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NUMERICAL WAYS FOR FRACTIONAL OPTIMAL CONTROL PROBLEMS WITH TIME DELAYS AND MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, a composite Chebyshev finite difference (ChFD) strategy is presented and applied for finding the solution of fractional optimal control problems (FOCPs) with time delays. The exhibited technique is an extension of the ChFD method and using the Chebyshev-Gauss-Lobatto (CGL) points. A numerical example is stated to exhibit the legitimacy and appropriateness of the recommended approach. Also, In this work one may use the Bezier curve strategy for solving of multi-point boundary value problems (BVPs). Numerical results are demonstrated the validity of the suggested approach for solving of multi-point BVPs.

1 Introduction

A tremendous use of fractional calculus is in engineering, (see [1, 2, 3, 4, 5, 6, 7, 8]). Recently, the applications have included solving various classes of nonlinear fractional differential equations (FDEs) numerically (see [1, 7]). Also, the Adomian decomposition method is an approach to solve the linear/nonlinear systems of FDEs (see [9, 10, 11, 12]).
In the present paper, we introduce ChFD method and apply it for finding the solution of FOCPs with time delays.
The paper is organized as follows: In Section 2, we give basic preliminaries. In Section 3, the Shifted Chebyshev polynomials are presented. In Sections 4 and 5, we introduce an approximation of the Left CFD and right RLFD, respectively. Section 6 is devoted to expansion of the delay function by the composite ChFD method. one may state an example in Section 7. Also, a remark is stated. Finally, In Section 8, the conclusion is stated.

2 Basic preliminaries

Definition 2.1. Let $x : [a, b] \to \mathcal{R}$ be a function, $\alpha > 0$ a real number, and $n = \alpha$, where α denotes the smallest integer greater than or equal to α (see [13]). The left (left RLFD) and right (right RLFD) Riemann-Liouville fractional derivatives are follow as

$$\begin{aligned} {}_aD_t^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)\,d\tau, \quad \text{(left RLFD)},\\ {}_tD_b^\alpha x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1}x(\tau)\,d\tau, \quad \text{(right RLFD)}, \end{aligned} \tag{2.1}$$

In addition, the left (left CFD) and right (right CFD) Caputo fractional derivatives are

$$\begin{aligned} {}_a^CD_t^\alpha x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}x^{(n)}(\tau)\,d\tau, \quad \text{(left CFD)},\\ {}_t^CD_b^\alpha x(t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}x^{(n)}(\tau)\,d\tau, \quad \text{(right CFD)}, \end{aligned} \tag{2.2}$$

3 Shifted Chebyshev polynomials

Chebyshev polynomials are defined in $[-1,1]$ as follows:

$$T_{n+1}(z)=2zT_n(z)-T_{n-1}(z),\ T_0(z)=1,\ T_1(z)=z,\ n=1,2,\dots.$$

For using these polynomials on $[0,L]$, one may utilize Chebyshev polynomials by introducing the change of variable $z=\frac{2t}{L}-1$. Now, one may have

$$T_n^*(t)=T_n(\frac{2t}{L}-1),\ T_0^*(t)=1.$$

A function $x\in L^2([0,L])$ can be defined as:

$$x(t)=\sum_{j=0}^{\infty}c_nT_n^*(t),$$

where

$$c_n=\frac{1}{h_n}\int_0^L x(t)T_n^*(t)w(t)\,dt,\ n=0,1,\dots. \tag{3.1}$$

4 Approximation of the Left CFD

In the sequel, some basic results for the approximation of the fractional derivative are given.

Theorem 4.1. *An approximation of the fractional derivative of order α in the Caputo sense of the function x at t_s is given by*

$${}_t^CD_t^\alpha x_N(t_s)\cong\sum_{r=0}^{N}x(t_r)d_{s,r}^\alpha,\ \alpha>0, \tag{4.1}$$

where

$$d_{s,r}^\alpha=\frac{4\theta_r}{N}\sum_{n=\lfloor\alpha\rfloor}^{N}\sum_{j=0}^{N}\sum_{k=\lfloor\alpha\rfloor}^{n}\frac{n\theta_n}{b_j}\frac{(-1)^{n-k}(n+k-1)!\Gamma(k-\alpha+\frac{1}{2}T_n^*(t_r)T_j^*(t_s)}{L^\alpha\Gamma(k+\frac{1}{2}(n-k)!\Gamma(k-\alpha-j+1)\Gamma(k-\alpha+j+1)} \tag{4.2}$$

and $s,r=0,1,\dots,N$ with $\theta_0=\theta_N=\frac{1}{2}$, $\theta_i=1$ for $i=1,2,\dots,N-1$.

Proof. See [14]. □

5 Approximation of the Right RLFD

Suppose that f be a sufficiently smooth function in $[0,b]$ and let $J(s;f)$ be defined as follows

$$J(s;f)=\int_s^b(t-s)^{-\alpha}f'(t)\,dt,\ 0<s<b. \tag{5.1}$$

By Eq. (2.2), one may have

$${}_sD_b^\alpha f(s)=\frac{f(b)}{\Gamma(1-\alpha)}(b-s)^{-\alpha}+\frac{J(s;f)}{\Gamma(1-\alpha)}.$$

By approximating $f(t)$, for $0\le t\le b$, we obtain

$$f(t)\approx p_N(t)=\sum_{k=0}^{N}{}''a_kT_k(\frac{2t}{b}-1),\ a_k=\frac{2}{N}\sum_{j=0}^{N}{}''f(t_j)T_k(\frac{2t_j}{b}-1), \tag{5.2}$$

where $t_j = \frac{b}{2} - \frac{b}{2}cos(\frac{\pi j}{N})$, $j = 0, 1, \ldots, N$, therefore

$$J(s; f) \approx J(s; p_N) = \int_s^b p'_N(t)(t-s)^{-\alpha}\, dt. \tag{5.3}$$

Also, ${}_sD_b^\alpha f(s)$ can be approximated by

$${}_sD_b^\alpha f(s) \approx \frac{f(b)}{\Gamma(1-\alpha)}(b-s)^{-\alpha} + \frac{J(s; p_N)}{\Gamma(1-\alpha)}. \tag{5.4}$$

6 Expansion of delay function by composite ChFD

For expanding the delay function $f(t-\tau)$ by the composite ChFD method, one may choose N_1, such that

$$N_1 = \begin{cases} \frac{t_f}{\tau}, & \text{if } \frac{t_f}{\tau} \in \mathcal{Z} \\ [\frac{t_f}{\tau}] + 1, & \text{otherwise,} \end{cases} \tag{6.1}$$

where $[\frac{t_f}{\tau}]$ denotes the greatest integer value less than or equal to $\frac{t_f}{\tau}$, and τ is time delay. It should be noted that N_1 is chosen in such a way so that the number of subintervals can be minimized. Therefore

$$f(t-\tau) \approx \sum_{n=2}^{N_1} \sum_{m=0}^{M}{}'' g_{nm} b_{nm}(t),$$

where

$$g_{nm} = \frac{2}{M} \sum_{j=0}^{M}{}'' f(t_{nj} - \tau) b_{nm}(t_{nj}). \tag{6.2}$$

it is obvious that

$$t_{nj} - \tau = t_{n-1,j},\ n = 2, \ldots, N_1,\ j = 0, 1, \ldots, M. \tag{6.3}$$

Now, utilizing Eqs. (6.2) and (6.3), one may get

$$g_{nm} = \frac{2}{M} \sum_{j=0}^{M}{}'' f(t_{n-1,j}) b_{nm}(t_{nj}),$$

because of

$$b_{nm}(t_{nj}) = b_{n-1,m}(t_{n-1,j}),$$

hence

$$g_{nm} = \frac{2}{M} \sum_{j=0}^{M} f(t_{n-1,j}) b_{n-1,m}(t_{n-1,j}). \tag{6.4}$$

The convergence of the composite ChFD method was presented in [15] for the linear OCPs with time delay.

Theorem 6.1. *For approximating $x_N(t)$, the error is the follow as:*

$$error = \|x_{exact} - x_N(t)\| \le \sum_{k=N+1}^{\infty} \|c_n\|,$$

where

$$x_N(t) = \sum_{n=0}^{N} c_n T_n(t),\ t \in [-1, 1].$$

Proof. see [16] □

7 Numerical example

Now, an example is solved to demonstrate the efficiency, the accuracy, and the applicability of the proposed technique.

Example 7.1. The following OCP with time delay is considered (see [15])

$$J = \frac{1}{2}x^2(1) + \frac{1}{2}\int_0^1 u^2(t)\,dt \tag{7.1}$$

$$s.t.\ \ \dot{x}(t) + {}_0^C D_t^{\alpha} x(t) = x(t - \frac{2}{5}) + u(t),\ 0 \le t \le 1, \tag{7.2}$$

$$x(t) = 2t^2 + 1,\ \ -\frac{2}{5} \le t \le 0,\ \alpha = 0.9,\ t_f = 1. \tag{7.3}$$

To solve this problem by the composite ChFD method, we choose $N = 2$ and $N_1 = 3$ in Eq. (6.1). Now, utilizing the proposed technique, one can obtain the following solutions $x(t)$ and $u(t)$ for this problem with $J_{approx} = 0.5273969950$.

$$x_{approx}(t) = -0.3695439090t + 1 - 0.6304560910t^2,$$
$$u_{approx}(t) = 0.9049153660t - 1.600000000 + 0.6950846360t^2.$$

The obtained solution for $u(t)$ is shown in Fig. 1.

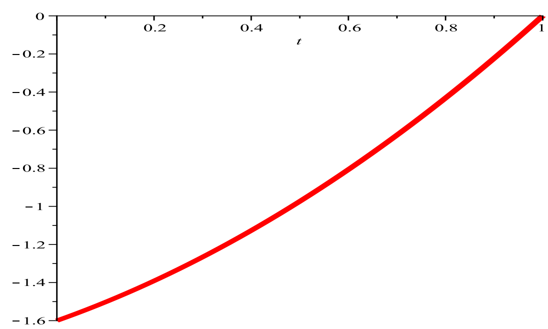

Figure 1. The graph of $u(t)$ for Example 7.1

Remark 7.2. Multi-point BVPs have considered for the numerical applications in various regions of science. In this remark, our objective is to obtain numerically solution by utilizing Bezier curves strategy. Numerical examples are stated to show the legitimacy and appropriateness of the proposed method.
Presently, the following multi-point BVP is considered

$$y^{(n_1)} = g(x, y, y', \dots, y^{(n_1-1)}),\ \ 0 \le x \le 1,$$

Now, Bezier curves method is introduced:

$$y(x) \cong y_{n+1}(x) = \sum_{i=0}^{n} c_i B_{i,n}(x),\ 0 \le x \le 1,\ n \ge 1, \tag{7.4}$$

where

$$B_{r,n}(\frac{x - x_0}{h}) = \binom{n}{r}\frac{1}{h^n}(x_f - x)^{n-r}(x - x_0)^r,\ \ x_0 \le x \le x_f,\ \ i = 0, 1, \dots, n,$$
$$h = x_f - x_0,\ x_0 = 0,\ x_f = 1,$$

the unknown control points are c_i, $i = 0, 1, \dots, n$. In this remark, the Bezier curve method is utilizing for solving the muli-point BVP. This technique is applied in [17, 18]. The convergence

of this method was proven when n tends infinity.
Now, substituting $y(x)$ in multi-point BVP, one may defined $f_{objective}$ for $x \in [x_0, x_f]$ as:

$$f_{objective} = \sum_{i=0}^{n} c_i^2, \tag{7.5}$$

with boundary conditions of the given problem. One may have:

$$y'(x) = n \sum_{i=0}^{n-1} B_{i,n}(x)(c_{i+1} - c_i), \tag{7.6}$$

$$y''(x) = n(n-1) \sum_{i=0}^{n-2} B_{i,n-2}(x)(c_{i+2} - 2c_{i+1} + c_i) \tag{7.7}$$

Now, one may utilize Lagrange multipliers technique for solving Eq. (7.5).

Example 7.3. The following four-point second-arrange nonlinear ordinary differential equation is considered (see [19])

$$y''(x) + (x^3 + x + 1)y^2(x) = f(x), \;\; 0 \le x \le 1,$$

with

$$y(0) = \frac{1}{6} y(\frac{2}{9}) + \frac{1}{3} y(\frac{7}{9}) - 0.286634,$$

where

$$f(x) = \frac{2}{9}[-6\cos(x - x^2) + \sin(x - x^2)(-3(1 - 2x)^2 + (1 + x + x^3)\sin(x - x^2))],$$

$$y_{exact}(x) = \frac{1}{3}\sin(x - x^2).$$

One may obtain $y_{approx}(x) = -0.0002714842857 + 0.3346606074x - 0.3419321979x^2 + 0.007137731959x^3$ with the proposed technique by $n = 3$. The approximate solution for $y(x)$ is shown in Fig. 2.

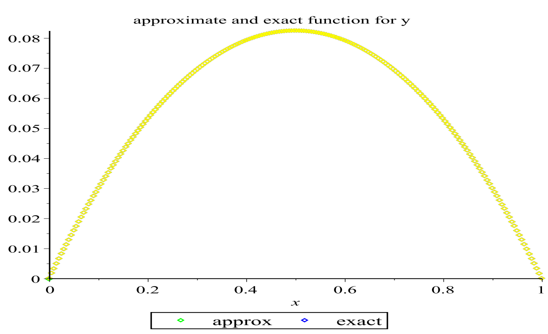

Figure 2. The graph of $y(x)$ for Example 7.3

Example 7.4. The following third-order linear differential equation is considered

$$y'''(x) - k^2 y' + a = 0, \;\; 0 \le x \le 1,$$

with

$$y'(0) = y'(1) = 0, \; y(\frac{1}{2}) = 0,$$

where

$$y_{exact}(x) = \frac{a}{k^3}(\sinh(\frac{k}{2}\sinh(kx)) + \frac{a}{k^2}(x-\frac{1}{2}) + \frac{a}{k^3}(\cosh(kx)-\cosh(\frac{k}{2}))\tanh\frac{k}{2},$$
$$k=5,\ a=1.$$

One may obtain $y_{approx}(x) = -0.01210708561 + 0.07264251371x^2 - 0.04842834258x^3 + 8.000000000 \times 10^{-11}x^4$ with the proposed technique by $n = 4$. The obtained solution for $y(x)$ is shown in Fig. 3.

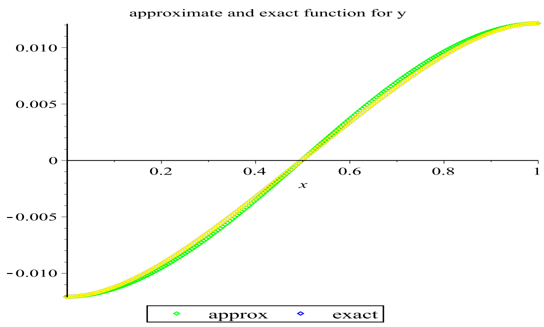

Figure 3. The graph of $y(x)$ for Example 7.4

Example 7.5. The following system is considered

$$y^{(4)}(x) + y(x)y'(x) - 4x^7 - 24 = 0, \quad 0 \le x \le 1,$$
$$y(0) = 0, y''(\frac{1}{2}) = 3,\ y'''(\frac{1}{4})) = 6,\ y(1) = 1,$$

where $y_{exact}(x) = x^4$. One may obtain $y_{approx}(x) = x^4$ with this method by $n = 4$ where the absolute error is zero. The approximate solution for $y(x)$ is shown in Fig. 4.

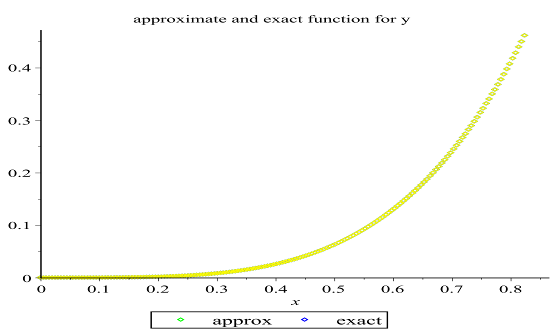

Figure 4. The graph of $y(x)$ for Example 7.5

8 Conclusions

A composite ChFD method as an extension of the ChFD scheme was applied for solving FOCPs with time delays. The composite ChFD method is based on Chebyshev polynomials using the well-known CGL points. One of the most important advantages of the proposed technique is good representation of smooth functions. Also, in this work one may utilize the Bezier curves method for solving the multi-point BVPs. Numerical examples are explained to show the appropriateness of the proposed technique.

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