Palestine Journal of Mathematics

Vol. 8(1)(2019) , 312–317

COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS OF TYPE(A-1) IN FUZZY METRIC SPACE

Dr. Varsha Sharma

Communicated by Ayman Badawi

MSC 2010 Classifications: 54H25, 47H10.

Keywords: Common fixed point, fuzzy metric space, Compatible mappings, Compatible mappings of type (A-1).

Abstract. The object of this paper is to establish a common fixed point theorem for compatible maps of type (A-1) on fuzzy metric space. Our result improves the result of Khan M. S. [7].

1 Introduction

The fuzzy theory has become an area of active research for the last fifty years. It has a wide range of applications in the field of science and engineering, for example, population dynamics, computer programming, nonlinear dynamical systems, medicine and so forth. The concept of fuzzy sets was introduced initially by Zadeh [12] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and application. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [6] and George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms.

Sessa [9] has introduced the concept of weakly commuting and Jungck [5] initiated the concept of compatibility. Cho [2] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. Singh et al. [10] proved fixed point theorems in a fuzzy metric space.

The concept of type A-compatible and S-compatible was given by Pathak and Khan [7]. Pathak et. al. [8] renamed A-compatible and S-compatible as compatible mappings of type (A-1) and compatible mappings of type (A-2) respectively.

B. Singh et. al. [10] proved fixed point theorems in fuzzy metric space and menger space using the concept of semicompatibility, weak compatibility and compatibility of type (β) respectively.

2 Preliminaries

Definition 2.1 (11). Let X be any set. A fuzzy set A in X is a function with domain X and Values in [0,1].

Definition 2.2 (3). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if ([0,1],$*$) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c and $d \in [0,1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.3 (3). The triplet $(X, M, *)$ is said to be a Fuzzy metric space if, X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions; for all x, y, z in X and $s, t > 0$,

(i) $M(x, y, 0) = 0, M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, t) : [0,\infty) \to [0,1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Remark 2.4. Every metric space (X, d) induces a fuzzy metric space $(X, M, *)$, where $a * b = \min\{a, b\}$ and for all $a, b \in X, M(x, y, t) = t/(t + d(x, y))$, for all $t > 0, M(x, y, 0) = 0$, which is called the fuzzy metric space induced by the metric d.

Definition 2.5 (3). A sequence $\{X_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be Convergent to x in X if, $\lim_{n\to\infty} M(X_n, X, t) = 1$, for each $t > 0$.

Definition 2.6 (3). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy Sequence if, $\lim_{n\to\infty} M(X_{n+p}, X_n, t) = 1$ for every $t > 0$ and for each $p > 0$.

A fuzzy metric space $(X, M, *)$ is Complete if, every Cauchy sequence in X converge to a point of X.

Definition 2.7 (4). Two self mappings P and Q of a fuzzy metric space $(X, M, *)$ are said to be Compatible, if $\lim_{n\to\infty} M(PQx_n, QPx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some z in X.

Definition 2.8 (1). Two self mappings P and Q of a fuzzy metric space $(X, M, *)$ are said to be Compatible of type (A), if $\lim_{n\to\infty} M(PQx_n, QQx_n, t) = \lim_{n\to\infty} M(QPx_n, PPx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some z in X.

Definition 2.9 (5). Two self mappings P and Q of a fuzzy metric space $(X, M, *)$ are said to be Compatible of type (A-1), if $\lim_{n\to\infty} M(QPx_n, PPx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some z in X.

Lemma 2.10 (10). *Let y_n is a sequence in an FM-space. If there exists a positive number $k < 1$ such that $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), t > 0, n \in N$, then $\{y_n\}$ is a Cauchy sequence in X.*

Lemma 2.11 (10). *If for two points x, y in X and a positive number $k < 1$ $M(x, y, kt) \geq M(x, y, t)$, then $x = y$.*

Next we give some properties of compatible mappings of type (A-1) which will be used in our main theorem.

Proposition 2.12 (7). *Let S and T be self maps of an FM-space X. If the pair (S, T) are Compatible of type (A-1) and $Sz = Tz$ for some z in X then $STz = TTz$.*

Proposition 2.13 (7). *Let S and T be self maps of an FM-space X with $t * t > t$ for all t in [0,1]. If the pair (S, T) are Compatible of type (A-1) and $Sx_n, Tx_n \to z$ for some z in X and a sequence $\{x_n\}$ in X then $TTx_n \to Sz$ if S is continuous at z.*

Proposition 2.14 (7). *Let S and T be self maps of an FM-space X. If the pair (S, T) are Compatible of type (A-1) and $Sz = Tz$ for some z in X then $TSz = SSz$.*

3 A Class of Implicit Relations

Let $K4$ be the set of all real continuous functions $F : R^4 \to R$, nondecreasing in first argument and satisfying the following conditions:

(a) for $u, v \geq 0, F(u, v, v, u) \geq 0$ or $F(u, v, u, v) \geq 0$ implies that $u \geq v$;

(b) $F(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

4 Main Result

We prove the following theorem

Theorem 4.1. *Let A, B, S and T be self mapping on a complete fuzzy metric space $(X, M, *)$, satisfying*
(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,
(ii) S and T are continuous,
(iii) For some $F \in K4$, there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,

$$F(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, kt)) \geq 0 \tag{1}$$

$$F(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, kt), M(By, Ty, t)) \geq 0 \tag{2}$$

If the pair (A, S) and (B, T) are compatible mappings of type (A-1), then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 0, 1, 2, ...$ Now putting in (1) $x = x_{2n}, y = x_{2n+1}$, we obtain

$$F\Big(M(Ax_{2n}, Bx_{2n+1}, kt), M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, kt)\Big) \geq 0$$

that is

$$F\Big(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, kt)\Big) \geq 0$$

Using (a), we get

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, t) \tag{3}$$

Analogously, putting $x = x_{2n+2}, y = x_{2n+1}$ in (2), we have

$$F\Big(M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+3}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+2}, t)\Big) \geq 0$$

Using (a), we get

$$M(y_{2n+3}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) \tag{4}$$

Thus, from (3) and (4), for n and t, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

Hence, by Lemma 2.9, $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore, $\{y_n\}$ converges to z in X. That is $\{Ax_{2n}\}\{Tx_{2n+1}\}, \{Bx_{2n+1}\}$ and $\{Sx_{2n}\}$ also converges to z in X.

Since the pair (A, S) and (B, T) are compatible mappings of type (A-1), then from proposition 2.13, we have

$$AAx_{2n} \to Sz \text{ and } BBx_{2n+1} \to Tz \tag{5}$$

By (1), putting $x = Ax_{2n}$ and $y = Bx_{2n+1}$, we get

$$F\Big(M(AAx_{2n}, BBx_{2n+1}, kt), M(SAx_{2n}, TBx_{2n+1}, t), M(AAx_{2n}, SAx_{2n}, t), M(BBx_{2n+1}, TBx_{2n+1}, kt)\Big) \geq 0$$

Taking limit $\to \infty$, using (5) and proposition 2.12 we get

$$F\Big(M(Sz, Tz, kt), M(Sz, Tz, t), M(Sz, Sz, t), M(Tz, Tz, kt)\Big) \geq 0$$

As F is non-decreasing in first argument, we have

$$F\Big(M(Sz,Tz,t),M(Sz,Tz,t),1,1\Big) \geq 0$$

Using (b), we have

$$M(Sz,Tz,t) \geq 1 \text{ for all } t > 0$$

Which gives $M(Sz,Tz,t) = 1$, that is

$$Sz = Tz \tag{6}$$

Again by inequality (1), putting $x = z$ and $y = Bx_{2n+1}$, we get

$$F\Big(M(Az,BBx_{2n+1},kt),M(Sz,TBx_{2n+1},t),M(Az,Sz,t),M(BBx_{2n+1},TBx_{2n+1},kt)\Big) \geq 0$$

Taking limit $n \to \infty$, using (5), (6) we get

$$F\Big(M(Az,Tz,kt),M(Sz,Tz,t),M(Az,Sz,t),M(Tz,Tz,kt)\Big) \geq 0$$
$$F\Big(M(Az,Sz,kt),M(Sz,Sz,t),M(Az,Sz,t),M(Tz,Tz,kt)\Big) \geq 0$$

As F is non-decreasing in first argument, we have

$$F\Big(M(Az,Sz,t),1,M(Az,Sz,t),1\Big) \geq 0$$

Using (a), we have $M(Az,Sz,t) \geq 1$ for all $t > 0$, Which gives $M(Az,Sz,t) = 1$. Thus

$$Az = Sz \tag{7}$$

Now by (iii) putting $x = z$ and $y = z$, we get

$$F\Big(M(Az,Bz,kt),M(Sz,Tz,t),M(Az,Sz,t),M(Bz,Tz,kt)\Big) \geq 0$$

Using (6) and (7)

$$F\Big(M(Az,Bz,kt),1,1,M(Bz,Az,kt)\Big) \geq 0$$

Using (a), we have $M(Az,Bz,kt) \geq 1$ for all $t > 0$, Which gives $M(Az,Bz,kt) = 1$. Thus

$$Az = Bz \tag{8}$$

Thus from (6), (7) and (8) we get

$$Az = Bz = Sz = Tz \tag{9}$$

Now we will prove that $Az = z$ By inequality (1), putting $x = z$ and $y = x_{2n+1}$,

$$F\Big(M(Az,Bx_{2n+1},kt),M(Sz,Tx_{2n+1},t),M(Az,Sz,t),M(Bx_{2n+1},Tx_{2n+1},kt)\Big) \geq 0$$

Taking limit $n \to \infty$, using (9) we get

$$F\Big(M(Az,z,kt),M(Sz,z,t),M(Az,Sz,t),M(z,z,kt)\Big) \geq 0$$
$$F\Big(M(Az,z,kt),M(Az,z,t),M(Az,Az,t),M(z,z,kt)\Big) \geq 0$$
$$F\Big(M(Az,z,kt),M(Az,z,t),1,1\Big) \geq 0$$

As F is non-decreasing in first argument, we have

$$F\Big(M(Az,z,t), M(Az,z,t), 1, 1\Big) \geq 0$$

Using (b), we have $M(Az,z,t) \geq 1$ for all $t > 0$, Which gives $M(Az,z,t) = 1$. Thus $Az = z$.

Combining all results , we get $z = Az = Bz = Sz = Tz$. From this we conclude that z is a common fixed point of A, B, S and T.

Uniqueness: Let z_1 be another common fixed point of A, B, S and T. Then

$$z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1$$
$$\text{and } z = Az = Bz = Sz = Tz$$

then by inequality (1), putting $x = z$ and $y = z_1$, we get

$$F\Big(M(Az, Bz, kt), M(Sz, Tz_1, t), M(Az, Sz, t), M(Bz_1, Tz_1, kt)\Big) \geq 0$$
$$F\Big(M(z, z_1, kt), M(z, z_1, t), 1, 1\Big) \geq 0$$

As F is non-decreasing in first argument, we have

$$F\Big(M(z, z_1, t), M(z, z_1, t), 1, 1\Big) \geq 0$$

Using (b), we have $M(z, z_1, t) \geq 1$ for all $t > 0$, Which gives $M(z, z_1, t) = 1$. Thus $z = z_1$. Thus z is the unique common fixed point of A, B, S and T.

If we increase the number of self maps from four to six then we have the following.

Corollary 4.2. *Let A, B, S, T, I and J be self mappings on a complete fuzzy metric space $(X, M, *)$, satisfying*
(i) $AB(X) \subseteq J(X)$ and $ST(X) \subseteq I(X)$,
(ii) I and J are continuous,
(iii) For some $F \in K_4$, there exists $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,

$$F\Big(M(ABx, STy, kt), M(Ix, Jy, t), M(ABx, Ix, t), M(STy, Jy, kt)\Big) \geq 0 \quad (1)$$

$$F\Big(M(ABx, STy, kt), M(Ix, Jy, t), M(ABx, Ix, kt), M(STy, Jy, t)\Big) \geq 0 \quad (2)$$

If the pair (AB, I) and (ST, J) are compatible mappings of type (A-1), then AB, ST, I and J have a unique common fixed point. Furthermore, if the pairs $(A, B), (A, I), (B, I), (S, T), (S, J)$ and (T, J) are commuting mapping then A, B, S, T, I and J have a unique common fixed point.

Proof. From theorem 3.1, z is the unique common fixed point of AB, ST, I and J. Finally, we need to show that z is also a common fixed point of A, B, S, T, I and J. For this, let z be the unique common fixed point of both the pairs (AB, I) and (ST, J). Then, by using commutativity of the pair $(A, B), (A, I)$ and (B, I), we obtain

$$Az = A(ABz) = A(BAz) = AB(Az), Az = A(Iz) = I(Az), \quad (3)$$
$$Bz = B(ABz) = B\big(A(Bz)\big) = BA(Bz) = AB(Bz), Bz = B(Iz) = I(Bz),$$

which shows that Az and Bz are common fixed point of (AB, I), yielding thereby

$$Az = z = Bz = Iz = ABz \quad (4)$$

in the view of uniqueness of the common fixed point of the pair (AB, I). Similarly, using the commutativity of $(S, T), (S, J), (T, J)$, it can be shown that

$$Sz = Tz = Jz = STz = z. \quad (5)$$

Now, we need to show that $Az = Sz(Bz = Tz)$ also remains a common fixed point of both the pairs (AB, I) and (ST, J). For this, put $x = z$ and $y = z$ in (1) and using (4) and (5), we get

$$F\Big(M(ABz, STz, kt), M(Iz, Jz, t), M(ABz, Iz, t), M(STz, Jz, kt)\Big) \geq 0,$$

that is,

$$F\Big(M(Az, Sz, kt), M(Az, Sz, t), M(Az, Az, t), M(Sz, Sz, kt)\Big) \geq 0,$$

As F is nondecreasing in first argument, we have

$$F\Big(M(Az, Sz, t), M(Az, Sz, t), 1, 1\Big) \geq 0,$$

Using (b), we obtain $M(Az, Sz, t) \geq 1$ for all $t > 0$ which gives $M(Az, Sz, t) = 1$, that is, $Az = Sz$. Similarly, it can be shown that $Bz = Tz$. Thus, z is the unique common fixed point of A, B, S, T, I and J. □

References

[1] Balasubramaniam P., Pant R. P., Common fixed point of four mappings in a fuzzy Metric spaces, *J. Fuzzy Math.* **10**, 2002, 379.

[2] Cho. Y. J., Pathak H. K., Kang S. M., *Fuzzy sets and systems*, **93**, 1998, 99-111.

[3] George A. and Veeramani P., "On some results in fuzzy metric spaces", *Fuzzy Sets and Systems*, **vol. 64**, no. 3, pp. 395-399, 1994.

[4] George A. and Veeramani P., *Fuzzy sets and systems*, 1997, 365-368.

[5] Jungck G., Compatible mapping and common fixed points, *Int. J. Math. Sci.*, **9**, 1986, 771-779.

[6] Kramosil O. and Michalek J., "Fuzzy metric and statistical metric spaces", *Kybernetika*, **vol. 11**, pp. 326-334, 1975.

[7] Khan M. S., Pathak H. K. and George Reny, *Int. Math. Forum* **2(11)**, 2007, 515-524.

[8] Pathak H. K. and Khan M. S., *Indian J. Pure appl. Math.*, **28(4)**, 1997, 477-485.

[9] Sessa S., On a weak commutativity condition of mapping in fixed point consideration *Publ. Inst. Math.*, **32(46)**, 1982, 149-153.

[10] Singh B. and Chauhan M. S., *Fuzzy sets and system*, 2000, 471-475.

[11] Vasuki R., "Common fixed points for R-weakly commuting maps in fuzzy metric spaces", *Indian Journal of Pure and Applied Mathematics*, **vol. 30**, no. 4, pp. 419-423, 1999.

[12] Zadeh L.A., Fuzzy Sets, *Inform. and Control*, **89**, 1965, 338-353.

Author information

Dr. Varsha Sharma, Department of Mathematics, IES, IPS Academy, Knowledge Village, AB Road, Indore (M.P.), 452012, India.
E-mail: math.varsha@gmail.com

Received: September 4, 2017.
Accepted: January 27, 2018