_\alpha$
Now $< rx,\ y > = \bar{r} \odot < x,\ y >$
$\Rightarrow < rx,\ y >^1_\alpha = r < x,\ y >^1_\alpha$ and $< rx,\ y >^2_\alpha = r < x,\ y >^2_\alpha$
Also $< x,\ x > \succeq \bar{0}$
$\Rightarrow < x,\ x >^1_\alpha \geq 0$ and $< x,\ x >^2_\alpha \geq 0$
Let $x = \underline{0}$ then $< x,\ x > = \bar{0}$
$\Rightarrow < x,\ x >^1_\alpha = 0$ and $< x,\ x >^2_\alpha = 0\ \ \forall\ \alpha \in (\ 0,\ 1]$
Choose $\alpha \in (\ 0,\ 1]$ arbitrary.
Now $< x,\ x >^1_\alpha = 0$
$\Rightarrow \inf\limits_{\alpha \in (0\ ,\ 1]} < x,\ x >^1_\alpha = 0$
$\Rightarrow x = \underline{0}$ by (IP5)
Also $< x,\ x >^2_\alpha = 0$
$\Rightarrow < x,\ x >^1_\alpha = 0$
$\Rightarrow x = \underline{0}$
Thus $< x,\ y >^1_\alpha$ and $< x,\ y >^2_\alpha$ are both crisp inner products on $X \times X$ and $\ \forall\ \alpha \in (\ 0,\ 1]$
. □

Remark 3.2. $\{< .,\ . >^1_\alpha\!:\ \ \alpha \in (\ 0,\ 1]\}$ and $\{< .,\ . >^2_\alpha\!:\ \ \alpha \in (\ 0,\ 1]\}$ are increasing and decreasing families of crisp inner products respectively.

Proof. Proof is obvious. □

Theorem 3.3. *[Riesz] Let H be a fuzzy Hilbert space and H^* be its first strong fuzzy dual space. Then for any $f \in H^*$ satisfying $N(f) = \{x \in H : f(x) = 0\}$ is complete w.r.t. $||\ ||^-_\alpha$, f can be represented as $||f(x)|| = < x, y > \ \ \forall x \in H$, where y is unique and depends on f such that $||f|| = ||y||$.*

Proof. Without loss of generality we may suppose that $f \neq 0$.
Note that $N(f) = \{x \in H : f(x) = 0\}$.
Since f is strongly fuzzy continuous, so by Lemma 2.12, $N(f)$ is a fuzzy closed subspace of H.
Again since $f \neq 0$ thus $N(f) \neq H$.
So by Theorem 2.11, $\exists z_0 \in H$ such that $z_0 \perp N(f)$.
Let $w = \frac{f(x)}{f(z_0)} z_0$.
Then $f(x - w) = f(x - \frac{f(x)}{f(z_0)} z_0) = 0$.
i. e. $x - w \in N(f)$.
So $z_0 \perp (x - w)$ i. e $\langle x - w,\ z_0 \rangle = \bar{0}$
i. e $\langle x,\ z_0 \rangle \ominus \langle w,\ z_0 \rangle = \bar{0}$
$\Rightarrow \langle x,\ z_0 \rangle^-_\alpha - \langle w,\ z_0 \rangle^+_\alpha = 0$ and

$\langle x,\ z_0\rangle_\alpha^+ - \langle w,\ z_0\rangle_\alpha^- = 0 \quad \forall\alpha \in (0,\ 1]$.
Now $\langle x,\ z_0\rangle_\alpha^- - \langle w,\ z_0\rangle_\alpha^+ = 0 \quad \forall\alpha \in (0,\ 1]$
$\Rightarrow \langle x,\ z_0\rangle_\alpha^- - \langle \frac{f(x)}{f(z_0)} z_0,\ z_0\rangle_\alpha^+ = 0$
$\Rightarrow \langle x,\ z_0\rangle_\alpha^- - \frac{f(x)}{f(z_0)} \langle z_0,\ z_0\rangle_\alpha^+ = 0$
$\Rightarrow f(x) = \frac{f(z_0)}{\langle z_0,\ z_0\rangle_\alpha^+} \langle x,\ z_0\rangle_\alpha^-$ (3.3.1).

Again from the relation $\langle x,\ z_0\rangle_\alpha^+ - \langle w,\ z_0\rangle_\alpha^- = 0 \quad \forall\alpha \in (0,\ 1]$,
we get $f(x) = \frac{f(z_0)}{\langle z_0,\ z_0\rangle_\alpha^-} \langle x,\ z_0\rangle_\alpha^+$ (3.3.2).

From (3.3.1) and (3.3.2) we have
$f(x) = \frac{f(z_0)}{||z_0||_*^2} \langle x,\ z_0\rangle^* \quad \forall x \in H$ (since L. H. S of 3.3.1 and 3.3.2 are independent on α),
where $\langle\ ,\ \rangle^*$, $||\ ||_*$ are crisp inner product and crisp norm respectively and
$\langle x,\ z_0\rangle_\alpha^+ = \langle x,\ z_0\rangle_\alpha^- = \langle x,\ z_0\rangle^*$,
$\langle z_0,\ z_0\rangle_\alpha^+ = \langle z_0,\ z_0\rangle_\alpha^- = ||z_0||_*^2$
Therefore $||f(x)|| = \overline{|f(x)|} = \langle x,\ y\rangle$, where $y = \frac{|f(z_0)|z_0}{||z_0||_*^2}$. For uniqueness, if possible suppose

that $y_1 (\neq y)$ such that
$\langle x,\ y\rangle = \langle x,\ y_1\rangle \quad \forall x \in H$
i. e $\langle x,\ y\rangle_\alpha^- = \langle x,\ y_1\rangle_\alpha^-$ and $\langle x,\ y\rangle_\alpha^+ = \langle x,\ y_1\rangle_\alpha^+ \quad \forall\alpha \in (0,\ 1],\ \forall x \in H$
i. e $\langle x,\ y - y_1\rangle_\alpha^- = \langle x,\ y - y_1\rangle_\alpha^+ = 0 \quad \forall\alpha \in (0,\ 1],\ \forall x \in H$
$\Rightarrow y - y_1 = 0$
$\Rightarrow y = y_1$. Finally we have to show that $||f||_\alpha^+ = ||y||_\alpha^+ \quad \forall\alpha \in (0,\ 1]$.

Note that $||f||(t) = \bigvee\{\alpha \in (0,\ 1] : t \in [||f||_\alpha^{*1},\ ||f||_\alpha^{*2}]\}$
where $||f||_\alpha^{*1} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{||f(x)||_\alpha^1}{||x||_\alpha^2}$ and $||f||_\alpha^{*2} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{||f(x)||_\alpha^2}{||x||_\alpha^1}$.
Recall that $[||f||_\alpha^{*1},\ ||f||_\alpha^{*2}] \subset [||f||_\alpha^-,\ ||f||_\alpha^+]$ and
for $\beta < \alpha$, $[||f||_\alpha^-,\ ||f||_\alpha^+] \subset [||f||_\beta^{*1},\ ||f||_\beta^{*2}]$.
Now $||f||_\alpha^+ \leq ||f||_\beta^{*2} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{||f(x)||_\beta^2}{||x||_\beta^1} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{|f(x)|}{||x||_\beta^1} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{< x,\ y >_\beta^1}{||x||_\beta^1}$.
i. e. $||f||_\alpha^+ \leq \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{||x||_\beta^1 ||y||_\beta^1}{||x||_\beta^1} = ||y||_\beta^1 \leq ||y||_\beta^2 \quad \forall\beta < \alpha$.
i. e. $||f||_\alpha^+ \leq \inf\limits_{\beta < \alpha} ||y||_\beta^2$.
i. e. $||f||_\alpha^+ \leq ||y||_\alpha^2$ (3.3.3).

Again we have $||f(x)|| = \langle x,\ y\rangle \quad \forall x \in H$.
Taking $x = y$ we get $||f(y)|| = \langle y,\ y\rangle = ||y||^2$.
i. e. $||f(y)||_\alpha^2 = (||y||_\alpha^2)^2$ and $||f(y)||_\alpha^1 = (||y||_\alpha^1)^2 \quad \forall\alpha \in (0,\ 1]$.
Now $(||y||_\alpha^2)^2 = ||f(y)||_\alpha^2 \leq ||f||_\alpha^{*2} ||y||_\alpha^1 \leq ||f||_\alpha^+ ||y||_\alpha^2$
$\Rightarrow ||f||_\alpha^+ \geq ||y||_\alpha^2 \quad \forall\alpha \in (0,\ 1]$ (3.3.4).

From (3.3.3) and (3.3.4), we have $||f||_\alpha^+ = ||y||_\alpha^2 \quad \forall\alpha \in (0,\ 1]$.

Now $||f||_\alpha^- \leq ||f||_\alpha^{*1} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{||f(x)||_\alpha^1}{||x||_\alpha^2} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{|f(x)|}{||x||_\alpha^2} \leq \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{|f(x)|}{||x||_\alpha^1} = \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{< x,\ y >_\alpha^1}{||x||_\alpha^1}$.
i. e. $||f||_\alpha^- \leq \sup\limits_{x \in H_{x \neq \underline{0}}} \dfrac{||x||_\alpha^1 ||y||_\alpha^1}{||x||_\alpha^1} = ||y||_\alpha^1$
i. e. $||f||_\alpha^- \leq ||y||_\alpha^1$ (3.3.5).

Now $||f||^{*1}_{\beta} \leq ||f||^{-}_{\alpha} \quad \forall \beta \leq \alpha, \forall \alpha, \beta \in (0,\ 1]$.

$\Rightarrow \sup_{x \in H_{x \neq \underline{0}}} \dfrac{||f(x)||^{1}_{\beta}}{||x||^{2}_{\beta}} \leq ||f||^{-}_{\alpha}$

$\Rightarrow \frac{||f(x)||^{1}_{\beta}}{||x||^{2}_{\beta}} \leq ||f||^{-}_{\alpha} \forall x \in H, x \neq \underline{0}$

So $\frac{|f(x)|}{||x||^{2}_{\beta}} \leq ||f||^{-}_{\alpha}$

$\Rightarrow \frac{\langle x,\ y \rangle^{2}_{\beta}}{||x||^{2}_{\beta}} \leq ||f||^{-}_{\alpha} \forall x \in H, x \neq \underline{0}$

Take $x = y$, then $\frac{\langle y,\ y \rangle^{2}_{\beta}}{||y||^{2}_{\beta}} \leq ||f||^{-}_{\alpha}$

$\Rightarrow \frac{||y||^{2}_{\beta}||y||^{2}_{\beta}}{||y||^{2}_{\beta}} \leq ||f||^{-}_{\alpha}$

$\Rightarrow ||y||^{2}_{\beta} \leq ||f||^{-}_{\alpha} \quad \forall \beta \leq \alpha, \forall \alpha, \beta \in (0,\ 1]$.

$\Rightarrow \inf_{\beta \leq \alpha} ||y||^{2}_{\beta} \leq ||f||^{-}_{\alpha}$

$\Rightarrow ||y||^{1}_{\alpha} \leq ||y||^{2}_{\alpha} \leq ||f||^{-}_{\alpha}$ (3.3.6).

Therefore from (3.3.5) and (3.3.6), we have $||f|| = ||y||$.

□

4 Fuzzy sesquilinear form

In this section, concept of fuzzy sesquilinear form is introduced and some properties are studied.

Definition 4.1. Let X and Y be vector spaces over the field R. Then a fuzzy sesquilinear form h on $X \times Y$ is a mapping $h\ :\ X \times Y \to F(R)$ such that for all $x,\ x_1,\ x_2 \in X$ and $y,\ y_1,\ y_2 \in Y$ and all scalars $\alpha,\ \beta$ the following conditions hold:
(a) $h(x_1 + x_2,\ y) = h(x_1,\ y) \oplus h(x_2,\ y)$
(b) $h(x,\ y_1 + y_2) = h(x,\ y_1) \oplus h(x,\ y_2)$
(c) $h(\alpha x,\ y) = \bar{\alpha} \odot h(x,\ y)$
(d) $h(x,\ \beta y) = \bar{\beta} \odot h(x,\ y)$
$\bar{\alpha}$ denotes the fuzzy real number corresponding to α.

Example 4.2. Let $(X,\ \langle\ ,\ \rangle)$ be a fuzzy inner product. Then $\langle\ ,\ \rangle$ is a fuzzy sesquilinear form on $X \times X$.

Proof. Let $x, y, z \in X$ and α, β are scalars.
Then (i) $\langle x + y,\ z \rangle = \langle x,\ z \rangle \oplus \langle y,\ z \rangle$
(ii) $\langle x,\ y + z \rangle = \langle y + z,\ x \rangle$
$= \langle y,\ x \rangle \oplus \langle z,\ x \rangle$
$= \langle x,\ y \rangle \oplus \langle x,\ z \rangle$
(iii) $\langle \alpha x,\ y \rangle = \bar{\alpha} \odot \langle x,\ y \rangle$
(iv) $\langle x,\ \beta y \rangle = \langle \beta y,\ x \rangle$
$= \bar{\beta} \odot \langle y,\ x \rangle$
$= \bar{\beta} \odot \langle x,\ y \rangle$
Hence from (i) to (iv), $(X,\ \langle\ ,\ \rangle)$ is a fuzzy sesquilinear form.

□

Example 4.3. Let X and Y be two vector spaces over the field R of real numbers and f be a real sesquilinear form on $X \times Y$. Define $h\ :\ X \times Y \to F(R)$ by

$$h(\ x,\ y)(t) = \begin{cases} \frac{f(x,\ y)}{t} & \text{if } 0\ < f(x,\ y) \leq\ t \\ 0 & \text{otherwise} \end{cases}$$

Then h is a fuzzy sesquilinear form on $X \times Y$

Proof. Let $\alpha \in (0,\ 1]$ then $h(x,\ y)(t) \geq \alpha$
$\Rightarrow f(x,\ y) \geq t\alpha \quad \Rightarrow t \leq \frac{h(x,\ y)}{\alpha}$
Therefore $[h(x,\ y)]_\alpha = [f(x,\ y),\ \frac{h(x,\ y)}{\alpha}] \quad \forall \alpha \in (0,\ 1]$.
Now for $f(x,\ y) < 0$ or $t < 0$ the proof is obvious.
So let $f(x,\ y) \geq 0$ and $t \geq 0$.
Let $x_1, x_2 \in X; y_1, y_2 \in Y$ and $a,\ b$ are real numbers.
Then (i) $h^1_\alpha(x_1 + x_2,\ y_1) = f(x_1 + x_2,\ y_1)$
$= f(x_1,\ y_1) + f(x_2,\ y_1)$
$= h^1_\alpha(x_1,\ y_1) + h^1_\alpha(x_2,\ y_1)$.....(a)
$h^2_\alpha(x_1 + x_2,\ y_1) = \frac{f(x_1+x_2,\ y_1)}{\alpha}$
$= \frac{f(x_1,\ y_1)}{\alpha} + \frac{f(x_2,\ y_1)}{\alpha}$
$= h^2_\alpha(x_1,\ y_1) + h^2_\alpha(x_2,\ y_1)$.....(b)
Then from (a) and (b) $h(x_1 + x_2,\ y_1) = h(x_1,\ y_1) \oplus h(x_2,\ y_1)$.
(ii) $h^1_\alpha(\ x_1,\ y_1 + y_2) = f(x_1,\ y_1 + y_2)$
$= f(x_1,\ y_1) + f(x_1,\ y_2)$
$= h^1_\alpha(x_1,\ y_1) + h^1_\alpha(\ x_1,\ y_2)$
Now $h^2_\alpha(\ x_1,\ y_1 + y_2) = \frac{f(x_1,\ y_1+y_2)}{\alpha}$
$= \frac{f(x_1,\ y_1)}{\alpha} + \frac{f(x_1,\ y_2)}{\alpha}$
$= h^2_\alpha(x_1,\ y_1) + h^2_\alpha(\ x_1,\ y_2)$
Therefore $h(x_1,\ y_1 + y_2) = h(x_1,\ y_1) \oplus h(x_1,\ y_2)$.
(iii)$h^1_\alpha(ax_1,\ y_1) = f(ax_1,\ y_1)$
$= af(x_1,\ y_1)$
$= ah^1_\alpha(x_1,\ y_1)$
$h^2_\alpha(ax_1,\ y_1) = \frac{f(ax_1,\ y_1)}{\alpha}$
$= \frac{af(x_1,\ y_1)}{\alpha}$
$= ah^2_\alpha(x_1,\ y_1)$
So $h(ax_1,\ y_1) = \bar{a} \odot h(x_1,\ y_1)$.
(iv) $h^1_\alpha(\ x_1,\ by_1) = f(x_1,\ by_1)$
$= bf(x_1,\ y_1)$
$= bh^1_\alpha(x_1,\ y_1)$
$h^2_\alpha(\ x_1,\ by_1) = \frac{f(x_1,\ by_1)}{\alpha}$
$= \frac{bf(x_1,\ y_1)}{\alpha}$
$= bh^2_\alpha(x_1,\ y_1)$
Therefore $h(x_1,\ by_1) = \bar{b} \odot h(x_1,\ y_1)$.
Hence from (i) to (iv), h is a fuzzy sesquilinear form.

□

5 Norm of fuzzy sesquilinear form

In this Section notion of norm of fuzzy sesquilinear form is introduced and Riesz representation theorem for sesquilinear form is established.

Definition 5.1. Let h be a fuzzy sesquilinear form on $X \times Y$, where X and Y are real fuzzy normed linear spaces. h is said to be bounded if $\exists$ a real number k such that
$|h(x,\ y)| \oslash (||x|| \odot ||y||) \preceq \bar{k},\ \forall\ (x,\ y) \in X \times Y - \{(0,\ 0)\}$
Here $[|h(x,\ y)|]_\alpha = [\ max\ \{0, h^1_\alpha(x,\ y), -h^2_\alpha(x,\ y)\},\ max\ \{|h^1_\alpha(x,\ y)|, |h^2_\alpha(x,\ y)|\}]$
$\forall\ \alpha \in (0,\ 1]$
Let h be a bounded sesquilinear form on $X \times Y$. Then $\exists\ k \in R$ such that
$|h(x,\ y)| \oslash (||x|| \odot ||y||) \preceq \bar{k},\ \forall\ (x,\ y) \in X \times Y - \{(0,\ 0)\}$
Let $A = max\ \{0, h^1_\alpha(x,\ y), -h^2_\alpha(x,\ y)\}$ and $B = max\ \{|h^1_\alpha(x,\ y)|, |h^2_\alpha(x,\ y)|\}$.
Then $\frac{A}{||x||^2_\alpha ||y||^2_\alpha} \leq k$
and $\frac{B}{||x||^1_\alpha ||y||^1_\alpha} \leq k\ \forall\ \alpha \in (0,\ 1]$

Define $||h||_\alpha^{*1} = \bigvee_{(x,\ y)\in X\times Y-\{(0,0)\}} \frac{A}{||x||_\alpha^2\, ||y||_\alpha^2}$
and $||h||_\alpha^{*2} = \bigvee_{(x,\ y)\in X\times Y-\{(0,0)\}} \frac{B}{||x||_\alpha^1\, ||y||_\alpha^1}$

Lemma 5.2. *Let $\eta \in F(R)$, then $|\eta|(t) = |-\eta|(t)\ \forall t \in R$.*

Proof. Let η be a fuzzy real number, so $\eta \in F(R)$.
Then $|\eta|(t) = max\ \{\ \eta(t),\ \eta(-t)\}\ if\ t \geq 0$
$\qquad = 0 \qquad\qquad otherwise$
Let $[|\eta|]_\alpha\ =\ [\ |\eta|_\alpha^1,\ |\eta|_\alpha^2]\ \ \forall\ \alpha\ \in\ (\ 0,\ 1]$.
Choose a fixed $\alpha_0\ \in\ (\ 0,\ 1]$ and let $|\eta|_{\alpha_0}^1\ =\ a,\ |\eta|_{\alpha_0}^2\ =\ b.$
Then either $a\ >\ 0,\ b\ >\ 0$ or $a\ =\ 0,\ b\ >\ 0$
In the trivial case when $a = 0,\ b = 0,\ \eta = \bar{0}$ proof is obvious.
Now let $A_\eta = max\ \{\ 0, |\eta|_{\alpha_0}^1,\ -|\eta|_{\alpha_0}^2\}$
$\qquad = max\ \{\ 0,\ a,\ -b\}\ =\ |\eta|_{\alpha_0}^1$
and $B_\eta = max\ \{\ |\eta|_{\alpha_0}^1,\ |\eta|_{\alpha_0}^2\}$
$\qquad = max\ \{\ |a|,\ |b|\}\ =\ |\eta|_{\alpha_0}^2.$
Case I. When $a\ > 0,\ b\ > 0.$
$A_\eta\ =\ max\ \{\ 0,\ a,\ -b\}\ =\ a,\quad A_{-\eta}\ =\ max\ \{\ 0,\ -b,\ a\}\ =\ a$
and $B_\eta\ =\ max\ \{\ |a|,\ |b|\}\ =\ b,\quad B_{-\eta}\ =\ max\ \{\ |-b|,\ |-a|\}\ =\ b$
Case II. When $a\ = 0,\ b\ > 0.$
$A_\eta\ =\ max\ \{\ 0,\ 0,\ -b\}\ =\ 0,\quad A_{-\eta}\ =\ max\ \{\ 0,\ -b,\ 0\}\ =\ 0$
and $B_\eta\ =\ max\ \{\ 0,\ |b|\}\ =\ b,\quad B_{-\eta}\ =\ max\ \{\ |-b|,\ 0\}\ =\ b$
Thus $A_\eta\ =\ A_{-\eta}$ and $B_\eta\ =\ B_{-\eta}$
Hence $[|\eta|]_{\alpha_0}\ =\ [|-\eta|]_{\alpha_0}$ for $\alpha_0\ \in\ (\ 0,\ 1]$. Since $\alpha_0\ \in\ (\ 0,\ 1]$ is arbitrary,
thus $|\eta|(t) = |-\eta|(t)\ \forall t \in R$.

□

Theorem 5.3. *Let h be a bounded fuzzy sesquilinear form on $X\times Y$ such that $h_\alpha^1(x,\ y).h_\alpha^2(x,\ y) \geq 0\ \ \forall\ \alpha \in (0,\ 1]$ and $\forall\ (x,\ y) \in X \times Y$, where X and Y are real fuzzy normed linear spaces. Then $\{\ ||h||_\alpha^{*1};\ \alpha \in (0,\ 1]\ \}$ forms a family of norms.*

Proof. Let $\alpha \in (0,\ 1]$.
Now we show that $||h||_\alpha^{*1}$ is a norm .
From definition it is clear that $||h||_\alpha^{*1}\ \geq 0$
Let $h = 0$, then $||h||_\alpha^{*1} = 0$
Conversely, let $||h||_\alpha^{*1} = 0$
Then $max\ \{0, h_\alpha^1(x,\ y), -h_\alpha^2(x,\ y)\} = 0$
$\Rightarrow h_\alpha^1(x,\ y) \leq 0 \leq h_\alpha^2(x,\ y)\ \ \Rightarrow\ h_\alpha^1(x,\ y).h_\alpha^2(x,\ y) \leq 0$
But since $h_\alpha^1(x,\ y).h_\alpha^2(x,\ y) > 0$ or $h = \bar{0}$.
Therefore $h = \bar{0}$.
Now let $\lambda = \lambda(x,\ y) = \bar{\lambda},\quad \lambda$ being a positive scalar.
Then $||\lambda\ h||_\alpha^{*1} = \bigvee_{(x,\ y)\in X\times Y-\{(0,0)\}} \frac{max\{0,\lambda h_\alpha^1(x,\ y),-\lambda h_\alpha^2(x,\ y)\}}{||x||_\alpha^2\,||y||_\alpha^2}$
$= |\lambda| \bigvee_{(x,\ y)\in X\times Y-\{(0,0)\}} \frac{max\{0,h_\alpha^1(x,\ y),-h_\alpha^2(x,\ y)\}}{||x||_\alpha^2\,||y||_\alpha^2}$
$= |\lambda|\ ||h||_\alpha^{*1}$
Thus $||\lambda\ h||_\alpha^{*1}\ =\ |\lambda|\ ||h||_\alpha^{*1}$ when $\lambda\ \geq\ 0.$
Now when $\lambda\ <\ 0$, let $p\ =\ -\lambda$, then $p\ >\ 0.$
Therefore $||\ p\ h||_\alpha^{*1}\ =\ |\ p\ |\ ||h||_\alpha^{*1}$
$\Rightarrow ||\ (-\lambda)\ h||_\alpha^{*1}\ =\ |\ -\lambda|\ ||h||_\alpha^{*1}$
$\Rightarrow ||\ -(\lambda\ h\ )||_\alpha^{*1}\ =\ |\ \lambda|\ ||h||_\alpha^{*1}$
$\Rightarrow ||\ \lambda\ h||_\alpha^{*1}\ =\ |\ \lambda|\ ||h||_\alpha^{*1}$ [From lemma 4.1]
Hence $||\ \lambda\ h||_\alpha^{*1}\ =\ |\ \lambda|\ ||h||_\alpha^{*1}\ \ \forall\ \lambda\ \in\ R$ and $h\ \in\ F(R).$
Thus $||\ \lambda\ h||_\alpha^{*1}\ =\ |\ \lambda|\ ||h||_\alpha^{*1}$ ‘for all scalar λ.
Let h_1, h_2 be two bounded fuzzy sesquilinear form on $X\ \times\ Y$
and let $[h_1]_\alpha = [a_\alpha,\ b_\alpha], [h_2]_\alpha = [c_\alpha,\ d_\alpha]$ where $a_\alpha, b_\alpha, c_\alpha, d_\alpha$ are reals with $a_\alpha.b_\alpha < 0$ or$a_\alpha =$

$b_\alpha = 0$ and $c_\alpha.d_\alpha < 0$ or $c_\alpha = d_\alpha = 0$.
Let $A = max\ \{0, (h_1+h_2)^1_\alpha, -(h_1+h_2)^2_\alpha\} = max\ \{0, a_\alpha + c_\alpha, -(b_\alpha + d_\alpha)\}$
and $B = max\ \{0, a_\alpha, -b_\alpha\} + max\ \{0, c_\alpha, -d_\alpha\}$
Case I : Let $a_\alpha < 0 < -b_\alpha$
Then $-d_\alpha < 0 < c_\alpha \Rightarrow B = -b_\alpha + c_\alpha$, then $B > 0, B > a_\alpha + c_\alpha, B > -b_\alpha - d_\alpha \Rightarrow A < B$.
$c_\alpha < 0 < -d_\alpha \Rightarrow B = -b_\alpha - d_\alpha$, then $B > 0, B > a_\alpha + c_\alpha, B = -b_\alpha - d_\alpha \Rightarrow A \leq B$.
Case II : Let $-b_\alpha < 0 < a_\alpha$
Then $-d_\alpha < 0 < c_\alpha \Rightarrow B = a_\alpha + c_\alpha$, then $B > 0, B = a_\alpha + c_\alpha, B > -b_\alpha - d_\alpha \Rightarrow A \leq B$.
$c_\alpha < 0 < -d_\alpha \Rightarrow B = a_\alpha - d_\alpha$, then $B > 0, B > a_\alpha + c_\alpha, B > -b_\alpha - d_\alpha \Rightarrow A < B$.
Also when $a_\alpha = b_\alpha = 0$ or $c_\alpha = d_\alpha = 0$ we can get $A \leq B$.
Thus in all the cases we have $A \leq B$.
Thus we get $||h_1 + h_2||^{*1}_\alpha \leq ||h_1||^{*1}_\alpha + ||h_2||^{*1}_\alpha$
Hence we have $||h||^{*1}_\alpha$ is a norm .
Since $\alpha \in (0,\ 1]$ is arbitrary, $\{\ ||h||^{*1}_\alpha;\ \alpha \in (0,\ 1]\ \}$ forms a family of norms.

□

Theorem 5.4. *Let h be a bounded fuzzy sesquilinear form on $X \times Y$, where X and Y are real fuzzy normed linear spaces.Then $\{\ ||h||^{*2}_\alpha;\ \alpha \in (0,\ 1]\ \}$ forms a family of norms.*

Proof. Let $\alpha \in (0,\ 1]$.
Now we show that $||h||^{*2}_\alpha$ is a norm .
From definition it is clear that $||h||^{*2}_\alpha \geq 0\ \ \forall (x,\ y) \in X \times X - \{(0,0)\}$
Let $h = 0$, then $||h||^{*2}_\alpha = 0$
Conversely, let $||h||^{*2}_\alpha = 0$
Then $max\ \{\ |h^1_\alpha(x,\ y)|,\ |h^2_\alpha(x,\ y)|\ \} = 0$
$\Rightarrow h^1_\alpha(x,\ y) = h^2_\alpha(x,\ y) = 0\ \ \Rightarrow h = 0$
Now let $\lambda = \lambda(x,\ y) = \bar{\lambda}\ \ \ \lambda$ being a scalar.
Then $||\lambda \odot h||^{*2}_\alpha = \bigvee_{(x,\ y) \in X \times Y - \{(0,0)\}} \frac{max\{|\lambda h^1_\alpha(x,\ y)|, |\lambda h^2_\alpha(x,\ y)|\}}{||x||^1_\alpha ||y||^1_\alpha}$
$= |\lambda| \bigvee_{(x,\ y) \in X \times Y - \{(0,0)\}} \frac{max\{|h^1_\alpha(x,\ y)|, |h^2_\alpha(x,\ y)|\}}{||x||^1_\alpha ||y||^1_\alpha}$
$= |\lambda|\ ||h||^{*2}_\alpha$
Let h_1, h_2 be two bounded fuzzy sesquilinear form on $X \times Y$.
and let $[h_1]_\alpha = [a_\alpha,\ b_\alpha], [h_2]_\alpha = [c_\alpha,\ d_\alpha]$
Case I : Let $|a_\alpha| + |c_\alpha| > |b_\alpha| + |d_\alpha|$
$\Rightarrow (|a_\alpha| - |b_\alpha|) + (|c_\alpha| - |d_\alpha|) > 0$
Then (a) $|a_\alpha| > |b_\alpha|\ and\ |c_\alpha| > |d_\alpha|$
(b) $|a_\alpha| > |b_\alpha|\ and\ |c_\alpha| < |d_\alpha|$
(c) $|a_\alpha| < |b_\alpha|\ and\ |c_\alpha| > |d_\alpha|$
Now for I(a) we have $max\ \{\ |a_\alpha + c_\alpha|,\ |b_\alpha + d_\alpha|\} \leq max\ \{\ |a_\alpha| + |c_\alpha|,\ |b_\alpha| + |d_\alpha|\}$
$= |a_\alpha| + |c_\alpha| = max\ \{\ |a_\alpha|,\ |b_\alpha\} + max\ \{\ |c_\alpha|,\ |d_\alpha|\}$
For I(b) we have $max\ \{\ |a_\alpha + c_\alpha|,\ |b_\alpha + d_\alpha|\} \leq max\ \{\ |a_\alpha| + |c_\alpha|,\ |b_\alpha| + |d_\alpha|\}$
$= |a_\alpha| + |c_\alpha| < |a_\alpha| + |d_\alpha| = max\ \{\ |a_\alpha|,\ |b_\alpha|\} + max\ \{\ |c_\alpha|,\ |d_\alpha|\}$
For I(c) we have $max\ \{\ |a_\alpha + c_\alpha|,\ |b_\alpha + d_\alpha|\} \leq max\ \{\ |a_\alpha| + |c_\alpha|,\ |b_\alpha| + |d_\alpha|\}$
$= |a_\alpha| + |c_\alpha| < |b_\alpha| + |c_\alpha| = max\{|a_\alpha|,\ |b_\alpha|\} + max\{|c_\alpha|,\ |d_\alpha|\}$
Similarly by interchanging $a_\alpha\ and\ b_\alpha; c_\alpha\ and\ d_\alpha$ we can get case II.
Thus in all cases we have $max\{|a_\alpha + c_\alpha|,\ |b_\alpha + d_\alpha|\} \leq max\{|a_\alpha|,\ |b_\alpha|\} + max\{|c_\alpha|,\ |d_\alpha|\}$.
Thus we get $||h_1 + h_2||^{*2}_\alpha \leq ||h_1||^{*2}_\alpha + ||h_2||^{*2}_\alpha$
Hence we have $||h||^{*2}_\alpha$ is a norm.
Since $\alpha \in (0,\ 1]$ is arbitrary, $\{||h||^{*2}_\alpha; \alpha \in (0,\ 1]\}$ forms a family of norms.

□

Definition 5.5. For $\alpha \leq \beta;\ \alpha, \beta \in (0,\ 1]$ we have,
$|h(x,\ y)|^1_\alpha \leq |h(x,\ y)|^1_\beta$ and $||x||^2_\alpha ||y||^2_\alpha \geq ||x||^2_\beta ||y||^2_\beta\ \forall (x,\ y) \in X \times Y - \{(0,0)\}$ and h being a bounded sesquilinear form such that $h^1_\alpha(x,\ y).h^2_\alpha(x,\ y) > 0$ or $h = \bar{o}\ \ \forall \alpha \in (0,\ 1]$.
Then $\frac{|h(x,\ y)|^1_\alpha}{||x||^2_\alpha ||y||^2_\alpha} \leq \frac{|h(x,\ y)|^1_\beta}{||x||^2_\beta ||y||^2_\beta} \Rightarrow \bigvee_{(x,\ y) \in X \times Y - \{(0,0)\}} \frac{|h(x,\ y)|^1_\alpha}{||x||^2_\alpha ||y||^2_\alpha} \leq \bigvee_{(x,\ y) \in X \times Y - \{(0,0)\}} \frac{|h(x,\ y)|^1_\beta}{||x||^2_\beta ||y||^2_\beta}$
$\Rightarrow ||h||^{*1}_\alpha \leq ||h||^{*1}_\beta$ Thus $\{||h||^{*1}_\alpha; \alpha \in (0,\ 1]\}$ forms an ascending family of norms.

Similarly we can show that $\{||h||_\alpha^{*2}; \alpha \in (0,\ 1]\}$ forms an descending family of norms.
Therefore $\{[||h||_\alpha^{*1}, ||h||_\alpha^{*2}]; \alpha \in (0,\ 1]\}$ is a family of nested bounded closed intervals of real numbers.
Define a function $||h||^* \ : \ R \ \to \ [0,\ 1]$ by
$||h||^*(t) = \vee\{\alpha \ \in \ (0,\ 1] \ : \ t \in [||h||_\alpha^{*1}, ||h||_\alpha^{*2}]\}$
Then from Proposition 2.1[2] $||h||^*$ is a fuzzy interval and it is a fuzzy norm .

Theorem 5.6. *[Riesz] Let $H_1,\ H_2$ be two fuzzy Hilbert spaces and $h \ : \ H_1 \times H_2 \to F(R)$ be a bounded fuzzy sesquilinear form such that $h_\alpha^1(x,\ y).h_\alpha^2(x,\ y) \geq 0 \ \forall \alpha \in (0,\ 1]$. Assume further that $\{y \in \ H_2; h(x,\ y) = 0\},\ \ \forall x \in H_1$ is complete w.r.t. $||\ ||_\alpha^1$. Then h can be represented as $h(x,\ y) =< Sx,\ y >$ where $S \ : \ H_1 \to H_2$ is a bounded linear operator. S is uniquely determined by h and has the norm $||h||^* = ||S||$*

Proof. Consider $h(x,\ y)$ and keep x fixed.
Now taking y as a variable we have from Theorem 2.1[13]

$$h(x,\ y) \ = \ < y,\ z > \ = \ < z,\ y > \tag{5.6.1}$$

Here $z \in H_2$ is unique but depends on fixed $x \in H_1$. It follows that for each x we get an unique $z \in H_2$. So we can define an operator $S \ : \ H_1 \ \to \ H_2$ given by $Sx \ = \ z$
Substituting $z \ = \ Sx$ in (i) we have $h(x,\ y) \ = \ < Sx,\ y >$
Now for $k_1, k_2 \ \in \ R$ and $\forall \alpha \ \in \ (0,\ 1]$ we have,

$$\begin{aligned}
&< S(k_1x_1 + k_2x_2),\ y >_\alpha^1 \\
&= h_\alpha^1(k_1x_1 + k_2x_2,\ y) \\
&= k_1h_\alpha^1(x_1,\ y) + k_2h_\alpha^1(x_2,\ y) \\
&= k_1 < Sx_1,\ y >_\alpha^1 + k_2 < Sx_2,\ y >_\alpha^1 \\
&=< k_1Sx_1 + k_2Sx_2),\ y >_\alpha^1 \\
&\Rightarrow \ S(k_1x_1 + k_2x_2) = \ k_1Sx_1 + k_2Sx_2
\end{aligned}$$

Therefore S is linear.

For $\beta < \ \alpha,\ \alpha, \beta \ \in \ (0,\ 1]$ we have,

$$\begin{aligned}
&||h||_\alpha^{*2} \leq \ ||h||_\beta^{*2} \\
&= \ \bigvee_{(x,\ y) \in H_1 \times H_2 - \{(0,0)\}} \frac{h_\beta^2(x,\ y)}{||x||_\beta^1 ||y||_\beta^1} \\
&= \ \bigvee_{(x,\ y) \in H_1 \times H_2 - \{(0,0)\}} \frac{h_\beta^1(x,\ y)}{||x||_\beta^1 ||y||_\beta^1} \quad \text{since } h(x,\ y) = \ \bar{h}(x,\ y). \\
&= \ \bigvee_{(x,\ y) \in H_1 \times H_2 - \{(0,0)\}} \frac{<Sx,\ y>_\beta^1}{||x||_\beta^1 ||y||_\beta^1} \\
&\leq \ \bigvee_{(x,\ y) \in H_1 \times H_2 - \{(0,0)\}} \frac{||Sx||_\beta^1 ||y||_\beta^1}{||x||_\beta^1 ||y||_\beta^1} \\
&= \ \bigvee_{x \in H_1 - \{0\}} \frac{||Sx||_\beta^1}{||x||_\beta^1} \\
&\leq \ \bigvee_{x \in H_1 - \{0\}} \frac{||Sx||_\beta^2}{||x||_\beta^1} \\
&= \ ||S||_\beta^2 \\
&\Rightarrow \ ||h||_\alpha^{*2} \leq \ ||S||_\beta^2 \quad \forall \beta < \ \alpha
\end{aligned}$$

Taking infimum we have

$$||h||_\alpha^{*2} \leq \ \bigwedge_{\beta < \ \alpha;\ \alpha, \beta \ \in \ (0,\ 1]} ||S||_\beta^2$$

$$\Rightarrow \ ||h||_\alpha^{*2} \leq \ ||S||_\alpha^2 \tag{5.6.2}$$

Now we have

$$\begin{aligned}
&h(x,\ Sx) = \ < Sx,\ Sx >= \ ||Sx||^2 \\
&\Rightarrow \ h_\alpha^2(x,\ Sx) = \ (||Sx||_\alpha^2)^2 \quad \forall \alpha \in \ (0,\ 1] \\
&\Rightarrow \ (||Sx||_\alpha^2)^2 = \ h_\alpha^2(x,\ Sx) \leq \ ||h||_\alpha^{*2} ||x||_\alpha^1 ||Sx||_\alpha^1 \\
&\leq \ ||h||_\alpha^{*2} ||x||_\alpha^1 ||Sx||_\alpha^2 \\
&\Rightarrow \ \frac{||Sx||_\alpha^2}{||x||_\alpha^1} \leq \ ||h||_\alpha^{*2}
\end{aligned}$$

Taking supremum we have

$$\bigvee_{x \in H_1 - \{0\}} \frac{||Sx||_\alpha^2}{||x||_\alpha^1} \leq \ ||h||_\alpha^{*2}$$

$\Rightarrow\ ||S||_\alpha^2 \leq\ ||h||_\alpha^{*2}$
$\Rightarrow\ ||h||_\alpha^{*2} \geq\ ||S||_\alpha^2$

(5.6.3)

Now from (5.6.2) and (5.6.3) we have $||h||_\alpha^{*2} =\ ||S||_\alpha^2 \quad \forall\alpha \in\ (0,\ 1]$

(5.6.4)

Again $h(x,\ Sx) =\ < Sx,\ Sx >= ||Sx||^2$
$\Rightarrow\ h_\alpha^1(x,\ Sx) =\ (||Sx||_\alpha^1)^2$ and $h_\alpha^1(x,\ Sx) = h_\alpha^2(x,\ Sx)\ \forall\alpha \in\ (0,\ 1]$
Therefore $||Sx||_\alpha^1 = ||Sx||_\alpha^2\ \ \forall\alpha \in\ (0,\ 1]$.
$\Rightarrow\ (||Sx||_\alpha^1)^2 =\ h_\alpha^1(x,\ Sx) \leq\ ||h||_\alpha^{*1}||x||_\alpha^2||Sx||_\alpha^2$
$=\ ||h||_\alpha^{*1}||x||_\alpha^2||Sx||_\alpha^1$
$\Rightarrow\ \frac{||Sx||_\alpha^1}{||x||_\alpha^2} \leq\ ||h||_\alpha^{*1}$
Taking supremum we have
$\bigvee_{x\in H_1-\{0\}} \frac{||Sx||_\alpha^1}{||x||_\alpha^2} \leq\ ||h||_\alpha^{*1}$
$\Rightarrow\ ||S||_\alpha^1 \leq\ ||h||_\alpha^{*1}$
$\Rightarrow\ ||h||_\alpha^{*1} \geq\ ||S||_\alpha^1$

(5.6.5)

Now $h(x,\ Sx) =\ < Sx,\ Sx >= ||Sx||^2$
$\Rightarrow\ h_\alpha^1(x,\ Sx) =\ (||Sx||_\alpha^1)^2 \quad \forall\alpha \in\ (0,\ 1]$
$\Rightarrow\ \frac{h_\alpha^1(x,\ Sx)}{||Sx||_\alpha^1} =\ ||Sx||_\alpha^1 \leq\ ||S||_\alpha^1||x||_\alpha^2$
$\Rightarrow\ \frac{h_\alpha^1(x,\ Sx)}{||Sx||_\alpha^2} \leq\ \frac{h_\alpha^1(x,\ Sx)}{||Sx||_\alpha^1} \leq\ ||S||_\alpha^1||x||_\alpha^2$
$\Rightarrow\ \frac{h_\alpha^1(x,\ Sx)}{||Sx||_\alpha^2\ ||x||_\alpha^2} \leq\ ||S||_\alpha^1$
$\Rightarrow\ ||h||_\alpha^{*1}\ \leq\ ||S||_\alpha^1$

(5.6.6)

Thus from (5.6.5) and (5.6.6) we get $||h||_\alpha^{*1} = ||S||_\alpha^1\ \ \forall\alpha \in\ (0,\ 1]$

(5.6.7)

Hence from (5.6.4) and (5.6.7)$||h||^* \ =\ ||S||$
Since h is bounded, so $h(x,\ y) \oslash (||x|| \odot ||y||) \preceq \bar{k}$ and therefore $||h||_\alpha^{*2} \leq\ k, k$ being a real number and $\alpha \in\ (0,\ 1]$
So $||S||_\alpha^2 \leq\ k$ and $||S||_\alpha^1 \leq\ k$
$\Rightarrow \frac{||Sx||_\alpha^1}{||x||_\alpha^2} \leq\ k$ and $\frac{||Sx||_\alpha^2}{||x||_\alpha^1} \leq\ k\forall\ x \in\ H_1 - \{0\}$
$\Rightarrow ||Sx|| \oslash ||x|| \preceq\ \bar{k}$.
Therefore S is fuzzy bounded.
For uniqueness if possible suppose that there is another bounded linear operator T such that $h(x,\ y) =< Sx,\ y >=< Tx,\ y >$
Therefore $< Sx,\ y >_\alpha^1 =\ < Tx,\ y >_\alpha^1 \forall\alpha \in\ (0,\ 1]$.
$\Rightarrow< Sx,\ y >_\alpha^1 -\ < Tx,\ y >_\alpha^1 =\ 0\forall\alpha \in\ (0,\ 1], \forall x \in H_1$ and $\forall y \in H_2$
$\Rightarrow< (S - T)(x),\ y >_\alpha^1 =\ 0\forall\alpha \in\ (0,\ 1], \forall x \in H_1$ and $\forall y \in H_2$
$\Rightarrow (S - T)(x) =\ \theta\forall x \in H_1$
$\Rightarrow S - T =\ 0$
$\Rightarrow S =\ T$
Thus S is unique and this comletes the proof.

□

6 Conclusion

In this paper, idea of fuzzy sesquilinear form on linear spaces is introduced and establish Riesz representation theorem for fuzzy sesquilinear form. This Riesz theorem for sesquilinear form can be applied for proving existence of fuzzy adjoint operators. We think that there is a wide scope of research in operator theory in fuzzy setting by using the results of this manuscript.

References

[1] C. Alsina, B. Schweizer, A. Sklar, On the definition of a probabilistic normed space, Aequationes Math. 46 (1993) 91-98.

[2] T. Bag, S. K. Samanta, Fuzzy bounded linear operators in Felbin's type fuzzy normed linear spaces, Fuzzy Sets and Systems, 159 (2008) 685-707.

[3] R. Biswas, Fuzzy inner product spaces and fuzzy norm functions, Information Sciences, 53(1991) 185-190.

[4] A. M. El-Abyad, H. M. El-Hamouly, Fuzzy inner product spaces, Fuzzy Sets and Systems, 44(2) (1991) 309-326.

[5] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets and Systems 48 (1992) 239-248.

[6] M. Goudarzi, S. M. Vaezpour, On the definition of fuzzy Hilbert spaces and its application, The Journal of Nonlinear Science and Applications 2(1) (2009) 46-59.

[7] A. Hasankhani, A. Nazari, M. Saheli, Some properties of fuzzy Hilbert spaces and norm of operators, Iranian Journal of Fuzzy Systems, 7(3) (2010) 129-157.

[8] O. Kaleva, S. Seikala, On fuzzy metric spaces, Fuzzy Sets and Systems, 12 (1984) 215-229.

[9] J. K. Kohli, R.Kumar, On fuzzy inner product spaces and fuzzy co-inner product spaces, Fuzzy Sets and Systems, 53(2) (1993) 227-232.

[10] Pinaki Majumdar, S. K. Samanta, On fuzzy inner product spaces, The Journal of Fuzzy Mathematics, 16(2) (2008) 377-392.

[11] Mizumoto, J.Tanaka, Some properties of fuzzy numbers in : M.M.Gupta et al. Editors, Advances in Fuzzy Set Theory and Applications (North-Holland, New-York, 1979) 153-164.

[12] S. Mukherjee, T. Bag, Fuzzy real inner product space and its properties, Ann. Fuzzy Math. Inform. 6(2) (2013) 377-389.

[13] S. Mukherjee, T. Bag, Some properties of Fuzzy Hilbert Spaces and Fixed Point Theorems in Such Spaces, The Journal of Fuzzy Mathematics, 20(3) 2012 539-550.

[14] J. Xiao and X. Zhu, On linearly topological structure and property of fuzzy normed linear space, FSS, 125 (2002) 153-161.

Author information

S. Ghosal and T. Bag, Department of Mathematics, Visva Bharati, Santiniketan-731235, W.Bengal, India.
E-mail: tarapadavb@gmail.com

Received: August 28, 2017.
Accepted: April 21, 2018.