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Partition Laplacian energy of a graph

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Abstract. Let $G = (V, E)$ be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of V. Recently we have introduced the partition energy of a graph $E_{P_k}(G)$ and computed partition energy of some families of graphs with respect to a given partition. In this paper, we introduce the concept of partition Laplacian energy $LE_{P_k}(G)$ which depends on the underlying graph G and the partition of the vertex set V of G. We obtain an upper bound and few lower bounds for partition Laplacian energy, also obtain partition Laplacian energy of some families of graphs, their internal-complements and show that k-partition Laplacian energy of a r-regular graph G is equal to its k-partition energy with respect to any partition P_k of V.

1 Introduction

Let $G = (V, E)$ be a graph of order n. The energy of a graph G was defined by I. Gutman in 1978 as the sum of the absolute values of eigenvalues of G [4]. The concept of graph energy has origin in chemistry which is used to estimate the total π-electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph whose eigenvalues with respect to adjacency matrix $A(G)$ represent the energy level of the electron in the molecule. In Hückel theory the sum of the energies of all the electrons in a molecule is called the π-electron energy of a molecule. In spectral graph theory, the energy-like quantities such as Laplacian energy, distance energy, color energy, color Laplacian energy of a graph etc., are studied in [1], [5], [6], [7].

E. Sampathkumar and M. A. Sriraj in [9] have introduced L-matrix with respect to a partition $P_k = \{V_1, V_2, \dots, V_k\}$ of the vertex set V of a graph $G = (V, E)$ of order n represented by a unique square symmetric matrix $P_k(G) = [a_{ij}]$ of order n, whose entries a_{ij} are defined as follows:
(i) Suppose for some $V_r \in P_k$, both v_i, $v_j \in V_r$. Then $a_{ij} = 2$ or -1 according as v_iv_j is an edge or not.
(ii) For $r \neq s$, suppose $v_i \in V_r$ and $v_j \in V_s$. Then $a_{ij} = 1$ or 0 according as v_iv_j is an edge or not.
The matrix $P_k(G)$ thus defined is called the L-matrix of the partition P_k of the graph $G = (V, E)$.

Recently in [10], we have defined k-partition eigenvalues of G as the eigenvalues of the matrix $P_k(G)$ and the k-partition energy $E_{P_k}(G)$ is defined as the sum of the absolute values of k-partition eigenvalues of G. In this paper we have determined partition energy of some known graphs, their k-complement and $k(i)$-complement. We have also obtained some bounds for $E_{P_k}(G)$.

The concept of color energy was introduced by C. Adiga et al. in [1]. In [7], Pradeep G Bhat and Sabitha D'Souza have studied the color Laplacian energy of a graph. Let G be a colored graph on n vertices and m edges. The color Laplacian matrix of G is defined as $L_c(G) = D(G) - A_c(G)$ where $D(G) = diag(d_1, d_2, \dots, d_n)$ represents the diagonal matrix with vertex degrees $d_1, d_2, \dots, d_n$ of $v_1, v_2, \dots, v_n$ of G and $A_c(G)$, the color matrix. The eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_n\}$ of $L_c(G)$ are called color Laplacian eigenvalues of the graph G. If auxiliary color eigenvalues γ_i, $i = 1$ to n are defined as $\gamma_i = \mu_i - \frac{2m}{n}$, then color Laplacian energy of G

is defined as $\sum_{i=1}^{n} | \gamma_i |$.

Now we state definitions of two types of complements of a partition graph called k-complement and $k(i)$-complements as follows:

Definition 1.1. [8] Let G be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set V. Then the k-complement of G is obtained as follows: For all V_i and V_j in P_k, $i \neq j$ remove the edges between V_i and V_j and add the edges between the vertices of V_i and V_j which are not in G and is denoted by $\overline{(G)_k}$.

The matrix of k-complement is obtained from L-matrix $P_k(G)$ as follows: In $P_k(G)$ interchange 1 and 0 in the non-principal diagonal entries. The matrix thus obtained is the matrix of $\overline{G_k}$ and denoted by $P_k(\overline{(G)_k})$.

Definition 1.2. [8] Let G be a graph and $P_k = \{V_1, V_2, ..., V_k\}$ be a partition of its vertex set V. Then the $k(i)$-complement of G is obtained as follows: For each set V_r in P_k, remove the edges of G joining the vertices within V_r and add the edges of $\overline{G}$ (complement of G) joining the vertices of V_r, and is denoted by $\overline{(G)_{k(i)}}$.

The matrix of $k(i)$-complement is obtained by interchanging 2 and -1 in the matrix $P_k(G)$ and is denoted by $P_k(\overline{(G)_{k(i)}})$.

2 Partition Laplacian energy

Consider a graph $G = (V, E)$ of order n and size m with a partition $P_k = \{V_1, V_2, \ldots, V_k\}$ of V. Let $P_k(G)$ be partition matrix and $D(G) = diag(d_1, d_2, \ldots, d_n)$ represents the diagonal matrix with vertex degrees $d_1, d_2, \ldots, d_n$ of $v_1, v_2, \ldots, v_n$ of G. Then we define the partition Laplacian matrix of G as $LP_k(G) = D(G) - P_k(G)$. The eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_n\}$ of this matrix $LP_k(G)$ are called *k-partition Laplacian eigenvalues*. We also define auxiliary partition eigenvalues γ_i, $i = 1, 2, \cdots, n$ as $\gamma_i = \mu_i - \frac{2m}{n}$. The *$k$-partition Laplacian energy* of G or *partition Laplacian energy* of G, denoted by $LE_{P_k}(G)$ is defined as $\sum_{i=1}^{n} | \gamma_i |$.

$$i, e., LE_{P_k}(G) = \sum_{i=1}^{n} | \mu_i - \frac{2m}{n} |.$$

If the vertex set of a graph G of order n is partitioned into n sets then the partition Laplacian energy coincides with the usual Laplacian energy of a graph. So partition Laplacian energy may be considered as a generalization of Laplacian energy of a graph.

In this paper, we define the partition Laplacian energy and establish an upper bound and some lower bounds for partition Laplacian energy. We obtain partition Laplacian energy of some family of graphs, its k-complement and $k(i)$-complement. Also prove that k-partition Laplacian energy of a r-regular graph $G = (V, E)$ is equal to its k-partition energy with respect to any partition P_k of V.

3 Some basic properties of partition Laplacian eigenvalues of a graph

Let $G = (V, E)$ be a graph with n vertices, m edges and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of V. For $1 \leq r \leq k$, let b_t denote the total number of edges joining the vertices of V_r and c_t be the total number of edges joining the vertices from V_r to V_s for $r \neq s, 1 \leq s \leq k$ and e_t be the number of non-adjacent pairs of vertices within V_r. Let $m_1 = \sum_{t=1}^{k} b_t$, $m_2 = \sum_{t=1}^{\frac{k(k-1)}{2}} c_t$ and $m_3 = \sum_{t=1}^{k} e_t$ and d_i represent the degree of v_i where $i = 1, 2, \cdots, n$. Let $LP_k(G)$ be the partition Laplacian matrix. If the characteristic polynomial $L\Phi_{P_k}(G,\mu) = det[\mu I - LP_k(G)] = a_0\mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots + a_n$, then the coefficient a_i can be interpreted using the principal minors of $LP_k(G)$.

The first three coefficients of the characteristic polynomial of $LP_k(G)$ are determined in the following proposition.

Proposition 3.1. *The first three coefficients of $L\Phi_{P_k}(G,\mu)$ are given as follows:*

$$(1)\ a_0 = 1, \qquad (2)\ a_1 = -2m, \qquad (3)\ a_2 = \sum_{1\le i<j\le n}^{k} d_i d_j - [4m_1 + m_2 + m_3].$$

Proof. (1) It follows from the definition $L\Phi_{P_k}(G,\lambda) = det[\mu I - LP_k(G)]$ that $a_0 = 1$.

(2) Note that for each $i \in \{1,2,3,\dots,n\}$, the number $(-1)^i a_i$ is the sum of those principal minors of $LP_k(G)$ which have i rows and i columns. Since the diagonal elements are d_i, $(-1)a_1 = \sum_{i=1}^n d_i = 2m$.
Hence $a_1 = -2m$.

(3)

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\le i<j\le n} d_i d_j - \sum_{1\le i<j\le n} a_{ij}^2 \\ &= \sum_{1\le i<j\le n} d_i d_j - [4m_1 + m_2 + m_3].\end{aligned}$$

Hence, $a_2 = \sum_{1\le i<j\le n}^{k} d_i d_j - [4m_1 + m_2 + m_3]$.□

We prove the following results to obtain the bounds for partition Laplacian energy of a graph G.

Proposition 3.2. *If $\mu_1, \mu_2, \cdots, \mu_n$ are partition Laplacian eigenvalues of $LP_k(G)$, then*

$$\sum_{i=1}^{n} {\mu_i}^2 = 2[4m_1 + m_2 + m_3] + \sum_{i=1}^{n} d_i^2.$$

Proof. We know that

$$\begin{aligned}\sum_{i=1}^{n} \mu_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\ &= 2\sum_{i<j} a_{ij}^2 + \sum_{i=1}^{n} a_{ii}^2 \\ &= 2[4m_1 + m_2 + m_3] + \sum_{i=1}^{n} d_i^2.\end{aligned}$$

Proposition 3.3. *Let G_1 and G_2 be two graphs of order n. Suppose that P_k and P'_k are partitions of vertex sets of G_1 and G_2 respectively. If $\mu_1, \mu_2, \dots, \mu_n$ and $\mu'_1, \mu'_2, \dots, \mu'_n$ are the eigenvalues of $LP_k(G_1)$ and $LP'_k(G_2)$ respectively, then*

$$\sum \mu_i \mu'_i \le \sqrt{\left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2\right)\left(2(4m'_1 + m'_2 + m'_3) + \sum_{i=1}^{n} (d'_i)^2\right)}$$

where m_1, m_2, m_3 *are as defined above for* G_1 *and* m'_1, m'_2, m'_3 *for* G_2 *and* d_i,d'_i *are degrees of an* i^{th} *vertex of corresponding graphs respectively.*

Proof. By Cauchy - Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Setting $a_i = \mu_i$ and $b_i = \mu'_i$ in the above inequality, we get

$$\left(\sum_{i=1}^{n} \mu_i \mu'_i\right)^2 \leq \left(\sum_{i=1}^{n} {\mu_i}^2\right)\left(\sum_{i=1}^{n} {\mu'_i}^2\right)$$

$$\sum_{i=1}^{n} \mu_i \mu'_i \leq \sqrt{\left(2(4m_1+m_2+m_3)+\sum_{i=1}^{n}(d_i)^2\right)\left(2(4m'_1+m'_2+m'_3)+\sum_{i=1}^{n}(d'_i)^2\right)}. \square$$

4 Some bounds for partition Laplacian energy of a graph

In the present section, we obtain an upper bound and some lower bounds for $LE_{P_k}(G)$.

Theorem 4.1. *Let* G *be a graph of order* n *and size* m *and* P_k *be a partition of vertex set of* G. *Then*

$$LE_{P_k}(G) \leq \sqrt{n\left(2(4m_1+m_2+m_3)+\sum_{i=1}^{n} d_i^2\right)-4m^2}$$

where m_1, m_2, m_3 *are as defined above for* G.

Proof. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $LP_k(G)$.
We know that Cauchy - Schwartz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} {a_i}^2\right)\left(\sum_{i=1}^{n} {b_i}^2\right).$$

Let $a_i = 1$, $b_i = \mid \gamma_i \mid$. Then

$$\begin{aligned}
(LE_{P_k}(G))^2 &= \left(\sum_{i=1}^{n} \mid \gamma_i \mid\right)^2 \\
&\leq n \sum_{i=1}^{n} \mid \gamma_i \mid^2 \\
&= n\sum_{i=1}^{n} {\gamma_i}^2 \\
&= n\sum_{i=1}^{n}(\mu_i - \frac{2m}{n})^2 \\
&= n\sum_{i=1}^{n} \mu_i^2 - 4m^2 \\
&= n\left(2(4m_1+m_2+m_3)+\sum_{i=1}^{n} d_i^2\right)-4m^2.
\end{aligned}$$

Thus, $LE_{P_k}(G) \leq \sqrt{n\left(2(4m_1+m_2+m_3)+\sum_{i=1}^{n} d_i^2\right)-4m^2}$. $\square$

Corollary 4.2. *If G is r-regular, then*

$$LE_{P_k}(G) = E_{P_k}(G) \leq \sqrt{2n(4m_1 + m_2 + m_3)}$$

Theorem 4.3. *Let G be a graph of order n and size m and P_k be a partition of vertex set of G. If D= det $[LP_k(G) - \frac{2m}{n}I]$, then*

$$LE_{P_k}(G) \geq \sqrt{2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} + n(n-1)D^{\frac{2}{n}}}.$$

Proof. We know that

$$\begin{aligned}
(LE_{P_k}(G))^2 &= \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 \\
&= \sum_{i=1}^{n} |\gamma_i| \sum_{j=1}^{n} |\gamma_j| \\
&= \left(\sum_{i=1}^{n} |\gamma_i|^2\right) + \sum_{i\neq j} |\gamma_i||\gamma_j| .
\end{aligned}$$

Now we use arithmetic mean and geometric mean inequality which is as follows.

$$\frac{1}{n(n-1)} \sum_{i\neq j} |\gamma_i||\gamma_j| \geq \left(\prod_{i\neq j} |\gamma_i||\gamma_j|\right)^{\frac{1}{n(n-1)}} .$$

$$\begin{aligned}
(LE_{P_k}(G))^2 &\geq \sum_{i=1}^{n} |\gamma_i|^2 + n(n-1)\left(\prod_{i\neq j} |\gamma_i||\gamma_j|\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n} |\gamma_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\gamma_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \sum_{i=1}^{n} \gamma_i^2 + n(n-1)D^{\frac{2}{n}} \\
&= \sum_{i=1}^{n} \mu_i^2 - \frac{4m^2}{n} + n(n-1)D^{\frac{2}{n}} \\
&= 2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} + n(n-1)D^{\frac{2}{n}}.
\end{aligned}$$

$Thus, LE_{P_k}(G) \geq \sqrt{2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} + n(n-1)D^{\frac{2}{n}}}$. □

Corollary 4.4. *If G is r-regular, then*

$$LE_{P_k}(G) = E_{P_k}(G) \geq \sqrt{2(4m_1 + m_2 + m_3) + n(n-1)D^{\frac{2}{n}}}.$$

We need the following two theorems to establish some more lower bounds for partition Laplacian energy of a graph.

Theorem 4.5. *[2] Let $a, a_1, a_2, \cdots, a_n, A$ and $b, b_1, b_2, \cdots, b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B$ for all $i = 1, 2, \cdots, n$ then the following inequality is valid. $\mid n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \mid \leq \alpha(n)(A-a)(B-b)$ where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and $[\frac{n}{2}]$ denotes the greatest integer part of $\frac{n}{2}$ and equality holds iff $a_1 = a_2 = \cdots = a_n$ and $b_1 = b_2 = \cdots = b_n$.*

Theorem 4.6. *[3] Let $a_i \neq 0, b_i, r$ and R are real numbers satisfying $ra_i \leq b_i \leq Ra_i$ then $\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_i b_i$.*

Theorem 4.7. *Let G be a graph with n vertices and m edges. If $\mid \gamma_1 \mid \geq \mid \gamma_2 \mid \geq, \cdots, \geq \mid \gamma_n \mid$ where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are auxiliary partition eigenvalues of G with respect to $P_k = \{V_1, V_2, \cdots, V_k\}$, Then*

$$LE_{P_k}(G) \geq \sqrt{n\left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2\right) - 4m^2 - \alpha(n)\left(\mid \gamma_1 \mid - \mid \gamma_n \mid\right)^2}.$$

Proof. Consider a graph G with n vertices and m edges.Given $\mid \gamma_1 \mid \geq \mid \gamma_2 \mid \geq, \cdots, \geq \mid \gamma_n \mid$. Put $a_i = \mid \gamma_i \mid, b_i = \mid \gamma_i \mid, a = \mid \gamma_n \mid, b = \mid \gamma_n \mid, A = B = \mid \gamma_1 \mid$ in Theorem 4.5 to get

$\left| n\sum_{i=1}^{n} \mid \gamma_i \mid^2 - (\sum_{i=1}^{n} \mid \gamma_i \mid)^2 \right| \leq \alpha(n)\left(\mid \gamma_1 \mid - \mid \gamma_n \mid\right)^2.$

But $\sum_{i=1}^{n} \mid \gamma_i \mid^2 = \left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2\right) - \frac{4m^2}{n}$ and

$LE_{P_k}(G) \leq \sqrt{n\left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} (d_i)^2\right) - 4m^2}$.

$\therefore \quad \left| n\left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2\right) - 4m^2 - \left(LE_{P_k}(G)\right)^2 \right| \leq \alpha(n)\left(\mid \gamma_1 \mid - \mid \gamma_n \mid\right)^2$

$\Rightarrow n\left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2\right) - 4m^2 - \left(LE_{P_k}(G)\right)^2 \leq \alpha(n)\left(\mid \gamma_1 \mid - \mid \gamma_n \mid\right)^2$
Hence,
$LE_{P_k}(G) \geq \sqrt{n\left(2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2\right) - 4m^2 - \alpha(n)\left(\mid \gamma_1 \mid - \mid \gamma_n \mid\right)^2}$. □

Theorem 4.8. *Let G be a graph with n vertices and m edges. If $\mid \gamma_1 \mid \geq \mid \gamma_2 \mid \geq, \cdots, \geq \mid \gamma_n \mid$ where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are auxiliary partition eigenvalues of G with respect to $P_k = \{V_1, V_2, \cdots, V_k\}$, Then*

$$LE_{P_k}(G) \geq \frac{2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} + n \mid \gamma_1 \mid\mid \gamma_n \mid}{\mid \gamma_1 \mid + \mid \gamma_n \mid}.$$

Proof. Consider a graph G with n vertices and m edges. Given $\mid \gamma_1 \mid \geq \mid \gamma_2 \mid \geq, \cdots, \geq \mid \gamma_n \mid$. Choose $b_i = \mid \gamma_i \mid$ and $a_i = 1$, $r = \mid \gamma_n \mid$ and $R = \mid \gamma_1 \mid$ Then $\mid \gamma_n a_i \mid \leq \mid \gamma_i \mid \leq \mid \gamma_1 a_i \mid$ and by Theorem 4.6
$\sum_{i=1}^{n} \mid \gamma_i \mid^2 + \mid \gamma_1 \mid\mid \gamma_n \mid \sum_{i=1}^{n} 1 \leq (\mid \gamma_1 \mid + \mid \gamma_n \mid)\sum_{i=1}^{n} \mid \gamma_i \mid$

$\Rightarrow 2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} + n \mid \gamma_1 \mid\mid \gamma_n \mid \leq (\mid \gamma_1 \mid + \mid \gamma_n \mid) LE_{P_k}(G)$.
Hence,
$LE_{P_k}(G) \geq \frac{2(4m_1 + m_2 + m_3) + \sum_{i=1}^{n} d_i^2 - \frac{4m^2}{n} + n \mid \gamma_1 \mid\mid \gamma_n \mid}{\mid \gamma_1 \mid + \mid \gamma_n \mid}$. □

5 Partition Laplacian energy of some family of graphs

In this section we prove that k-partition Laplacian energy of a r-regular graph G is equal to its k-partition energy with respect to any partition P_k of V. We obtain partition Laplacian energy of complete product of two circulant graphs, its k-complement, $k(i)$-complement. Also we determine partition Laplacian energy of star graph, its $(k+1)(i)$-complement, multipartite graphs and its $k(i)$-complements.

Theorem 5.1. *Let $G = (V, E)$ be r-regular graph with n vertices, m edges and $P_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of V then, $LE_{P_k}(G) = E_{P_k}(G)$.*

Proof. We know that $L\Phi_k(G,x) = det[xI - LP_k(G)] = det[xI - rI + P_k(G)] = det[(x-r)I + P_k(G)] = (-1)^n det[(r-x)I - P_k(G)] = (-1)^n \Phi_k(G, r-x)$.
Thus, if $\lambda_1, \lambda_2, \cdots, \lambda_n$ represent the k-partition eigenvalues of a r-regular graph G, then $r-\lambda_1, r-\lambda_2, \cdots, r-\lambda_n$ represent the k-partition Laplacian eigenvalues of G. Also for a r-regular graph G, $\gamma_i = \mu_i - \frac{2m}{n} = \mu_i - r = -\lambda_i$.
Hence $LE_{P_k}(G) = E_{P_k}(G)$. □

Theorem 5.2. *[10] Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two r_1, r_2 regular graphs of order n_1, n_2 with $\Phi_1(G_1 : \lambda)$, $\Phi_1(G_2 : \lambda)$ as characteristic polynomials respectively. Then the characteristic polynomial of $G_1 \nabla G_2$ with respect to the partition $P_2 = \{V_1, V_2\}$ is*

$$\Phi_2(G_1 \nabla G_2 : \lambda) = \frac{\Phi_1(G_1 : \lambda)\Phi_1(G_2 : \lambda)[(\lambda - (3r_1 - n_1 + 1))(\lambda - (3r_2 - n_2 + 1)) - n_1 n_2]}{[\lambda - (3r_1 - n_1 + 1)][\lambda - (3r_2 - n_2 + 1)]},$$

where $3r_1 - n_1 + 1$ and $3r_2 - n_2 + 1$ are the 1-partition eigenvalues of G_1 and G_2 respectively.

Theorem 5.3. *If $G_i = (V_i, E_i)$ is a circulant graph of degree r_i with n_i vertices $i = 1, 2$ with $P_2 = \{V_1, V_2\}$ and $S_1 = LP_1(G_1) + n_2 I_{n_1}$, $S_2 = LP_1(G_2) + n_1 I_{n_2}$ then*

(1) $LE_{P_2}(G_1 \nabla G_2) = \sum_{t=1}^{n_1 - 1} \left| r_1 + n_2 - \lambda_t - \frac{2m_1}{n} \right| + \sum_{l=1}^{n_2 - 1} \left| r_2 + n_1 - \xi_l - \frac{2m_1}{n} \right| +$
$\left| n - (r_1 + r_2) - 1 + \sqrt{[n - (r_1 + r_2) - 1]^2 - (n - 2r_1 - 1)(n - 2r_2 - 1) + n_1 n_2} - \frac{2m_1}{n} \right| +$
$\left| n - (r_1 + r_2) - 1 - \sqrt{[n - (r_1 + r_2) - 1]^2 - (n - 2r_1 - 1)(n - 2r_2 - 1) + n_1 n_2} - \frac{2m_1}{n} \right|$ *where*
$m_1 = \frac{n_1 r_1 + n_2 r_2 + n_1 n_2}{2}$, *$\lambda_t$ and ξ_l are 1-partition eigenvalues of G_1 and G_2 respectively .*

(2) $LE_{P_2}(\overline{(G_1 \nabla G_2)_2}) = LE_{P_1}(G_1) + LE_{P_1}(G_2)$.

(3) $LE_{P_2}\overline{(G_1 \nabla G_2)_{2(i)}} = \sum_{t=1}^{n_1 - 1} \left| n - r_1 - 1 - \lambda'_t - \frac{2m_2}{n} \right| + \sum_{l=1}^{n_2 - 1} \left| n - r_2 - 1 - \xi'_l - \frac{2m_2}{n} \right| +$
$\left| r_1 + r_2 + 1 + \sqrt{(r_1 + r_2 + 1)^2 - (n_2 - n_1 + 2r_1 + 1)(n_1 - n_2 + 2r_1 + 1) + n_1 n_2} - \frac{2m_2}{n} \right| +$
$\left| r_1 + r_2 + 1 - \sqrt{(r_1 + r_2 + 1)^2 - (n_2 - n_1 + 2r_1 + 1)(n_1 - n_2 + 2r_1 + 1) + n_1 n_2} - \frac{2m_2}{n} \right|$ *where*
$n = n_1 + n_2$, $m_2 = \frac{n^2 - n - n_1 r_1 - n_2 r_2}{2}$ *and λ'_t and ξ'_l are 1-partition eigenvalues of $\overline{(G_1)_{1(i)}}$ and $\overline{(G_2)_{1(i)}}$ respectively .*

Proof. (1) The Laplacian partition matrix of $G_1 \nabla G_2$ with respect to $P_2 = \{V_1, V_2\}$ is

$$LP_2(G_1 \nabla G_2) = \left(\begin{array}{c|c} S_1 & B \\ \hline B^T & S_2 \end{array} \right)$$

where B is an $n_1 \times n_2$ matrix in which all the entries are $1's$.
Since G_1 and G_2 are regular graphs, it follows that row sum of $S_1 = LP_1(G_1) + n_2 I_{n_1}$ and $S_2 = LP_1(G_2) + n_1 I_{n_2}$ are $n_1 + n_2 - 2r_1 - 1$ and $n_1 + n_2 - 2r_2 - 1$ which represent eigenvalues of the matrices S_1 and S_2 respectively.
Hence by Theorem 4.2, we get the following.
$L\Phi_{P_2}(G_1 \nabla G_2, \mu)$=
$$\frac{\Phi(S_1, \mu)\Phi(S_2, \mu)[\mu^2 - 2\mu[n - (r_1 + r_2) - 1] + (n - 2r_1 - 1)(n - 2r_2 - 1) - n_1 n_2]}{(\mu - (n - 2r_1 - 1))(\mu - (n - 2r_2 - 1))}$$
where $\Phi(S_1, \mu)$ and $\Phi(S_2, \mu)$ represent the characteristic polynomials of S_1 and S_2 respectively.
Therefore the 2-partition Laplacian eigenvalues of $G_1 \nabla G_2$ are the roots of
$\frac{\Phi(S_1, \mu)}{\mu - (n - 2r_1 - 1)} = 0$, $\frac{\Phi(S_2, \mu)}{\mu - (n - 2r_2 - 1)} = 0$ and
$\mu^2 - 2\mu[n - (r_1 + r_2) - 1] + (n - 2r_1 - 1)(n - 2r_2 - 1) - n_1 n_2] = 0$.

Also G_1 and G_2 are circulant implies that the matrices S_1 and S_2 are circulant.
Hence the 2-partition Laplacian eigenvalues of $G_1\nabla G_2$ are

$$\begin{cases} \mu_1 & once \\ \mu_2 & once \\ r_1+n_2-\lambda_t & for\ \ t=1,2,\cdots,n_1-1 \\ r_2+n_1-\xi_l & for\ \ l=1,2,\cdots,n_2-1 \end{cases}$$

where $\mu_1 = n-(r_1+r_2)-1+\sqrt{[n-(r_1+r_2)-1]^2-(n-2r_1-1)(n-2r_2-1)+n_1n_2}$,

$\mu_2 = n-(r_1+r_2)-1-\sqrt{[n-(r_1+r_2)-1]^2-(n-2r_1-1)(n-2r_2-1)+n_1n_2}$,
$\lambda_t = \sum_{s=2}^{n_1} a_s e^{\frac{2\pi it(s-1)}{n_1}}$ and $\xi_l = \sum_{s=2}^{n_1} b_s e^{\frac{2\pi il(s-1)}{n_2}}$.
Here $-a_s$ and $-b_s$ are first row entries(except principal diagonal) of S_1 and S_2 respectively.
It can also be observed that λ_t and ξ_l are 1-partition eigenvalues of G_1 and G_2.
Consider $\gamma_i = \mu_i - \frac{2m_1}{n}$ where $m_1 = \dfrac{n_1r_1+n_2r_2+n_1n_2}{2}$ is the number of edges in $G_1\nabla G_2$.

Thus $LE_{P_2}(G_1\nabla G_2) = \sum_{t=1}^{n_1-1}\left|r_1+n_2-\lambda_t-\frac{2m_1}{n}\right| + \sum_{l=1}^{n_2-1}\left|r_2+n_1-\xi_l-\frac{2m_1}{n}\right| +$
$\left|n-(r_1+r_2)-1+\sqrt{[n-(r_1+r_2)-1]^2-(n-2r_1-1)(n-2r_2-1)+n_1n_2}-\frac{2m_1}{n}\right| +$
$\left|n-(r_1+r_2)-1-\sqrt{[n-(r_1+r_2)-1]^2-(n-2r_1-1)(n-2r_2-1)+n_1n_2}-\frac{2m_1}{n}\right|$.

(2) It can be easily observed that

$$LP_2\overline{(G_1\nabla G_2)_2} = \left(\begin{array}{c|c} LP_1(G_1) & \mathbf{0} \\ \hline \mathbf{0} & LP_1(G_2) \end{array}\right)$$

Hence $L\Phi_{P_2}(\overline{(G_1\nabla G_2)_2},\mu) = L\Phi_{P_1}(G_1,\mu)L\Phi_{P_1}(G_2,\mu)$

Thus $LE_{P_2}\overline{(G_1\nabla G_2)_2} = LE_{P_1}(G_1) + LE_{P_1}(G_2)$.

(3) The Laplacian partition matrix of $\overline{(G_1\nabla G_2)}_{2(i)}$ is

$$LP_2(\overline{(G_1\nabla G_2)}_{2(i)}) = \left(\begin{array}{c|c} H_1 & B \\ \hline B^T & H_2 \end{array}\right)$$

where $H_1 = LP_1\overline{(G_1)}_{1(i)} + n_2I_{n_1}$ and $H_2 = LP_1\overline{(G_2)}_{1(i)} + n_1I_{n_2}$.
Row sum of H_1 and H_2 are $n_2-n_1+2r_1+1$ and $n_1-n_2+2r_2+1$ which represent eigenvalues of H_1 and H_2 respectively. With simplification similar to (1), we get the partition Laplacian eigenvalues of $\overline{(G_1\nabla G_2)}_{2(i)}$ as follows.

$$\begin{cases} \mu_1 & once \\ \mu_2 & once \\ n-r_1-1-\lambda'_t & for\ \ t=1,2,\cdots,n_1-1 \\ n-r_2-1-\xi'_l & for\ \ l=1,2,\cdots,n_2-1 \end{cases}$$

where $\mu_1 = r_1+r_2+1+\sqrt{(r_1+r_2+1)^2-(n_2-n_1+2r_1+1)(n_1-n_2+2r_1+1)+n_1n_2}$

$\mu_2 = r_1+r_2+1-\sqrt{(r_1+r_2+1)^2-(n_2-n_1+2r_1+1)(n_1-n_2+2r_1+1)+n_1n_2}$,
$\lambda'_t = \sum_{s=2}^{n_1} c_s e^{\frac{2\pi it(s-1)}{n_1}}$ $\xi'_l = \sum_{s=2}^{n_2} d_s e^{\frac{2\pi il(s-1)}{n_2}}$ are 1-partition eigenvalues of $\overline{(G_1)}_{1(i)}$ and $\overline{(G_2)}_{1(i)}$

respectively and $-c_s$, $-d_s$ are first row entries(except principal diagonal) of H_1 and H_2 respectively. Thus

$$LE_{P_2}\overline{(G_1\nabla G_2)_{2(i)}} = \sum_{t=1}^{n_1-1}\left|n-r_1-1-\lambda'_t-\frac{2m_2}{n}\right| + \sum_{l=1}^{n_1-1}\left|n-r_2-1-\xi'_l-\frac{2m_2}{n}\right| +$$
$$\left|r_1+r_2+1+\sqrt{[(r_1+r_2+1)^2+(n_2-n_1+2r_1+1)(n_1-n_2+2r_1+1)+n_1n_2}-\frac{2m_2}{n}\right| +$$
$$\left|r_1+r_2+1-\sqrt{[(r_1+r_2+1)^2-(n_2-n_1+2r_1+1)(n_1-n_2+2r_1+1)+n_1n_2}-\frac{2m_2}{n}\right|$$ where

$n = n_1+n_2$ and $m_2 = \dfrac{n^2-n-n_1r_1-n_2r_2}{2}$ is the number of edges in $\overline{(G_1\nabla G_2)}_{2(i)}$. □

Theorem 5.4. *Let v_0 be the vertex of degree $n-1$ of $K_{1,n-1} = (V,E)$ and $P_{k+1} = \{V_0, V_1, V_2, \cdots, V_k\}$ be a partition of V where $V_0 = \{v_0\}$ and $V_j = \{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$ for $j = 1,2,\cdots,k$ and $n_1+n_2+\cdots+n_k = n-1$. Then*

(1) 0 is a $k+1$- partition Laplacian eigenvalue of $K_{1,n-1}$ repeated $n-k-1$ times.

(2) $$LE_{P_{k+1}}(K_{1,n-1}) = \left|\frac{2(n-1)}{n}\right|(n-k-1)+(k-1)\left|p-\frac{2(n-1)}{n}\right|$$
$$+\left|\frac{(n+p-1)+\sqrt{(n+p-1)^2-4(n-1)(p-1)}}{2}-\frac{2(n-1)}{n}\right|$$
$$+\left|\frac{(n+p-1)-\sqrt{(n+p-1)^2-4(n-1)(p-1)}}{2}-\frac{2(n-1)}{n}\right|$$
if $n_1 = n_2 = \cdots = n_k = p$.

Proof. The matrix $LP_k(K_{1,n-1})$ with respect to $P_{k+1} = \{V_0, V_1, V_2, \cdots, V_k\}$ is

$$= \begin{array}{c} - \\ v_0 \\ v_{11} \\ v_{12} \\ \vdots \\ v_{1n_1} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2n_2} \\ \vdots \\ v_{k1} \\ v_{k2} \\ \vdots \\ v_{kn_k} \end{array}
\begin{array}{c} \begin{array}{cccccccccccccc} v_0 & v_{12} & v_{13} & \cdots & v_{1n_1} & v_{21} & v_{22} & \cdots & v_{2n_2} & \cdots & v_{k1} & v_{k2} & \cdots & v_{kn_k} \end{array} \\
\begin{pmatrix}
n-1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 & \dots & -1 & -1 & \dots & -1 \\
-1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
-1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
-1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\
-1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 1 & \dots & 1 \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 1 & \dots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 & 1 & \dots & 1
\end{pmatrix} \end{array}$$

Consider the characteristic equation $\det[\mu I - LP_{k+1}(G)] = 0$ and perform the following operations.

(1) Subtract the row corresponding to the vertex v_{i1} from the rows corresponding to the vertices v_{ij} where $j = 2,3,\cdots,n_i$ for each $i = 1,2,3,\cdots,k$ and

(2) Add the columns corresponding to the vertices $v_{i2},\cdots,v_{in_i}$ to the column corresponding to v_{i1} for $i = 1,2,\cdots,k$.

On further simplification we get,

$$\mu^{n_1+n_2+\cdots+n_k-k}\begin{vmatrix} \mu-(n-1) & n_1 & n_2 & \ldots & n_k \\ 1 & \mu-n_1 & 0 & \ldots & 0 \\ 1 & 0 & \mu-n_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & \mu-n_k \end{vmatrix}=0.$$

This proves that $\mu=0$ is a root repeated $n_1+n_2+\cdots+n_k-k$ times. Further if $n_1=n_2=n_3=\cdots=n_k=p$ in the above determinant then it reduces to

$$\mu^{n-1-k}\begin{vmatrix} \mu-(n-1) & p & p & \ldots & p \\ 1 & \mu-p & 0 & \ldots & 0 \\ 1 & 0 & \mu-p & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & \mu-p \end{vmatrix}=0.$$

Multiply first column by $(p-\mu)$ and add all the other columns to it. This on further simplification gives $\mu^{n-1-k}(\mu-p)^{k-1}[\mu^2-\mu(n+p-1)+(n-1)(p-1)]=0$.
Hence its partition Laplacian spectrum is

$$\begin{cases} 0 & (n-k-1)\ times \\ p & k-1\ times \\ \dfrac{(n+p-1)+\sqrt{(n+p-1)^2-4(n-1)(p-1)}}{2} & once \\ \dfrac{(n+p-1)-\sqrt{(n+p-1)^2-4(n-1)(p-1)}}{2} & once \end{cases}$$

Consider $\gamma_i=\mu_i-\dfrac{2(n-1)}{n}$.
Thus $LE_{P_{k+1}}(K_{1,n-1})=\left|\dfrac{2(n-1)}{n}\right|(n-k-1)+(k-1)\left|p-\dfrac{2(n-1)}{n}\right|$
$+\left|\dfrac{(n+p-1)+\sqrt{(n+p-1)^2-4(n-1)(p-1)}}{2}-\dfrac{2(n-1)}{n}\right|$
$+\left|\dfrac{(n+p-1)-\sqrt{(n+p-1)^2-4(n-1)(p-1)}}{2}-\dfrac{2(n-1)}{n}\right|$
if $n_1=n_2=\cdots=n_k=p$. □

Theorem 5.5. *Let v_0 be the vertex of degree $n-1$ of $K_{1,n-1}=(V,E)$ and $P_{k+1}=\{V_0,V_1,V_2,\cdots,V_k\}$ be a partition of V where $V_0=\{v_0\}$ and $V_j=\{v_{j1},v_{j2},\cdots,v_{jn_j}\}$ for $j=1,2,\cdots,k$ and $n_1+n_2+\cdots+n_k=n-1$. Then*

(1) $2+p$ is a $k+1$- partition Laplacian eigenvalue of $\overline{(K_{1,n-1})_{(k+1)(i)}}$ repeated $n-k-1$ times.

(2) $LE_{P_{k+1}}(\overline{(K_{1,n-1})_{(k+1)(i)}})=\left|\dfrac{n+p+1}{n}\right|(n-k-1)+\left|\dfrac{n-2np+p+1}{n}(k-1)\right|$
$+\left|\dfrac{(n-p+1)+\sqrt{(n-p+1)^2+4(n-1)(p-1)}}{2}-\dfrac{(n-1)(p+1)}{n}\right|$
$+\left|\dfrac{(n-p+1)-\sqrt{(n-p+1)^2+4(n-1)(p-1)}}{2}-\dfrac{(n-1)(p+1)}{n}\right|$
if $n_1=n_2=\cdots=n_k=p$.

Proof. The matrix $LP_{k+1}(\overline{(K_{1,n-1})_{(k+1)(i)}})$ can be obtained by replacing the non diagonal entries 1 by -2 and 1 in the principal diagonal corresponding to $d(v_i)$ by n_i in $LP_{k+1}(K_{1,n-1})$. With operations similar to those in Theorem 5.4 we get $n_1+2,n_2+2,\cdots,n_k+2$ as partition laplacian eigenvalues of $\overline{(K_{1,n-1})_{(k+1)(i)}}$ repeated $n_1-1, n_2-1, \cdots, n_k-1$ times respectively. Further if $n_1=n_2=n_3=\cdots=n_k=p$, we get

$$(\mu-2-p)^{n-1-k}\begin{vmatrix} \mu-(n-1) & p & p & \dots & p \\ 1 & \mu+p-2 & 0 & \dots & 0 \\ 1 & 0 & \mu+p-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & \mu+p-2 \end{vmatrix} = 0.$$

Multiply first column by $-(\mu+p-2)$ and add all the other columns to it. Further simplification gives
$(\mu-2-p)^{n-1-k}(\mu+p-2)^{k-1}[\mu^2-\mu(n-p+1)-(n-1)(p-1)]=0.$
Hence its partition Laplacian spectrum is

$$\begin{cases} 2+p & (n-k-1) \ times \\ 2-p & k-1 \ times \\ \dfrac{(n-p+1)+\sqrt{(n-p+1)^2+4(n-1)(p-1)}}{2} & once \\ \dfrac{(n-p+1)-\sqrt{(n-p+1)^2+4(n-1)(p-1)}}{2} & once \end{cases}$$

Consider $\gamma_i=\mu_i-\dfrac{(n-1)(p+1)}{n}$. Then

$$LE_{P_{k+1}}(\overline{(K_{1,n-1})_{(k+1)(i)}}) = \left|\frac{n+p+1}{n}\right|(n-k-1)+\left|\frac{n-2np+p+1}{n}(k-1)\right|$$
$$+\left|\frac{(n-p+1)+\sqrt{(n-p+1)^2+4(n-1)(p-1)}}{2}-\frac{(n-1)(p+1)}{n}\right|$$
$$+\left|\frac{(n-p+1)-\sqrt{(n-p+1)^2+4(n-1)(p-1)}}{2}-\frac{(n-1)(p+1)}{n}\right|. \square$$

Theorem 5.6. *Let $P_k=\{V_1,V_2,\cdots,V_k\}$ be a partition of the complete multipartite graph $K_{n_1,n_2,\cdots,n_k}$ where $V_j=\{v_{j1},v_{j2},\cdots,v_{jn_j}\}$ where $j=1,2,\cdots,k$ and $n=n_1+n_2+\cdots+n_k$. Then $LE_{P_k}(K_{n_1,n_2,\cdots,n_k})=4n-4p-2k+2$ if $n_1=n_2=\cdots=n_k=p$.*

Proof. The partition Laplacian matrix of $K_{n_1,n_2,\cdots,n_k}$ with respect to $P_k=\{V_1,V_2,\cdots,V_k\}$ is

$$\begin{array}{c|cccccccccccccc} - & v_{11} & v_{12} & \cdots & v_{1n_1} & v_{21} & v_{22} & \cdots & v_{2n_2} & \cdots & v_{k1} & v_{k2} & \cdots & v_{kn_k} \\ \hline v_{11} & x_1 & 1 & \dots & 1 & -1 & -1 & \dots & -1 & \dots & -1 & -1 & \dots & -1 \\ v_{12} & 1 & x_1 & \dots & 1 & -1 & -1 & \dots & -1 & \dots & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{1n_1} & 1 & 1 & \dots & x_1 & -1 & -1 & \dots & -1 & \dots & -1 & -1 & \dots & -1 \\ v_{21} & -1 & -1 & \dots & -1 & x_2 & 1 & \dots & 1 & \dots & -1 & -1 & \dots & -1 \\ v_{22} & -1 & -1 & \dots & -1 & 1 & x_2 & \dots & 1 & \dots & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{2n_2} & -1 & -1 & \dots & -1 & 1 & 1 & \dots & x_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{k1} & -1 & -1 & \dots & -1 & -1 & -1 & \dots & 0 & \dots & x_k & 1 & \dots & 1 \\ v_{k2} & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 & \dots & 1 & x_k & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{kn_k} & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 & \dots & 1 & 1 & \dots & x_k \end{array}$$

where $x_i=n-n_i$ for $i=1$ to k. Consider the characteristic equation $\det[\mu I-LP_k(K_{n_1,n_2,\cdots,n_k})]=0$. By using operations similar to those in Theorem 5.4 we get the roots $\mu=(n-n_i-1)$ repeated

$n_i - 1$ times for $i = 1$ to k and

$$\begin{vmatrix} \mu-(n-1) & n_2 & n_3 & \dots & n_k \\ n_1 & \mu-(n-1) & n_3 & \dots & n_k \\ n_1 & n_2 & \mu-(n-1) & \dots & n_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n_1 & n_2 & n_3 & \dots & \mu-(n-1) \end{vmatrix} = 0.$$

If $n_1 = n_2 = \cdots = n_k = p$ then

$$\begin{vmatrix} \mu-(n-1) & p & p & \dots & p \\ p & \mu-(n-1) & p & \dots & p \\ p & p & \mu-(n-1) & \dots & p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p & p & p & \dots & \mu-(n-1) \end{vmatrix} = 0.$$

On simplification we get $(\mu - p + 1)[\mu - (n + p - 1)]^{k-1} = 0$.
Hence if $n_1 = n_2 = \cdots = n_k = p$, the k-partition Laplacian eigenvalues of $K_{n_1,n_2,\cdots,n_k}$ are

$$\begin{cases} n-p-1 & (n-k)\ times \\ n+p-1 & k-1\ times \\ p-1 & once \end{cases}$$

Also $\gamma_i = \mu_i - (k-1)p$.
Hence $LE_{P_k}(K_{n_1,n_2,\cdots,n_k}) = \mid -1 \mid (n-k) + \mid 2p-1 \mid (k-1) + \mid 2p-n-1 \mid$
$= 4n - 4p - 2k + 2.$ □

Theorem 5.7. *[10] The k-partition energy of $K_{n_1} \nabla K_{n_2} \nabla \cdots \nabla K_{n_k}$ in which each of k partitions contains $n_1, n_2, \ldots, n_k$ vertices respectively where $2 \le k \le \lfloor \frac{n}{2} \rfloor$, $n_1 = n_2 = \cdots = n_k = p$ and $n = kp$ is $4(n_1 + n_2 + \cdots + n_k - k)$.*

Theorem 5.8. *Let $P_k = \{V_1, V_2, \cdots, V_k\}$ be a partition of the vertex set of $\overline{(K_{n_1,n_2,\cdots,n_k})_{k(i)}}$ where $V_j = \{v_{j1}, v_{j2}, \cdots, v_{jn_j}\}$ where $j = 1, 2, \cdots, k$ and $n = n_1 + n_2 + \cdots + n_k$. Then $LE_{P_k}\overline{(K_{n_1,n_2,\cdots,n_k})_{k(i)}} = 4(n_1 + n_2 + \cdots + n_k - k)$ if $n_1 = n_2 = \cdots = n_k = p$.*

Proof. The matrix $LP_k\overline{(K_{n_1,n_2,\cdots,n_k})_{k(i)}}$ can be obtained from $P_k(K_{n_1,n_2,\cdots,n_k})$ by interchanging 1 and -2 in the non principal diagonal entries and replacing $x_i = n - n_i$ by $y_i = n - 1$ for all $i = 1$ to n which is nothing but the matrix $LP_k(K_{n_1} \nabla K_{n_2} \nabla \cdots \nabla K_{n_k})$. Since this is regular, from Theorem 5.1 and 5.7

$$LE_{P_k}\overline{(K_{n_1,n_2,\cdots,n_k})_{k(i)}} = E_{P_k}(K_{n_1} \nabla K_{n_2} \nabla \cdots \nabla K_{n_k}) = 4(n_1 + n_2 + \cdots + n_k - k)$$

where $n_1 = n_2 = \cdots = n_k = p$. □

References

[1] C. Adiga, E. Sampathkumar, M. A. Sriraj, Shrikanth A. S, Color energy of a graph, *Proc. Jangjeon Math. Soc.*, 16 (2013), No. 3, 335-351.

[2] M. Biernacki, H. Pidek, C.Ryll-Nardzewki, Sur une inégalité entre des intégrales définies, *Univ. Marie Curie-Sklodowaka* **A4**(1950) 1-4.

[3] J. B. Diaz and F. T. Metcalf, Stronger forms of a class of inequalities of G. Pólya-G. Szegő and L. V. Kantorovich, *Bull. Amer. Math. Soc.* 69 (1963) 415-418.

[4] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, 103(1978), 1-22.

[5] I. Gutman, B. Zhou, The Laplacian energy of a graph, *Lin. Algebra Appl.*, 414(2006), 29-37.

[6] G. Indulal, I. Gutman, A. Vijayakumar, On distance energy of graphs, *Match Commun. Math. Comput. Chem.*, 60(2008), 461-472.

[7] Pradeep G Bhat, Sabitha D'Souza, Color Laplacian energy of a graph, *Proc. Jangjeon Math. Soc.*, 18(2015), No.3, 321-330.

[8] E. Sampathkumar, L. Pushpalatha, C. V. Venkatachalam and Pradeep Bhat, Generalized complements of a graph, *Indian J. Pure and Appl. Math.*, 29(6)(1998), 625-639.

[9] E. Sampathkumar and M. A. Sriraj, Vertex labeled/colored graphs, matrices and signed Graphs, *J. of Combinatorics, Information and System Sciences*, 38(2014), 113-120.

[10] E. Sampathkumar, S. V. Roopa, K. A. Vidya and M. A. Sriraj, Partition Energy of a graph, *Proc.Jangjeon Math. Soc.*, 18(2015), No.4, 473-493.

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