Palestine Journal of Mathematics

Vol. 8(1)(2019) , 318–326

ON TRIBONACCI AND TRIBONACCI-LUCAS QUATERNIONS POLYNOMIALS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B39; Secondary 11B37, 11R52.

Keywords and phrases: Tribonacci quaternion, Tribonacci-Lucas quaternion, Tribonacci quaternion polynomial, Tribonacci-Lucas quaternion polynomials, quaternion, recurrence relation.

Abstract. In this paper, we introduce the Tribonacci and Tribonacci-Lucas quaternion polynomials. We obtain the Binet formulas, generating functions and exponential generating functions of these quaternions. Moreover, we give some properties and identities for the Tribonacci and Tribonacci-Lucas quaternions.

1 Introduction

For any positive real number x, the Tribonacci and Tribonacci-Lucas polynomials, $\{T_n(x)\}_{n\in\mathbb{N}}$ and $\{t_n(x)\}_{n\in\mathbb{N}}$, are defined by, for $n \geq 3$,

$$T_n(x) = x^2 T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x) \tag{1.1}$$

and

$$t_n(x) = x^2 t_{n-1}(x) + xt_{n-2}(x) + t_{n-3}(x), \tag{1.2}$$

respectively, where $T_0(x) = 0$, $T_1(x) = 1$, $T_2(x) = x^2$, $t_0(x) = 3$, $t_1(x) = x^2$ and $t_2(x) = x^4 + 2x$.

Let $\alpha(x)$, $\omega_1(x)$ and $\omega_2(x)$ be the roots of the characteristic equation $\lambda^3 - x^2\lambda^2 - x\lambda - 1 = 0$. Then, the Binet formulas for the Tribonacci and Tribonacci-Lucas polynomials are given by

$$\begin{aligned} T_n(x) = {} & \frac{\alpha^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))} - \frac{\omega^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))} \\ & + \frac{\omega_2^{n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))},\ n \geq 0 \end{aligned}$$

and

$$t_n(x) = \alpha^n(x) + \omega_1^n(x) + \omega_2^n(x),\ n \geq 0$$

with $\alpha(x) = \frac{x^2}{3} + A(x) + B(x)$, $\omega_1(x) = \frac{x^2}{3} + \epsilon A(x) + \epsilon^2 B(x)$ and $\omega_2(x) = \frac{x^2}{3} + \epsilon^2 A(x) + \epsilon B(x)$, where

$$A(x) = \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} + \sqrt{\frac{x^6}{37} + \frac{7x^3}{54} + \frac{1}{4}}},$$

$$B(x) = \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} - \sqrt{\frac{x^6}{37} + \frac{7x^3}{54} + \frac{1}{4}}},$$

with $\epsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

One can easily see that

$$\alpha(x) + \omega_1(x) + \omega_2(x) = x^2 \text{ and } \alpha(x)\omega_1(x)\omega_2(x) = 1. \tag{1.3}$$

The generating functions of the Tribonacci and Tribonacci-Lucas polynomials are given by

$$G(y) = \sum_{n=0}^{\infty} T_n(x)y^n = \frac{y}{1 - x^2y - xy^2 - y^3} \tag{1.4}$$

and

$$g(y)=\sum_{n=0}^{\infty} t_n(x)y^n=\frac{3-2x^2y-xy^2}{1-x^2y-xy^2-y^3}, \tag{1.5}$$

where $T_n(x)$ is the n-th Tribonacci polynomial and $t_n(x)$ is the n-th Tribonacci-Lucas polynomial. For more details and properties related to the Tribonacci and Tribonacci-Lucas polynomials, we refer to [7, 16]. Taking $x=1$ in (1.4) and (1.5), we obtain the generating function of the Tribonacci and Tribonacci-Lucas numbers, respectively, for more of this type of numbers [2, 19].

On the other hand, a quaternion q, with real components q_r, q_i, q_j, q_k and basis $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, is an element of the form

$$q=q_r+q_i\mathbf{i}+q_j\mathbf{j}+q_k\mathbf{k},\ (q_r\mathbf{1}=q_r),$$

where

$$\begin{gathered}\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\\ \mathbf{ij}=-\mathbf{ji}=\mathbf{k},\ \mathbf{jk}=-\mathbf{kj}=\mathbf{i},\mathbf{ki}=-\mathbf{ik}=\mathbf{j}.\end{gathered} \tag{1.6}$$

In [9], Horadam defined the n-th Fibonacci and n-th Lucas quaternions as

$$Q_n=F_n+F_{n+1}\mathbf{i}+F_{n+2}\mathbf{j}+F_{n+3}\mathbf{k},\ n\geq 0$$

and

$$K_n=L_n+L_{n+1}\mathbf{i}+L_{n+2}\mathbf{j}+L_{n+3}\mathbf{k},\ n\geq 0$$

respectively, where F_n and L_n are the n-th Fibonacci number and the n-th Lucas number and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the multiplication rules (1.6).

Recently, in [6], we defined the n-th generalized Tribonacci quaternion as

$$Q_{v,n}=V_n+V_{n+1}\mathbf{i}+V_{n+2}\mathbf{j}+V_{n+3}\mathbf{k},\ n\geq 0,$$

where V_n is the n-th generalized Tribonacci number and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the multiplication rules (1.6).

The Fibonacci and Lucas quaternions have been studied in several papers. For instance, Iyer [13, 14] gave some relations connecting the Fibonacci and Lucas quaternions. Iakin [11, 12] introduced the concept of a higher order quaternion and generalized quaternions with quaternion components. In [10], Horadam studied the quaternion recurrence relations. Swamy [20] derived the relations of generalized Fibonacci quaternions. Halici [8] derived the generating functions and many other identities for the Fibonacci and Lucas quaternions. Akyiğit et al. [1] introduced the Fibonacci generalized quaternions. Cerda-Morales [5] gave some properties of third order Jacobsthal quaternions in a generalized quaternion algebra. Ramírez [17] has obtained some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions. Catarino [4] has derived some properties of the $h(x)$-Fibonacci quaternion polynomials. Polatli et al. [15] gave some properties of Split k-Fibonacci and Split k-Lucas quternions. Szynal-Liana and Włoch [21] have introduced Pell quaternions and Pell octonions. Bolat and İpek [3] have given various identities related to Pell quaternions and Pell-Lucas quaternions.

Inspired by these, in this paper, we introduce the Tribonacci and Tribonacci-Lucas quaternion polynomials. We obtain the Binet formulas, generating functions and exponential generating functions of these quaternions. Moreover, we give some properties for the Tribonacci and Tribonacci-Lucas quaternions.

2 Some properties of the Tribonacci and Tribonacci-Lucas quaternion polynomials

Definition 2.1. For $n\geq 0$, the Tribonacci and Tribonacci-Lucas quaternion polynomials are defined by

$$Q_{T,n}(x)=T_n(x)+T_{n+1}(x)\mathbf{i}+T_{n+2}(x)\mathbf{j}+T_{n+3}(x)\mathbf{k} \tag{2.1}$$

and

$$Q_{t,n}(x)=t_n(x)+t_{n+1}(x)\mathbf{i}+t_{n+2}(x)\mathbf{j}+t_{n+3}(x)\mathbf{k} \tag{2.2}$$

where $T_n(x)$ and $t_n(x)$ are the n-th Tribonacci polynomial and the n-th Tribonacci-Lucas polynomial. Here $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are quaternionic units which satisfy the multiplication rules (1.6).

Taking $x=1$ in (2.1) and (2.2), we obtain the usualTribonacci and Tribonacci-Lucas quaternions, respectively, for more of this type of numbers [6].

Proposition 2.2. *For $n \geq 0$, the following identities hold:*

$$Q_{T,n+3}(x)=x^2Q_{T,n+2}(x)+xQ_{T,n+1}(x)+Q_{T,n}(x), \tag{2.3}$$

$$Q_{t,n+3}(x)=x^2Q_{t,n+2}(x)+xQ_{t,n+1}(x)+Q_{t,n}(x). \tag{2.4}$$

Proof. (2.3): From equations (1.1) and (2.1), we obtain

$$\begin{aligned}
&x^2Q_{T,n+2}(x)+xQ_{T,n+1}(x)+Q_{T,n}(x)\\
&\quad= x^2(T_{n+2}(x)+T_{n+3}(x)\mathbf{i}+T_{n+4}(x)\mathbf{j}+T_{n+5}(x)\mathbf{k})\\
&\qquad+x(T_{n+1}(x)+T_{n+2}(x)\mathbf{i}+T_{n+3}(x)\mathbf{j}+T_{n+4}(x)\mathbf{k})\\
&\qquad+T_n(x)+T_{n+1}(x)\mathbf{i}+T_{n+2}(x)\mathbf{j}+T_{n+3}(x)\mathbf{k}\\
&\quad=(x^2T_{n+2}(x)+xT_{n+1}(x)+T_n(x))+(x^2T_{n+3}(x)+xT_{n+2}(x)+T_{n+1}(x))\mathbf{i}\\
&\qquad+(x^2T_{n+4}(x)+xT_{n+3}(x)+T_{n+2}(x))\mathbf{j}+(x^2T_{n+5}(x)+xT_{n+4}(x)+T_{n+3}(x))\mathbf{k}\\
&\quad=T_{n+3}(x)+T_{n+4}(x)\mathbf{i}+T_{n+5}(x)\mathbf{j}+T_{n+6}(x)\mathbf{k}=Q_{T,n+3}(x).
\end{aligned}$$

(2.4): The proof is similar to (2.3), using the equations (1.2) and (2.2). □

Theorem 2.3 (Binet formulas). *For $n \geq 0$, we have*

$$T_n(x)=\left\{\begin{array}{c}\frac{\underline{\alpha}\alpha^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\underline{\omega_1}\omega_1^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\+\frac{\underline{\omega_2}\omega_2^{n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\end{array}\right\} \tag{2.5}$$

and

$$t_n(x)=\underline{\alpha}\alpha^n(x)+\underline{\omega_1}\omega_1^n(x)+\underline{\omega_2}\omega_2^n(x), \tag{2.6}$$

*where $\underline{\alpha}=1+\alpha(x)\boldsymbol{i}+\alpha^2(x)\boldsymbol{j}+\alpha^3(x)\boldsymbol{k}$, $\underline{\omega_1}=1+\omega_1(x)\boldsymbol{i}+\omega_1^2(x)\boldsymbol{j}+\omega_1^3(x)\boldsymbol{k}$, $\underline{\omega_2}=1+\omega_2(x)\boldsymbol{i}+\omega_2^2(x)\boldsymbol{j}+\omega_2^3(x)\boldsymbol{k}$ and **i**, **j**, **k** are quaternion units which satisfy the multiplication rules (1.6).*

Proof. (2.5): Using the Definition 2.1, $Q_{T,n}(x)=T_n(x)+T_{n+1}(x)\mathbf{i}+T_{n+2}(x)\mathbf{j}+T_{n+3}(x)\mathbf{k}$ and the classic Binet formulas for the Tribonacci and Tribonacci-Lucas polynomials, we obtain

$$\begin{aligned}
Q_{T,n}(x)&=\left(\begin{array}{c}\frac{\alpha^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\omega_1^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\+\frac{\omega_2^{n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\end{array}\right)\\
&+\left(\begin{array}{c}\frac{\alpha^{n+2}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\omega_1^{n+2}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\+\frac{\omega_2^{n+2}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\end{array}\right)\mathbf{i}\\
&+\left(\begin{array}{c}\frac{\alpha^{n+3}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\omega_1^{n+3}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\+\frac{\omega_2^{n+3}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\end{array}\right)\mathbf{j}\\
&+\left(\begin{array}{c}\frac{\alpha^{n+4}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\omega_1^{n+4}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\+\frac{\omega_2^{n+4}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\end{array}\right)\mathbf{k}\\
&=\left\{\begin{array}{c}\frac{\alpha^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}(1+\alpha(x)\mathbf{i}+\alpha^2(x)\mathbf{j}+\alpha^3(x)\mathbf{k})\\-\frac{\omega_1^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}(1+\omega_1(x)\mathbf{i}+\omega_1^2(x)\mathbf{j}+\omega_1^3(x)\mathbf{k})\\+\frac{\omega_2^{n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}(1+\omega_2(x)\mathbf{i}+\omega_2^2(x)\mathbf{j}+\omega_2^3(x)\mathbf{k})\end{array}\right\}\\
&=\left\{\begin{array}{c}\frac{\underline{\alpha}\alpha^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\underline{\omega_1}\omega_1^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\+\frac{\underline{\omega_2}\omega_2^{n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\end{array}\right\}
\end{aligned}$$

and

$$\begin{aligned}
Q_{t,n}(x) &= t_n(x) + t_{n+1}(x)\mathbf{i} + t_{n+2}(x)\mathbf{j} + t_{n+3}(x)\mathbf{k} \\
&= \alpha^n(x) + \omega_1^n(x) + \omega_2^n(x) + (\alpha^{n+1}(x) + \omega_1^{n+1}(x) + \omega_2^{n+1}(x))\mathbf{i} \\
&\quad + (\alpha^{n+2}(x) + \omega_1^{n+2}(x) + \omega_2^{n+2}(x))\mathbf{j} + (\alpha^{n+3}(x) + \omega_1^{n+3}(x) + \omega_2^{n+3}(x))\mathbf{k} \\
&= \alpha^n(x)(1 + \alpha(x)\mathbf{i} + \alpha^2(x)\mathbf{j} + \alpha^3(x)\mathbf{k}) + \omega_1^n(x)(1 + \omega_1(x)\mathbf{i} + \omega_1^2(x)\mathbf{j} + \omega_1^3(x)\mathbf{k}) \\
&\quad + \omega_2^n(x)(1 + \omega_2(x)\mathbf{i} + \omega_2^2(x)\mathbf{j} + \omega_2^3(x)\mathbf{k}) \\
&= \underline{\alpha}\alpha^n(x) + \underline{\omega_1}\omega_1^n(x) + \underline{\omega_2}\omega_2^n(x).
\end{aligned}$$

□

Theorem 2.4. *The generating functions for the Tribonacci and Tribonacci-Lucas quaternion polynomials are*

$$G_T(y) = \frac{y + \boldsymbol{i} + (x^2 + xy + y^2)\boldsymbol{j} + (x^4 + x + x^3y + y + x^2y^2)\boldsymbol{k}}{1 - x^2y - xy^2 - y^3} \tag{2.7}$$

and

$$g_t(y) = \frac{\left\{\begin{array}{c} 3 - 2x^2y - xy^2 + (x^2 + 2xy + 3y^2)\boldsymbol{i} + (x^4 + 2x + x^3y + 3y + x^2y^2)\boldsymbol{j} \\ +(x^6 + 3x^3 + 3 + x^5y + 3x^2y + x^4y^2 + 2xy^2)\boldsymbol{k} \end{array}\right\}}{1 - x^2y - xy^2 - y^3}, \tag{2.8}$$

respectively.

Proof. Let $G_T(y) = \sum_{n=0}^{\infty} Q_{T,n}(x)y^n$ and $g_t(y) = \sum_{n=0}^{\infty} Q_{t,n}(x)y^n$. Then we get the following equation

$$\begin{aligned}
&(1-x^2y - xy^2 - y^3)G_T(y) \\
&\quad = Q_{T,0}(x) + (Q_{T,1}(x) - x^2Q_{T,0}(x))y + (Q_{T,2}(x) - x^2Q_{T,1}(x) - xQ_{T,0}(x))y^2 \\
&\quad + \sum_{n=3}^{\infty}(Q_{T,n}(x) - x^2Q_{T,n-1}(x) - xQ_{T,n-2}(x) - Q_{T,n-3}(x))y^n.
\end{aligned}$$

Since, for each $n \geq 3$, the coefficient of y^n is zero in the right-hand side of this equation, we obtain

$$\begin{aligned}
G_T(y) &= \frac{Q_{T,0}(x) + (Q_{T,1}(x) - x^2Q_{T,0}(x))y + (Q_{T,2}(x) - x^2Q_{T,1}(x) - xQ_{T,0}(x))y^2}{1 - x^2y - xy^2 - y^3} \\
&= \frac{y + \mathbf{i} + (x^2 + xy + y^2)\mathbf{j} + (x^4 + x + x^3y + y + x^2y^2)\mathbf{k}}{1 - x^2y - xy^2 - y^3}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
g_t(y) &= \frac{Q_{t,0}(x) + (Q_{t,1}(x) - x^2Q_{t,0}(x))y + (Q_{t,2}(x) - x^2Q_{t,1}(x) - xQ_{t,0}(x))y^2}{1 - x^2y - xy^2 - y^3} \\
&= \frac{\left\{\begin{array}{c} 3 - 2x^2y - xy^2 + (x^2 + 2xy + 3y^2)\mathbf{i} + (x^4 + 2x + x^3y + 3y + x^2y^2)\mathbf{j} \\ +(x^6 + 3x^3 + 3 + x^5y + 3x^2y + x^4y^2 + 2xy^2)\mathbf{k} \end{array}\right\}}{1 - x^2y - xy^2 - y^3}.
\end{aligned}$$

The proof is completed. □

Theorem 2.5. *For $m \geq 2$, the generating functions for the Tribonacci quaternion polynomial $\{Q_{T,n+m}(x)\}_{n\geq 0}$ and the Tribonaci-Lucas quaternion polynomial $\{Q_{t,n+m}(x)\}_{n\geq 0}$ are*

$$\sum_{n=0}^{\infty} Q_{T,n+m}(x)y^n = \frac{Q_{T,m}(x) + (xQ_{T,m-1}(x) + Q_{T,m-2})y + Q_{T,m-1}(x)y^2}{1 - x^2y - xy^2 - y^3} \tag{2.9}$$

and

$$\sum_{n=0}^{\infty} Q_{t,n+m}(x)y^n = \frac{Q_{t,m}(x) + (xQ_{t,m-1}(x) + Q_{t,m-2})y + Q_{t,m-1}(x)y^2}{1 - x^2y - xy^2 - y^3}, \tag{2.10}$$

respectively.

Proof. Here we will just prove (2.10) since the proof of (2.9) can be done in a similar way. Using Theorem 2.3 and Theorem 2.4, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{t,n+m}(x)y^n &= \sum_{n=0}^{\infty}(\underline{\alpha}\alpha^{n+m}(x) + \underline{\omega_1}\omega_1^{n+m}(x) + \underline{\omega_2}\omega_2^{n+m}(x))y^n \\
&= \underline{\alpha}\alpha^m(x)\sum_{n=0}^{\infty}(\alpha(x)y)^n + \underline{\omega_1}\omega_1^m(x)\sum_{n=0}^{\infty}(\omega_1(x)y)^n + \underline{\omega_2}\omega_2^m(x)\sum_{n=0}^{\infty}(\omega_2(x)y)^n \\
&= \underline{\alpha}\alpha^m(x)\frac{1}{1-\alpha(x)y} + \underline{\omega_1}\omega_1^m(x)\frac{1}{1-\omega_1(x)y} + \underline{\omega_2}\omega_2^m(x)\frac{1}{1-\omega_2(x)y} \\
&= \frac{Q_{t,m}(x) + (xQ_{t,m-1}(x) + Q_{t,m-2})y + Q_{t,m-1}(x)y^2}{1 - x^2y - xy^2 - y^3}.
\end{aligned}$$

□

Theorem 2.6. *For $n \in \mathbb{N}$, the exponential generating functions for the Tribonacci and Tribonacci-Lucas quaternion polynomials are*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{Q_{T,n}}{n!}y^n = &\frac{\underline{\alpha}\alpha(x)e^{\alpha(x)y}}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))} - \frac{\underline{\omega_1}\omega_1(x)e^{\omega_1(x)y}}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))} \\
&+ \frac{\underline{\omega_2}\omega_2(x)e^{\omega_2(x)y}}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}
\end{aligned} \tag{2.11}$$

and

$$\sum_{n=0}^{\infty} \frac{Q_{t,n}}{n!}y^n = \underline{\alpha}e^{\alpha(x)y} + \underline{\omega_1}e^{\omega_1(x)y} + \underline{\omega_2}e^{\omega_2(x)y}, \tag{2.12}$$

respectively, where $\underline{\alpha} = 1 + \alpha(x)\boldsymbol{i} + \alpha^2(x)\boldsymbol{j} + \alpha^3(x)\boldsymbol{k}$, $\underline{\omega_1} = 1 + \omega_1(x)\boldsymbol{i} + \omega_1^2(x)\boldsymbol{j} + \omega_1^3(x)\boldsymbol{k}$, $\underline{\omega_2} = 1+\omega_2(x)\boldsymbol{i}+\omega_2^2(x)\boldsymbol{j}+\omega_2^3(x)\boldsymbol{k}$ *and* $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ *are quaternion units which satisfy the multiplication rules (1.6).*

Proof. By considering the Binet formulas for the Tribonacci and Tribonacci-Lucas quaternion polynomials given in Theorem 2.3, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{Q_{T,n}}{n!}y^n &= \sum_{n=0}^{\infty}\left(\begin{array}{c} \frac{\underline{\alpha}\alpha^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))} - \frac{\underline{\omega_1}\omega_1^{n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))} \\ + \frac{\underline{\omega_2}\omega_2^{n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))} \end{array}\right)\frac{y^n}{n!} \\
&= \frac{\underline{\alpha}\alpha(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}\sum_{n=0}^{\infty}\frac{(\alpha(x)y)^n}{n!} \\
&\quad - \frac{\underline{\omega_1}\omega_1(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\sum_{n=0}^{\infty}\frac{(\omega_1(x)y)^n}{n!} \\
&\quad + \frac{\underline{\omega_2}\omega_2(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\sum_{n=0}^{\infty}\frac{(\omega_2(x)y)^n}{n!} \\
&= \frac{\underline{\alpha}\alpha(x)e^{\alpha(x)y}}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))} - \frac{\underline{\omega_1}\omega_1(x)e^{\omega_1(x)y}}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))} \\
&\quad + \frac{\underline{\omega_2}\omega_2(x)e^{\omega_2(x)y}}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}
\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{Q_{t,n}}{n!}y^n &= \sum_{n=0}^{\infty}(\underline{\alpha}\alpha^n(x)+\underline{\omega_1}\omega_1^n(x)+\underline{\omega_2}\omega_2^n(x))\frac{y^n}{n!}\\ &= \underline{\alpha}\sum_{n=0}^{\infty}\frac{(\alpha(x)y)^n}{n!}+\underline{\omega_1}\sum_{n=0}^{\infty}\frac{(\omega_1 y)^n}{n!}+\underline{\omega_2}\sum_{n=0}^{\infty}\frac{(\omega_2 y)^n}{n!}\\ &= \underline{\alpha}e^{\alpha(x)y}+\underline{\omega_1}e^{\omega_1(x)y}+\underline{\omega_2}e^{\omega_2(x)y}.\end{aligned}$$

The proof is completed. □

Theorem 2.7. *For $n \geq 0$ and related with Tribonacci and Tribonacci-Lucas quaternion polynomials, we have*

$$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^{r+s}Q_{T,r+s}(x) = Q_{T,3n}(x) \tag{2.13}$$

and

$$\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^{r+s}Q_{t,r+s}(x) = Q_{t,3n}(x), \tag{2.14}$$

respectively.

Proof. In contrast, here we will just prove (2.13) since the proof of (2.14) can be done in a similar way. By the Binet formulas for the Tribonacci and Tribonacci-Lucas quaternion polynomials, we have

$$\begin{aligned}&\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^{r+s}Q_{T,r+s}(x)\\ &= \sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^{r+s}\left(\frac{\underline{\alpha}\alpha^{r+s+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}\right)\\ &-\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^{r+s}\left(\frac{\underline{\omega_1}\omega_1^{r+s+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\right)\\ &+\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}x^{r+s}\left(\frac{\underline{\omega_2}\omega_2^{r+s+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\right)\\ &= \frac{\underline{\alpha}\alpha(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}\sum_{r=0}^{n}\binom{n}{r}(x\alpha(x)+x^2\alpha^2(x))^r\\ &-\frac{\underline{\omega_1}\omega_1(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\sum_{r=0}^{n}\binom{n}{r}(x\omega_1(x)+x^2\omega_1^2(x))^r\\ &+\frac{\underline{\omega_2}\omega_2(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))}\sum_{r=0}^{n}\binom{n}{r}(x\omega_2(x)+x^2\omega_2^2(x))^r\\ &= \frac{\underline{\alpha}\alpha^{3n+1}(x)}{(\alpha(x)-\omega_1(x))(\alpha(x)-\omega_2(x))}-\frac{\underline{\omega_1}\omega_1^{3n+1}(x)}{(\alpha(x)-\omega_1(x))(\omega_1(x)-\omega_2(x))}\\ &+\frac{\underline{\omega_2}\omega_2^{3n+1}(x)}{(\alpha(x)-\omega_2(x))(\omega_1(x)-\omega_2(x))} = Q_{T,3n}(x).\end{aligned}$$

□

3 Matrix Representation of Tribonacci Quaternion Polynomials

The most useful technique for generating $\{Q_{T,n}(x)\}$ is by means of what we call the $S(x)$-matrix which has been defined and used in [18] and is a generalization of the R-matrix defined in [22].

We defined the $S(x)$-matrix by

$$\begin{bmatrix} T_{n+2}(x) \\ T_{n+1}(x) \\ T_n(x) \end{bmatrix} = \begin{bmatrix} x^2 & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} T_2(x) \\ T_1(x) \\ T_0(x) \end{bmatrix} \tag{3.1}$$

and

$$S^n(x) = \begin{bmatrix} x^2 & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} T_{n+1}(x) & xT_n(x)+T_{n-1}(x) & T_n(x) \\ T_n(x) & xT_{n-1}(x)+T_{n-2}(x) & T_{n-1}(x) \\ T_{n-1}(x) & xT_{n-2}(x)+T_{n-3}(x) & T_{n-2}(x) \end{bmatrix}, \tag{3.2}$$

where $T_{-1}(x) = 0$, $T_{-2}(x) = 1$ and $T_{-3}(x) = -x$ for convenience.

Now, let us define the following matrix as

$$Q_S(x) = \begin{bmatrix} Q_{T,4}(x) & xQ_{T,3}(x)+Q_{T,2}(x) & Q_{T,3}(x) \\ Q_{T,3}(x) & xQ_{T,2}(x)+Q_{T,1}(x) & Q_{T,2}(x) \\ Q_{T,2}(x) & xQ_{T,1}(x)+Q_{T,0}(x) & Q_{T,1}(x) \end{bmatrix}. \tag{3.3}$$

This matrix can be called as the *Tribonacci quaternion polynomial matrix*. Then, we can give the next theorem to the $Q_S(x)$-matrix.

Theorem 3.1. *If $Q_{T,n}(x)$ be the n-th Tribonacci quaternion polynomial. Then, for $n \geq 0$:*

$$Q_S(x) \cdot \begin{bmatrix} x^2 & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} Q_{T,n+4}(x) & P_{T,n+3}(x) & Q_{T,n+3}(x) \\ Q_{T,n+3}(x) & P_{T,n+2}(x) & Q_{T,n+2}(x) \\ Q_{T,n+2}(x) & P_{T,n+1}(x) & Q_{T,n+1}(x) \end{bmatrix}, \tag{3.4}$$

where $P_{T,n}(x) = xQ_{T,n}(x) + Q_{T,n-1}(x)$.

Proof. (By induction on n) If $n = 0$, then the result is obvious. Now, we suppose it is true for $n = m$, that is

$$Q_{S(x)} \cdot S^m(x) = \begin{bmatrix} Q_{T,m+4}(x) & P_{T,m+3}(x) & Q_{T,m+3}(x) \\ Q_{T,m+3}(x) & P_{T,m+2}(x) & Q_{T,m+2}(x) \\ Q_{T,m+2}(x) & P_{T,m+1}(x) & Q_{T,m+1}(x) \end{bmatrix},$$

with $P_{T,n}(x) = xQ_{T,n}(x) + Q_{T,n-1}(x)$. Using the Proposition 2.2, for $m \geq 0$, $Q_{T,m+3}(x) = x^2Q_{T,m+2}(x) + xQ_{T,m+1}(x) + tQ_{T,m}(x)$. Then, by induction hypothesis we obtain

$$\begin{aligned} Q_S(x) \cdot S^{m+1}(x) &= (Q_S(x) \cdot S^m(x)) \cdot S(x) \\ &= \begin{bmatrix} Q_{T,m+4}(x) & P_{T,m+3}(x) & Q_{T,m+3}(x) \\ Q_{T,m+3}(x) & P_{T,m+2}(x) & Q_{T,m+2}(x) \\ Q_{T,m+2}(x) & P_{T,m+1}(x) & Q_{T,m+1}(x) \end{bmatrix} \begin{bmatrix} x^2 & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_{T,m+5}(x) & P_{T,m+4}(x) & Q_{T,m+4}(x) \\ Q_{T,m+4}(x) & P_{T,m+3}(x) & Q_{T,m+3}(x) \\ Q_{T,m+3}(x) & P_{T,m+2}(x) & Q_{T,m+2}(x) \end{bmatrix}. \end{aligned}$$

Hence, the Eq. (3.4) holds for all $n \geq 0$. □

Corollary 3.2. *For $n \geq 0$,*

$$Q_{T,n+2}(x) = Q_{T,2}(x)T_{n+1}(x) + (xQ_{T,1}(x) + Q_{T,0}(x))T_n(x) + Q_{T,1}(x)T_{n-1}(x), \tag{3.5}$$

with $T_{-1}(x) = 0$ for convenience.

Proof. The proof can be easily seen by the coefficient (3,1) of the matrix $Q_S(x) \cdot S^n(x)$ and the Eq. (3.2). □

4 Conclusions

This study examines and studied Tribonacci and Tribonacci-Lucas quaternion polynomials with the help of a simple formula. For this purpose, Tribonacci and Tribonacci-Lucas polynomials was used and examined in detail particularly in Section 1, and it was shown that this sequences to generalize the Tribonacci and Tribonacci-Lucas numbers on quaternions. In this study, Binet formulas, generating functions, matrix representation and some properties of Tribonacci and Tribonacci-Lucas quaternion polynomials were obtained. Quaternions have great importance as they are used in quantum physics, applied mathematics and differential equations. Thus, in our future studies we plan to examine Tribonacci octonion polynomials and their key features.

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Received: September 4, 2017.
Accepted: November 29, 2017.