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Multiplicative Lie Maps on $\mathfrak{s}u(2)$ Lie Algebra

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Abstract. Let $\mathfrak{s}u(2)$ be a Lie algebra and $\mathfrak{h}$ be another Lie algebra. Suppose that a bijective map $\Phi : \mathfrak{s}u(2) \to \mathfrak{h}$ is a multiplicative Lie map. In this paper, we prove that Φ is additive.

1 The $\mathfrak{s}u(2)$ Lie algebra and multiplicative Lie maps

A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{K}$ with an operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ which we call a Lie bracket, such that the following axioms are satisfied:

- It is bilinear.
- It is skew symmetric: $[x, x] = 0$ which implies $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.
- It satisfies the Jacobi Identitiy: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Any associative algebra $\mathfrak{A}$ can be made into a Lie algebra by taking commutator as the Lie bracket:

$$[x, y] = xy - yx$$

for all $x, y \in \mathfrak{A}$.

The Lie algebra $\mathfrak{s}u(2)$ is $\mathfrak{s}u(2) = \left\{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} \mid a \in \mathbb{R}, z \in \mathbb{C} \right\}$.

It is easily verified that matrices of this form have trace zero and are anti-hermitian. This Lie algebra is then generated by the following matrices,

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which are easily seen to have the form of the general element specified above.

These satisfy $u_3u_2 = -u_2u_3 = -u_1$ and $u_2u_1 = -u_1u_2 = -u_3$. The commutator bracket is therefore specified by

$$[u_3, u_1] = 2u_2, \quad [u_1, u_2] = 2u_3, \quad [u_2, u_3] = 2u_1.$$

In the $\mathfrak{s}u(2)$ Lie algebra we have the decomposition

$$\mathfrak{s}u(2) = \mathfrak{s}u(2)_0 \oplus \mathfrak{s}u(2)_1$$

where $\mathfrak{s}u(2)_0$ is the span space by u_3 and $\mathfrak{s}u(2)_1$ is the span space by u_1 and u_2.

The above generators are related to the Pauli matrices by $u_1 = i\sigma_1$, $u_2 = -i\sigma_2$ and $u_3 = i\sigma_3$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This representation is routinely used in quantum mechanics to represent the spin of fundamental particles such as electrons.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras and $\Phi : \mathfrak{g} \to \mathfrak{h}$ a map of $\mathfrak{g}$ in $\mathfrak{h}$. We call Φ a *multiplicative Lie map* of $\mathfrak{g}$ in $\mathfrak{h}$ if for all $x, y \in \mathfrak{g}$

$$\Phi\big([x, y]\big) = [\Phi(x), \Phi(y)].$$

The following argument, which shall be named "*standard argument*", is true for multiplicative Lie maps:

Remark 1.1. *(Standard argument).* Let $x, y, z \in \mathfrak{g}$, and $\Phi(z) = \Phi(x) + \Phi(y)$. Then for any $a \in \mathfrak{g}$, we have $\Phi([z, a]) = \Phi([x, a]) + \Phi([y, a])$. In deed,

$$\begin{aligned} \Phi([z,a]) &= [\Phi(z), \Phi(a)] \\ &= [\Phi(x) + \Phi(y), \Phi(a)] \\ &= [\Phi(x), \Phi(a)] + [\Phi(y), \Phi(a)] \\ &= \Phi([x,a]) + \Phi([y,a]). \end{aligned}$$

Recently, the additivity of maps on rings and algebras has attracted the attentions of many researchers. One of the first results ever recorded was given by Martindale III which in his condition requires that the ring possess idempotents, see [10]. For the interested reader in additivity of maps defined in non-associative rings and algebras we can cite the recent works [2], [3], [4], [5], [6], [7], [8] and [9].

More specifically for the case of alternative rings we can cite [1] where the authors use a standard argument to prove their results and the math involved is sufficiently substantial to this line of research.

According to [10], "An interesting feature of this problem is that the conclusion of the theorem obviously fails if the ring $\mathcal{R}$ is either too "well behaved " or too "badly behaved"". This motivated us to ask the question: When is a multiplicative Lie map additive on $\mathfrak{s}u(2)$ Lie algebra? Does the $\mathfrak{s}u(2)$ Lie algebra is too well behaved or too badly behaved ? It is worth noting that as the $\mathfrak{s}u(2)$ Lie algebra it does not have idempotent elements so we can not use the Martindale's conditions but fortunately, in this paper we give a full answer for this question.

2 Main theorem

We shall prove as follows the main result of this paper.

Theorem 2.1. *Let $\mathfrak{s}u(2)$ and $\mathfrak{h}$ be Lie algebras. Consider a bijective multiplicative Lie mapping $\Phi : \mathfrak{s}u(2) \to \mathfrak{h}$, that is,*

$$\Phi([a, b]) = [\Phi(a), \Phi(b)]$$

for all $a, b \in \mathfrak{s}u(2)$. Then $\Phi(a + b) = \Phi(a) + \Phi(b)$ for all $a, b \in \mathfrak{s}u(2)$.

The following lemmas has the same hypotheses of Theorem 2.1 and we need these lemmas for the proof of this theorem.

Lemma 2.2. $\Phi(0) = 0$.

Proof. $\Phi(0) = \Phi([0, 0]) = [\Phi(0), \Phi(0)] = \Phi(0)^2 - \Phi(0)^2 = 0$. □

Lemma 2.3. *If $a \in \mathbb{R}u_1 \subset \mathfrak{s}u(2)_1$ and $b \in \mathfrak{s}u(2)_0$ then $\Phi(a + b) = \Phi(a) + \Phi(b)$.*

Proof. Since Φ is surjective, we may find an element $E = \alpha u_1 + \beta u_3 + \gamma u_2 \in \mathfrak{s}u(2)$ such that $\Phi(E) = \Phi(a) + \Phi(b)$. By Lemma 2.2 and Remark 1.1, we obtain

$$\Phi([E, u_3]) = \Phi([a, u_3]) + \Phi([b, u_3]) = \Phi([a, u_3]).$$

Since Φ is injective, we have $[E, u_3] = [a, u_3]$. Consider $a = \alpha_a u_1$ and $b = \beta_b u_3$ with $\alpha_a, \beta_b \in \mathbb{R}$ it follows that $\alpha_a = \alpha$ and $\gamma = 0$. For $u_1 \in \mathfrak{s}u(2)_1$, by Lemma 2.2 and Remark 1.1, we have

$$\Phi([E, u_1]) = \Phi([a, u_1]) + \Phi([b, u_1]) = \Phi([b, u_1]).$$

Again, since Φ is injective, we have $[E, u_1] = [b, u_1]$. It follows that $\beta_b = \beta$. Thus, $\Phi(E) = \Phi(\alpha u_1 + \beta u_3 + \gamma u_2) = \Phi(\alpha_a u_1 + \beta_b u_3) = \Phi(a + b)$.

□

Lemma 2.4. *If $c \in \mathbb{R}u_2 \subset \mathfrak{s}u(2)_1$ and $b \in \mathfrak{s}u(2)_0$ then $\Phi(b + c) = \Phi(b) + \Phi(c)$.*

Proof. Since Φ is surjective, we may find an element $E = \alpha u_1 + \beta u_3 + \gamma u_2 \in \mathfrak{su}(2)$ such that $\Phi(E) = \Phi(b) + \Phi(c)$. By Lemma 2.2 and Remark 1.1, we obtain

$$\Phi([E, u_3]) = \Phi([b, u_3]) + \Phi([c, u_3]) = \Phi([c, u_3]).$$

Since Φ is injective, we have $[E, u_3] = [c, u_3]$. Consider $c = \gamma_c u_2$ and $b = \beta_b u_3$ with $\gamma_c, \beta_b \in \mathbb{R}$ it follows that $\gamma_c = \gamma$ and $\alpha = 0$. For $u_2 \in \mathfrak{su}(2)_1$, by Lemma (2.2) and Remark (1.1), we have

$$\Phi([E, u_2]) = \Phi([b, u_2]) + \Phi([c, u_2]) = \Phi([b, u_2]).$$

Again, since Φ is injective, we have $[E, u_2] = [b, u_2]$. It follows that $\beta_b = \beta$. Thus, $\Phi(E) = \Phi(\alpha u_1 + \beta u_3 + \gamma u_2) = \Phi(\beta_b u_3 + \gamma_c u_2) = \Phi(a + c)$. □

Lemma 2.5. *If $a \in \mathbb{R}u_1 \subset \mathfrak{su}(2)_1$ and $c \in \mathbb{R}u_2 \subset \mathfrak{su}(2)_1$ then $\Phi(a + c) = \Phi(a) + \Phi(c)$.*

Proof. Note that $a + c = [\frac{1}{4}u_3, [u_3 + a, u_3 - c]]$. Let say $a = \alpha u_1$ and $c = \beta u_2$ with $\alpha, \beta \in \mathbb{R}$. By Lemma 2.3 and 2.4, we compute that

$$\begin{aligned}
\Phi(a + c) &= \Phi([\frac{1}{4}u_3, [u_3 + a, u_3 - c]]) \\
&= [\Phi(\frac{1}{4}u_3), \Phi([u_3 + a, u_3 - c])] \\
&= [\Phi(\frac{1}{4}u_3), [\Phi(u_3 + a), \Phi(u_3 - c)]] \\
&= [\Phi(\frac{1}{4}u_3), [\Phi(u_3) + \Phi(a), \Phi(u_3) + \Phi(-c)]] \\
&= [\Phi(\frac{1}{4}u_3), \Phi(2\beta u_1)] + [\Phi(\frac{1}{4}u_3), \Phi(-2\alpha u_2)] \\
&= \Phi([\frac{1}{4}u_3, 2\beta u_1]) + \Phi([\frac{1}{4}u_3, -2\alpha u_2]) \\
&= \Phi(a) + \Phi(c).
\end{aligned}$$

□

Lemma 2.6. *If $d \in \mathfrak{su}(2)_1$ and $b \in \mathfrak{su}(2)_0$ then $\Phi(d + b) = \Phi(d) + \Phi(b)$.*

Proof. Consider $b = \beta_b u_3$ and $d = \alpha_d u_1 + \gamma_d u_2$, with $\alpha_d, \beta_b, \gamma_d \in \mathbb{K}$. Since Φ is surjective, we can find an element $E = \alpha u_1 + \beta u_3 + \gamma u_2 \in \mathfrak{su}(2)$ such that $\Phi(E) = \Phi(b) + \Phi(d)$. By Lemma 2.2 and Remark 1.1, we have

$$\Phi([E, u_3]) = \Phi([b, u_3]) + \Phi([d, u_3]) = \Phi([d, u_3]).$$

Since Φ is injective, we obtain $[E, u_3] = [d, u_3]$. It follows that $\alpha_d = \alpha$ and $\gamma_d = \gamma$. Now for $u_1 \in \mathfrak{su}(2)_1$ and $u_3 \in \mathfrak{su}(2)_0$, by Lemmas 2.2, 2.5 and Remark 1.1, we have

$$\Phi([[E, u_1], u_3]) = \Phi([[b, u_1], u_3]) + \Phi([[d, u_1], u_3]) = \Phi([2\beta_b u_2, u_3]).$$

Again, since Φ is injective, we have $[[E, u_1], u_3] = [2\beta_b u_2, u_3] = 4\beta_b u_1$. It follows that $\beta_b = \beta$. Thus, $\Phi(E) = \Phi(\alpha u_1 + \beta u_3 + \gamma u_2) = \Phi(\alpha_d u_1 + \gamma_d u_2 + \beta_b u_3) = \Phi(d + b)$. □

Lemma 2.7. *If $a_1, a_2 \in \mathbb{R}u_1 \subset \mathfrak{su}(2)_1$ then $\Phi(a_1 + a_2) = \Phi(a_1) + \Phi(a_2)$.*

Proof. Let say $a_1 = \alpha u_1$ and $a_2 = \beta u_1$ with $\alpha, \beta \in \mathbb{R}$. Note that $a_1 + a_2 = [[\frac{1}{2}u_3 - a_2, \frac{1}{2}u_3 + a_1], \frac{1}{2}u_3]$. So by Lemmas 2.2, 2.3 and Remark 1.1 we get,

$$\begin{aligned}
\Phi(a_1 + a_2) &= \Phi([[\frac{1}{2}u_3 - a_2, \frac{1}{2}u_3 + a_1], \frac{1}{2}u_3) \\
&= [[\Phi(\frac{1}{2}u_3 - a_2), \Phi(\frac{1}{2}u_3 + a_1)], \Phi(\frac{1}{2}u_3)] \\
&= [[\Phi(\frac{1}{2}u_3) + \Phi(-a_2), \Phi(\frac{1}{2}u_3) + \Phi(a_1)], \Phi(\frac{1}{2}u_3)] \\
&= [\Phi([\frac{1}{2}u_3, a_1]) + \Phi([-a_2, \frac{1}{2}u_3]), \Phi(\frac{1}{2}u_3)] \\
&= \Phi(a_1) + \Phi(a_2).
\end{aligned}$$

□

Lemma 2.8. *If* $c_1, c_2 \in \mathbb{R}u_2 \subset \mathfrak{su}(2)_1$ *then* $\Phi(c_1 + c_2) = \Phi(c_1) + \Phi(c_2)$.

Proof. Let say $c_1 = \alpha u_2$ and $c_2 = \beta u_2$ with $\alpha, \beta \in \mathbb{R}$. Note that $c_1 + c_2 = [[\frac{1}{2}u_3 - c_2, \frac{1}{2}u_3 + c_1], \frac{1}{2}u_3]$. So by Lemmas 2.2, 2.4 and Remark 1.1 we get,

$$\begin{aligned}
\Phi(c_1 + c_2) &= \Phi([[\frac{1}{2}u_3 - c_2, \frac{1}{2}u_3 + c_1], \frac{1}{2}u_3]) \\
&= [[\Phi(\frac{1}{2}u_3 - c_2), \Phi(\frac{1}{2}u_3 + c_1)], \Phi(\frac{1}{2}u_3)] \\
&= [[\Phi(\frac{1}{2}u_3) + \Phi(-c_2), \Phi(\frac{1}{2}u_3) + \Phi(c_1)], \Phi(\frac{1}{2}u_3)] \\
&= \Phi([[\frac{1}{2}u_3, c_1], \frac{1}{2}u_3]) + \Phi([[-c_2, \frac{1}{2}u_3], \frac{1}{2}u_3]) \\
&= \Phi(c_1) + \Phi(c_2).
\end{aligned}$$

□

Lemma 2.9. *If* $b_1, b_2 \in \mathfrak{su}(2)_0$ *then* $\Phi(b_1 + b_2) = \Phi(b_1) + \Phi(b_2)$.

Proof. Consider $b_1 = \alpha_1 u_3$ and $b_2 = \alpha_2 u_3$ with $\alpha_1, \alpha_2 \in \mathbb{R}$. Note that $b_1 + b_2 = [\frac{1}{2}u_1, [\frac{1}{2}u_1 + b_2, \frac{1}{2}u_1 - b_1]]$. So by Lemmas 2.2, 2.3 and Remark 1.1 we get,

$$\begin{aligned}
\Phi(b_1 + b_2) &= \Phi([\frac{1}{2}u_1, [\frac{1}{2}u_1 + b_2, \frac{1}{2}u_1 - b_1]]) \\
&= [\Phi(\frac{1}{2}u_1), [\Phi(\frac{1}{2}u_1 + b_2), \Phi(\frac{1}{2}u_1 - b_1)]] \\
&= [\Phi(\frac{1}{2}u_1), [\Phi(\frac{1}{2}u_1) + \Phi(b_2), \Phi(\frac{1}{2}u_1) + \Phi(-b_1)]] \\
&= \Phi([\frac{1}{2}u_1, [\frac{1}{2}u_1, -b_1]]) + \Phi([\frac{1}{2}u_1, [b_2, \frac{1}{2}u_1]]) \\
&= \Phi(b_1) + \Phi(b_2).
\end{aligned}$$

□

Proof of Theorem 2.1. Suppose that $a, b \in \mathfrak{su}(2)$ with $a = a_0 + a_1$, $b = b_0 + b_1$ where $a_0, b_0 \in \mathfrak{su}(2)_0$ and $a_1, b_1 \in \mathfrak{su}(2)_1$. Consider $a_1 = \alpha_a u_1 + \beta_a u_2$ and $b_1 = \alpha_b u_1 + \beta_b u_2$ where $\alpha_a, \alpha_b, \beta_a, \beta_b \in \mathbb{R}$. Soon by Lemmas 2.5, 2.6, 2.7, 2.8 and 2.9 we get

$$\begin{aligned}
\Phi(a + b) &= \Phi(a_0 + a_1 + b_0 + b_1) \\
&= \Phi(a_0 + \alpha_a u_1 + \beta_a u_2 + b_0 + \alpha_b u_1 + \beta_b u_2) \\
&= \Phi((a_0 + b_0) + (\alpha_a u_1 + \alpha_b u_1) + (\beta_a u_2 + \beta_b u_2)) \\
&= \Phi(a_0 + b_0) + \Phi((\alpha_a u_1 + \alpha_b u_1) + (\beta_a u_2 + \beta_b u_2)) \\
&= \Phi(a_0 + b_0) + \Phi(\alpha_a u_1 + \alpha_b u_1) + \Phi(\beta_a u_2 + \beta_b u_2) \\
&= \Phi(a_0) + \Phi(b_0) + \Phi(\alpha_a u_1) + \Phi(\alpha_b u_1) + \Phi(\beta_a u_2) + \Phi(\beta_b u_2) \\
&= \Phi(a_0) + \Phi(\alpha_a u_1) + \Phi(\beta_a u_2) + \Phi(b_0) + \Phi(\alpha_b u_1) + \Phi(\beta_b u_2) \\
&= \Phi(a_0) + \Phi(\alpha_a u_1 + \beta_a u_2) + \Phi(b_0) + \Phi(\alpha_b u_1 + \beta_b u_2) \\
&= \Phi(a_0 + \alpha_a u_1 + \beta_a u_2) + \Phi(b_0 + \alpha_b u_1 + \beta_b u_2) \\
&= \Phi(a_0 + a_1) + \Phi(b_0 + b_1) \\
&= \Phi(a) + \Phi(b),
\end{aligned}$$

which shows that the multiplicative Lie mapping Φ is additive.

3 Final remarks

It is easy to see that $\mathfrak{su}(2)$ is a subalgebra of $\mathcal{M}_{2\times 2}(\mathbb{R})$, where

$$\mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.$$

In this section we show by way of example that the concept of hereditary of the additivity is not inherited from the algebra $\mathcal{M}_{2\times 2}(\mathbb{R})$. First we show the result which appear in [11] to be useful in the construction of our example.

Proposition 3.1. (Zhaofang, B. et al. [11]) *Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. Consider $\Phi : \mathcal{R} \longrightarrow \mathcal{R}'$ a multiplicative Lie isomorphism, then for any $T \in \mathcal{R}$ and $Z \in \mathcal{Z}(\mathcal{R})$, the center of $\mathcal{R}$, there exists $Z' \in \mathcal{Z}(\mathcal{R}')$, the center of $\mathcal{R}'$, such that $\Phi(A + Z) = \Phi(A) + Z'$.*

Example 3.2. Let $\mathcal{M}_{2\times 2}(\mathbb{R})$ be an matrix algebra with basis $\{E_{ij}\}$, $i, j \in \{1, 2\}$, where E_{ij} be matrices with 1 at position (i, j) and zeros everywhere else. Let us show that not necessarily get

$$\Phi(E_{11} + E_{12}) = \Phi(E_{11}) + \Phi(E_{12}).$$

Indeed, since Φ is surjective, we may find an element $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathcal{M}_{2\times 2}(\mathbb{R})$ where $A_{ij} \in \mathbb{R}E_{ij}$ such that $\Phi(A) = \Phi(E_{11}) + \Phi(E_{12})$. By Lemma 2.2 and Remark 1.1, we obtain

$$\Phi([A, E_{11}]) = \Phi([E_{11}, E_{11}]) + \Phi([E_{12}, E_{11}]) = \Phi([E_{12}, E_{11}]).$$

Since Φ is injective, we have $[A, E_{11}] = [E_{12}, E_{11}]$. It follows that $A_{12} = E_{12}$ and $A_{21} = 0$. With a similar argument, we have $\Phi([A, E_{12}]) = \Phi([E_{11}, E_{12}])$ and by Lemma 2.4 we have too $\Phi([A, E_{21}]) = \Phi(E_{11} - (E_{22} + E_{21}))$. With this we conclude that $Z = A_{11} - E_{11} + A_{22} \in \mathcal{Z}(\mathcal{M}_{2\times 2}(\mathbb{R}))$ the center of $\mathcal{M}_{2\times 2}(\mathbb{R})$. Therefore by Proposition 3.1 we get $\Phi(E_{11} + E_{12}) = \Phi(E_{11}) + \Phi(E_{12}) + Z'$, where Z' is not necessarily zero.

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