Palestine Journal of Mathematics

Vol. 8(1)(2019) , 294–298

A note on q-Fekete-Szeg$\ddot{O}$ problem

HAMID SHAMSAN , FUAD. S. M. ALSARARI , S. LATHA

Communicated by Ayman Badawi

2010 AMS Subject Classification: 30C45.

Keywords and phrases: Coefficient estimates, Fekete-Szeg$\ddot{o}$ problem, q-starlike functions, q-convex functions.

Abstract. In this present investigation, we obtain Fekete-Szeg$\ddot{o}$ inequalities for the classes of q-starlike and q-convex functions.

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}, \tag{1.1}$$

and suppose $\mathcal{S}$ is the set of all functions in $\mathcal{A}$ that are univalent in the unit disk $\mathbb{D}$.

A classical result of Fekete and Szeg$\ddot{o}$ [1] determined the maximum value of $|a_3 - \mu a_2^2|$ as a function of the real parameter μ, for the class of univalent function f. There are several results of this type in the literature, dealing with estimates for various classes of functions [2, 3, 5]. In [4], Jackson introduced and studied the concept of the q-derivative ∂_q as follows:

For $(0 < q < 1)$, we define

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{1.2}$$

Equivalently (1.2), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Note that as $q \to 1$, $[n]_q \to n$.

Definition 1.1. For $(0 < q < 1)$ let S_q^* and K_q denote classes of q-starlike and q-convex univalent functions, respectively, i.e.

$$S_q^* = \left\{ f(z) \in \mathcal{S} : Re\left\{ \frac{z\partial_q f(z)}{f(z)} \right\} > 0, \ \ z \in \mathbb{D} \right\} \tag{1.3}$$

and

$$K_q = \left\{ f(z) \in \mathcal{S} : Re\left\{ \frac{\partial_q \left(z\partial_q f(z)\right)}{\partial_q f(z)} \right\} > 0, \ \ z \in \mathbb{D} \right\}. \tag{1.4}$$

Observe that $S_1^* = S^*$ and $K_1 = K$, where S^* and K are the classes of starlike and convex univalent functions, respectively.

2 Main results

We denote by $\mathcal{P}$ a class of the analytic functions in $\mathbb{D}$ with $p(0) = 1$ and $Re\{p(z)\} > 0$. We need the following Lemma:

Lemma 2.1. *[6], Let$p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + ...$, then*

$$|c_n| \leq 2, \quad n \geq 1.$$

If $|c_1| = 2$ then $p(z) \equiv p_1(z) = \frac{(1+\gamma_1 z)}{(1-\gamma_1 z)}$ with $\gamma_1 = \frac{c_1}{2}$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ and $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 \gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1 + \bar{\gamma}_1 \gamma_2 z}},$$

and $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$. Conversely if $p(z) = p_2(z)$ for some $|\gamma_1| < 1$ and $|\gamma_2| = 1$, then $\gamma_1 = \frac{c_1}{2}$, $\gamma_2 = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$.

Theorem 2.2. *If f of the form (1.1) is in S_q^*, then*

$$|a_2| \leq \frac{2}{[2]_q - 1}, \tag{2.1}$$

and

$$|a_3| \leq \frac{2}{[3]_q - 1} max \left\{ 1, 1 + \frac{2}{[2]_q - 1} \right\}, \tag{2.2}$$

and

$$\left| a_3 - \frac{[2]_q - 1}{[3]_q - 1} a_2^2 \right| \leq \frac{2}{[3]_q - 1}. \tag{2.3}$$

Equality in (2.1) holds if $\frac{z\partial_q f(z)}{f(z)} = p_1(z)$ and in (2.2) if $\frac{z\partial_q f(z)}{f(z)} = p_2(z)$ where p_1, p_2 are given as in Lemma 2.1.

Proof. For a function f given by (1.1) and by the definition of the class S_q^* there exists $p \in \mathcal{P}$ such that

$$\frac{z\partial_q f(z)}{f(z)} = p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...,$$

so that

$$\frac{(z + [2]_q a_2 z^2 + [3]_q a_3 z^3 + ...)}{(z + a_2 z^2 + a_3 z^3 + ...)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...,$$

which implies the equality

$$z + [2]_q a_2 z^2 + [3]_q a_3 z^3 + ... = z + (a_2 + c_1) z^2 + (a_3 + c_2 + a_2 c_1) z^3 + ...$$

Equating the coefficients on both sides we have

$$a_2 = \frac{c_1}{[2]_q - 1}, \quad a_3 = \frac{c_1^2 + c_2([2]_q - 1)}{([3]_q - 1)([2]_q - 1)}. \tag{2.4}$$

Using Lemma 2.1, we obtain

$$|a_2| = |\frac{c_1}{[2]_q - 1}| \leq \frac{2}{[2]_q - 1},$$

and

$$\begin{aligned}
|a_3| &= \left| \frac{([2]_q - 1)}{([3]_q - 1)([2]_q - 1)} \left\{ c_2 + \frac{c_1^2}{[2]_q - 1} \right\} \right| \\
&= \left| \frac{1}{[3]_q - 1} \left\{ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}(1 + \frac{2}{[2]_q - 1}) \right\} \right| \\
&\leq \frac{1}{[3]_q - 1} \left\{ 2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2} \left(|1 + \frac{2}{[2]_q - 1}| \right) \right\} \\
&= \frac{1}{[3]_q - 1} \left\{ 2 + \frac{|c_1|^2}{2} \left(|1 + \frac{2}{[2]_q - 1}| - 1 \right) \right\} \\
&\leq \frac{2}{[3]_q - 1} max\{1, |1 + \frac{2}{[2]_q - 1}|\}.
\end{aligned}$$

Thus

$$|a_3| \leq \frac{2}{[3]_q - 1} max\{1, |1 + \frac{2}{[2]_q - 1}|\}.$$

Moreover

$$\begin{aligned}
\left| a_3 - \frac{[2]_q - 1}{[3]_q - 1} a_2^2 \right| &= \left| \frac{1}{[3]_q - 1} [c_2 + \frac{c_1^2}{[2]_q - 1}] - \frac{c_1^2}{([2]_q - 1)^2} \frac{[2]_q - 1}{[3]_q - 1} \right| \\
&= \left| \frac{c_2}{[3]_q - 1} \right| \\
&\leq \frac{2}{[3]_q - 1},
\end{aligned}$$

as desired. $\square$

Remark. The above Theorem is a special case of Fekete-Szegö problem for real $\mu = \frac{([2]_q - 1)}{([3]_q - 1)}$ occurred very naturally and simple estimate was obtained. Now, we consider functional $|a_3 - \mu a_2^2|$ for complex μ.

Theorem 2.3. *Let $f \in S_q^*$. Then for $\mu \in \mathbb{C}$*

$$|a_3 - \mu a_2^2| \leq \frac{2}{[3]_q - 1} max \left\{ 1, \left| 1 + \frac{2}{[2]_q - 1} - 2\mu \frac{[3]_q - 1}{([2]_q - 1)^2} \right| \right\}.$$

Proof. Applying (2.5) we have

$$\begin{aligned}
a_3 - \mu a_2^2 &= \frac{1}{[3]_q - 1} \left\{ c_2 + \frac{c_1^2}{[2]_q - 1} \right\} - \mu \frac{c_1^2}{([2]_q - 1)^2} \\
&= \frac{1}{[3]_q - 1} \left\{ c_2 + \frac{c_1^2}{[2]_q - 1} - \mu \frac{[3]_q - 1}{([2]_q - 1)^2} c_1^2 \right\} \\
&= \frac{1}{[3]_q - 1} \left\{ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} \left(1 + \frac{2}{[2]_q - 1} - 2\mu \frac{[3]_q - 1}{([2]_q - 1)^2} \right) \right\}
\end{aligned}$$

Then by Lemma 2.1,

$$\begin{aligned}
|a_3 - \mu a_2^2| &\leq \frac{1}{[3]_q - 1} \left\{ 2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2} \left(\left| 1 + \frac{2}{[2]_q - 1} - 2\mu \frac{[3]_q - 1}{([2]_q - 1)^2} \right| \right) \right\} \\
&= \frac{1}{[3]_q - 1} \left\{ 2 + \frac{|c_1|^2}{2} \left(\left| 1 + \frac{2}{[2]_q - 1} - 2\mu \frac{[3]_q - 1}{([2]_q - 1)^2} \right| - 1 \right) \right\} \\
&\leq \frac{2}{[3]_q - 1} max \left\{ 1, \left| 1 + \frac{2}{[2]_q - 1} - 2\mu \frac{[3]_q - 1}{([2]_q - 1)^2} \right| \right\}.
\end{aligned}$$

$\square$

We next consider the case, when μ is real.

Theorem 2.4. *Let $f \in S_q^*$, then for $\mu \in \mathbb{R}$ we have*

$$\left|a_3 - \mu a_2^2\right| = \begin{cases} \frac{2}{[3]_q-1}\left\{1 + \frac{2}{[2]_q-1}\left(1 - \mu\frac{[3]_q-1}{[2]_q-1}\right)\right\}, & \text{if } \mu \le \sigma_1, \\ \frac{2}{[3]_q-1}, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ \frac{2}{[3]_q-1}\left\{2\mu\frac{[3]_q-1}{([2]_q-1)^2} - 1 - \frac{2}{[2]_q-1}\right\}, & \text{if } \mu \ge \sigma_2. \end{cases}$$

where $\sigma_1 = \frac{[2]_q-1}{[3]_q-1}, \qquad \sigma_2 = \frac{[3]_q([2]_q-1)}{[3]_q([2]_q-1)}.$

Proof. Let $\mu \le \sigma_1 \le \sigma_2$. In this case (2.5) and Lemma 2.1 give

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\le \frac{1}{[3]_q - 1}\left\{2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(1 + \frac{2}{[2]_q - 1} - 2\mu\frac{[3]_q - 1}{([2]_q - 1)^2}\right)\right\} \\ &\le \frac{2}{[3]_q - 1}\left\{1 + \frac{2}{[2]_q - 1}\left(1 - \mu\frac{[3]_q - 1}{([2]_q - 1)}\right)\right\}. \end{aligned}$$

Let, now $\sigma_1 \le \mu \le \sigma_2$. Then we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{2}{[3]_q - 1}.$$

Finally, if $\mu \ge \sigma_2$, then

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\le \frac{1}{[3]_q - 1}\left\{2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(2\mu\frac{[3]_q - 1}{([2]_q - 1)^2} - 1 - \frac{2}{[2]_q - 1}\right)\right\} \\ &= \frac{1}{[3]_q - 1}\left\{2 + \frac{|c_1|^2}{2}\left(2\mu\frac{[3]_q - 1}{([2]_q - 1)^2} - 2 - \frac{2}{[2]_q - 1}\right)\right\} \\ &\le \frac{2}{[3]_q - 1}\left\{\left(2\mu\frac{[3]_q - 1}{([2]_q - 1)^2} - 1 - \frac{2}{[2]_q - 1}\right)\right\}. \end{aligned}$$

Thus the proof is complete. □

It can be easily seen that

$$f(z) \in k_q \Leftrightarrow z\partial_q f(z) \in S_q^*. \tag{2.5}$$

Using (2.5) we easily obtain bounds of coefficients and a solution of the Fekete-Szegö problem in K_q.

Theorem 2.5. *If f of the form (1.1) is in K_q then*

$$|a_2| \le \frac{2}{[2]_q([2]_q - 1)}, \tag{2.6}$$

and

$$|a_3| \le \frac{2([2]_q + 1)}{[3]_q([3]_q - 1)}, \tag{2.7}$$

and

$$\left|a_3 - \frac{([2]_q)^2}{[3]_q([3]_q - 1)}a_2^2\right| \le \frac{2}{[3]_q([3]_q - 1)}. \tag{2.8}$$

Reasoning in the same lines as in the proof of Theorem 2.3 we obtain

Theorem 2.6. *Let $f \in K_q$. Then for $\mu \in \mathbb{C}$ holds*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{[3]_q([3]_q - 1)} max\left\{1, \left|1 + \frac{2}{[2]_q - 1} - 2\mu\frac{[3]_q([3]_q - 1)}{([2]_q)^2([2]_q - 1)^2}\right|\right\}.$$

References

[1] Fekete M. and Szegö G., *Eine Bermerkung Über ungerade schlichte Funktionen,* J. London Math. Soc., 8 (1933) 85-89.

[2] Keogh F. R. and Merkes E. P., *A coefficient inequality for certain classes of analytic functions,* Proc. Amer. Math. Soc., 20 (1969) 8-12.

[3] Koepf W., *On the Fekete-Szegö problem for close-to-convex functions,* Proc. Amer. Math. Soc., 101 (1987) 89-95.

[4] Jackson F. H., *On q-functions and a certain difference operator,* Trans. Royal Soc. Edinburgh, 46 (1909) 253-281.

[5] Pfluger A., *The fekete-szegö inequality for complex parameters,* Complex Variables and Elliptic Equations, 7 (1-3) (1986) 149-160.

[6] Pommerenke C. and Gensen G., *Univalent Functions,* Vandenhoeck und Ruprecht, Vol. 25. 1975.

Author information

HAMID SHAMSAN , FUAD. S. M. ALSARARI , S. LATHA, Department of Mathematics, Yuvaraja's College, University of Mysore, Mysore 570 005, INDIA.
E-mail: hmas19771@gmail.com; alsrary@yahoo.com; drlatha@gmail.com

Received: June 8, 2017.
Accepted: December 28, 2017