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GENERALIZED SYMMETRIC MEIR-KEELER CONTRACTION FOR HYBRID PAIR OF MAPPINGS WITH APPLICATION

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Abstract. We establish some common coupled fixed point theorems for hybrid pair of mappings under generalized symmetric Meir-Keeler contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. Moreover, an example and an application to integral equations are given here to illustrate the usability of the obtained results. We improve, extend and generalize several known results.

1 Introduction

Let $(X,\ d)$ be a metric space. We denote by 2^X the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. A functional $H : CL(X)\times CL(X) \to \mathbb{R}_+\cup\{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d is given by

$$H(A,\ B) = \begin{cases} \max\left\{\sup_{a\in A} D(a,\ B),\ \ \sup_{b\in B} D(b,\ A)\right\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A,\ B \in CB(X)$, where $D(x,\ A) = \inf_{a\in A} d(x,\ a)$ denote the distance from x to $A \subset X$. For simplicity, if $x \in X$, we denote $g(x)$ by gx.

The existence of fixed points for various multivalued contractions and non-expansive mappings has been studied by many authors under different conditions which was initiated by Markin [25]. The theory of multivalued mappings has found application in control theory, convex optimization, differential inclusions and economics.

In [19], Guo and Lakshmikantham given the notion of coupled fixed point. In [7], Gnana-Bhaskar and Lakshmikantham established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [22] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Gnana-Bhaskar and Lakshmikantham [7], which was later generalized by Ding et al. [18]. Many authors focused on coupled fixed point theory including [5, 6, 8, 9, 10, 18, 22, 24, 28].

Samet et al. [29] claimed that most of the coupled fixed point theorems in the setting of single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

The concepts related to coupled fixed point theory in the setting of multivalued mappings were extended by Abbas et al.[3] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces. Very few papers were devoted to coupled fixed point problems for hybrid pair of mappings including [2, 3, 11, 13, 14, 15, 16, 17, 23, 27].

In [3], Abbas et al. introduced the following for multivalued mappings:

Definition 1. Let X be a non-empty set, $F : X \times X \to 2^X$ and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

(2) a coupled coincidence point of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.

(3) a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 2. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.

Definition 3. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

Lemma 1 [27]. Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in K(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b\in B} d(a, b)$.

Aamri and ElMoutawakil [1] defined (EA) property for self-mappings which contained the class of non-compatible mappings. Kamran [21] extended the (EA) property for hybrid pair $f : X \to X$ and $T : X \to 2^X$. In [20], Jungck and Rhoades introduced the notion of occasionally weakly compatibility for self mappings. Abbas and Rhoades [4] extended the concept of occasionally weakly compatible mappings for hybrid pair $f : X \to X$ and $T : X \to 2^X$. Deshpande and Handa [12] introduced the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $f : X \to X$ and $F : X \times X \to 2^X$.

In [12], Deshpande and Handa introduced the following:

Definition 4. Mappings $g : X \to X$ and $F : X \times X \to CB(X)$ are said to satisfy the (EA) property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some s, t in X and A, B in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n &= s \in A = \lim_{n\to\infty} F(x_n, y_n), \\ \lim_{n\to\infty} gy_n &= t \in B = \lim_{n\to\infty} F(y_n, x_n). \end{aligned}$$

Definition 5. Mappings $F : X \times X \to 2^X$ and $g : X \to X$ are said to be occasionally $w-$compatible if and only if there exists some point $(x, y) \in X \times X$ such that $gx \in F(x, y)$, $gy \in F(y, x)$ and $gF(x, y) \subseteq F(gx, gy)$.

Occasionally $w-$compatibility is weaker condition than $w-$compatibility, see Example 8 in Deshpande and Handa [12].

Let (X, d) be a metric space and $T : X \to X$ a self mapping. If (X, d) is complete and T is a contraction, that is, there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \le kd(x, y), \text{ for all } x, y \in X, \tag{1.1}$$

then, by Banach contraction mapping principle, which is a classical and powerful tool in nonlinear analysis, we know that T has a unique fixed point p and, for any $x_0 \in X$, the Picard iteration $\{T^n x_0\}$ converges to p. The Banach contraction mapping principle has been generalized in several directions. One of these generalizations known as the Meir-Keeler fixed point theorem [26], has been obtained by replacing the contraction condition (1) by the following more general assumption: for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$x, y \in X, \varepsilon \le d(x, y) < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx, Ty) < \varepsilon. \tag{1.2}$$

In [28], Samet established the coupled fixed points of mixed strict monotone generalized Meir-Keeler operators and also established the existence and uniqueness results for coupled fixed point. Berinde and Pecurar [6] obtained more general coupled fixed point theorems for mixed monotone operators $F : X\times X \to X$ satisfying a generalized symmetric Meir-Keeler contractive condition.

In this paper, we establish a common coupled fixed point theorem for hybrid pair of mappings under generalized symmetric Meir-Keeler contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. An example and an application to integral equation demonstrate the effectiveness of our generalization. We improve, extend and generalize the results of Berinde and Pecurar [6], Gnana-Bhaskar and Lakshmikantham [7], Meir and Keeler [26], Samet [28] and many other results in the existing literature.

2 Main results

Theorem 2.1. *(X, d) be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Suppose for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(gx,\ gu) + d(gy,\ gv)}{2} \leq \varepsilon + \delta(\varepsilon),$$

implies

$$\frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon, \tag{2.1}$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

Proof. Proof. Let $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$, because $F(X \times X) \subseteq g(X)$. Since $F : X \times X \to K(X)$, therefore by Lemma 1, there exist $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that

$$\begin{aligned} d(gx_1,\ z_1) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)), \\ d(gy_1,\ z_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, there exist $x_2, y_2 \in X$ such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$\begin{aligned} d(gx_1,\ gx_2) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)), \\ d(gy_1,\ gy_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n, x_n)$ such that

$$\begin{aligned} d(gx_{n+1},\ gx_{n+2}) &\leq H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1})), \\ d(gy_{n+1},\ gy_{n+2}) &\leq H(F(y_n,\ x_n),\ F(y_{n+1},\ x_{n+1})). \end{aligned}$$

Now, by (2.1), for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{d(gx,\ gu) + d(gy,\ gv)}{2} \leq \varepsilon + \delta(\varepsilon),$$

implies

$$\frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon. \tag{2.2}$$

Condition (2.2) implies the strict contractive condition

$$\frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2} < \frac{d(gx,\ gu) + d(gy,\ gv)}{2}. \tag{2.3}$$

Thus, by (2.3), we have

$$\begin{aligned} & \frac{d(gx_{n+1},\ gx_n)+d(gy_{n+1},\ gy_n)}{2} \\ \leq\ & \frac{H(F(x_n,\ y_n),\ F(x_{n-1},\ y_{n-1}))+H(F(y_n,\ x_n),\ F(y_{n-1},\ x_{n-1}))}{2} \\ <\ & \frac{d(gx_n,\ gx_{n-1})+d(gy_n,\ gy_{n-1})}{2}, \end{aligned}$$

which shows that the sequence of nonnegative numbers $\{\alpha_n\}_{n=0}^{\infty}$ given by

$$\alpha_n = \frac{d(gx_n,\ gx_{n-1})+d(gy_n,\ gy_{n-1})}{2}, \tag{2.4}$$

is non-increasing. Therefore, there exists some $\varepsilon \geq 0$ such that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{d(gx_n,\ gx_{n-1})+d(gy_n,\ gy_{n-1})}{2} = \varepsilon.$$

We shall prove that $\varepsilon = 0$. Suppose, to the contrary, that $\varepsilon > 0$. Then there exists a positive integer p such that

$$\varepsilon < \alpha_p < \varepsilon + \delta(\varepsilon),$$

which, by (2.2), implies

$$\frac{H(F(x_p,\ y_p),\ F(x_{p-1},\ y_{p-1}))+H(F(y_p,\ x_p),\ F(y_{p-1},\ x_{p-1}))}{2} < \varepsilon,$$

it follows that

$$\alpha_{p+1} = \frac{d(gx_{p+1},\ gx_p)+d(gy_{p+1},\ gy_p)}{2} < \varepsilon,$$

which is a contradiction. Thus $\varepsilon = 0$ and hence

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{d(gx_n,\ gx_{n-1})+d(gy_n,\ gy_{n-1})}{2} = 0. \tag{2.5}$$

Let now $\varepsilon > 0$ be arbitrary and $\delta(\varepsilon)$ the corresponding value from the hypothesis of our theorem. By (2.5), there exists a positive integer k such that

$$\alpha_{k+1} = \frac{d(gx_{k+1},\ gx_k)+d(gy_{k+1},\ gy_k)}{2} < \delta(\varepsilon). \tag{2.6}$$

For this fixed number k, consider now the set $A_k = \{(x,\ y) : gx_k \leq gx,\ gy_k \geq gy,\ \frac{1}{2}[d(gx_k,\ gx)+d(gy_k,\ gy)] < \varepsilon + \delta(\varepsilon)\}$. By (2.6), $A_k \neq \phi$. We claim that

$$(x,\ y) \in A_k \Rightarrow (F(x,\ y),\ F(y,\ x)) \in A_k. \tag{2.7}$$

Let $(x,\ y) \in A_k$. Then

$$\frac{d(gx_k,\ gx)+d(gy_k,\ gy)}{2} < \varepsilon + \delta(\varepsilon), \tag{2.8}$$

which, by (2.1), implies

$$\frac{H(F(x_k,\ y_k),\ F(x,\ y))+H(F(y_k,\ x_k),\ F(y,\ x))}{2} < \varepsilon. \tag{2.9}$$

Now, by (2.6), (2.9) and by triangle inequality, we have

$$\begin{aligned}
& \frac{D(gx_k,\ F(x,\ y))+D(gy_k,\ F(y,\ x))}{2} \\
\leq\ & \frac{D(gx_k,\ F(x_k,\ y_k))+D(gy_k,\ F(y_k,\ x_k))}{2} \\
& +\frac{H(F(x_k,\ y_k),\ F(x,\ y))+H(F(y_k,\ x_k),\ F(y,\ x))}{2} \\
\leq\ & \frac{d(gx_k,\ gx_{k+1})+d(gy_k,\ gy_{k+1})}{2} \\
& +\frac{H(F(x_k,\ y_k),\ F(x,\ y))+H(F(y_k,\ x_k),\ F(y,\ x))}{2} \\
<\ & \varepsilon+\delta(\varepsilon).
\end{aligned}$$

Thus $(F(x, y), F(y, x)) \in A_k$. Again

$$\begin{aligned}
& \frac{d(gx_k,\ gx_{k+1})+d(gy_k,\ gy_{k+1})}{2} \\
\leq\ & \frac{D(gx_k,\ F(x,\ y))+d(gy_k,\ F(y,\ x))}{2} \\
& +\frac{D(F(x,\ y),\ gx_{k+1})+D(F(y,\ x),\ gy_{k+1})}{2} \\
<\ & 2(\varepsilon+\delta(\varepsilon)).
\end{aligned}$$

Thus $(gx_{k+1}, gy_{k+1}) \in A_k$ and by induction,

$$(gx_n,\ gy_n) \in A_k,\ \text{for all } n > k.$$

This implies that for all $n, m > k$, we have

$$\begin{aligned}
& \frac{d(gx_n,\ gx_m)+d(gy_n,\ gy_m)}{2} \\
\leq\ & \frac{d(gx_n,\ gx_k)+d(gy_n,\ gy_k)}{2}+\frac{d(gx_k,\ gx_m)+d(gy_k,\ gy_m)}{2} \\
<\ & 2(\varepsilon+\delta(\varepsilon))=4\varepsilon.
\end{aligned}$$

This shows that $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, therefore there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \text{ and } \lim_{n\to\infty} gy_n = gy. \tag{2.10}$$

Now, since $gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n, x_n)$, therefore by using condition (2.3), we get

$$\begin{aligned}
& \frac{D(gx_{n+1},\ F(x,\ y))+D(gy_{n+1},\ F(y,\ x))}{2} \\
\leq\ & \frac{H(F(x_n,\ y_n),\ F(x,\ y))+H(F(y_n,\ x_n),\ F(y,\ x))}{2} \\
<\ & \frac{d(gx_n,\ gx)+d(gy_n,\ gy)}{2}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (2.10), we get

$$D(gx,\ F(x,\ y)) = 0 \text{ and } D(gy,\ F(y,\ x)) = 0,$$

it follows that

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, (x, y) is a coupled coincidence point of F and g. Hence $C(F, g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x, y) \in C(F, g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{2.11}$$

where $u, v \in X$. Since g is continuous at u and v. We have, by (2.11), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{2.12}$$

As F and g are $w-$compatible, so

$$(g^n x, g^n y) \in C(F, g), \text{ for all } n \geq 1, \tag{2.13}$$

that is, for all $n \geq 1$,

$$g^n x \in F(g^{n-1}x, g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y, g^{n-1}x). \tag{2.14}$$

Now, by using (2.3) and (2.14), we obtain

$$\begin{aligned} & \frac{D(g^n x, F(u, v)) + D(g^n y, F(v, u))}{2} \\ \leq & \frac{H(F(g^{n-1}x, g^{n-1}y), F(u, v)) + H(F(g^{n-1}y, g^{n-1}x), F(v, u))}{2} \\ < & \frac{d(g^n x, gu) + d(g^n y, gv)}{2}. \end{aligned}$$

On taking limit as $n \to \infty$ in above inequality, by using (2.11) and (2.12), we get

$$D(gu, F(u, v)) = 0 \text{ and } D(gv, F(v, u)) = 0,$$

it follows that

$$gu \in F(u, v) \text{ and } gv \in F(v, u). \tag{2.15}$$

Now, from (2.12) and (2.15), we have

$$u = gu \in F(u, v) \text{ and } v = gv \in F(v, u),$$

that is, (u, v) is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x, y) \in C(F, g)$, g is $F-$weakly commuting, that is $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$ and $g^2x = gx$ and $g^2y = gy$. Thus $gx = g^2x \in F(gx, gy)$ and $gy = g^2y \in F(gy, gx)$, that is, (gx, gy) is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x, y) \in C(F, g)$ and for some $u, v \in X$,

$$\lim_{n\to\infty} g^n u = x \text{ and } \lim_{n\to\infty} g^n v = y.$$

Since g is continuous at x and y. Therefore x and y are fixed points of g, that is,

$$gx = x \text{ and } gy = y. \tag{2.16}$$

Since $(x, y) \in C(F, g)$. Therefore, by using (2.16), we obtain

$$x = gx \in F(x, y) \text{ and } y = gy \in F(y, x),$$

that is, (x, y) is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F, g)) = \{(x, x)\}$. Then $\{x\} = \{gx\} = F(x, x)$. Hence (x, x) is a common coupled fixed point of F and g. □

If we put $g = I$ (the identity mapping) in the Theorem 2.1, we get the following result:

Corollary 2.2. *Let (X, d) be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Suppose for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(x,\ u) + d(y,\ v)}{2} \leq \varepsilon + \delta(\varepsilon),$$

implies

$$\frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon, \tag{2.17}$$

for all $x, y, u, v \in X$. Then F has a coupled fixed point.

If we take F to be a singleton set in Theorem 2.1, then we get the following result:

Corollary 2.3. *Let (X, d) be a metric space, $F : X \times X \to X$ and $g : X \to X$ be two mappings. Suppose for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(gx,\ gu) + d(gy,\ gv)}{2} \leq \varepsilon + \delta(\varepsilon),$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon, \tag{2.18}$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point.

If we put $g = I$ (the identity mapping) in the Corollary 2.3, we get the following result:

Corollary 2.4. *Let (X, d) be a complete metric space, $F : X \times X \to X$ be a mapping. Suppose for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\varepsilon \leq \frac{d(x,\ u) + d(y,\ v)}{2} \leq \varepsilon + \delta(\varepsilon),$$

implies

$$\frac{d(F(x,\ y),\ F(u,\ v)) + d(F(y,\ x),\ F(v,\ u))}{2} < \varepsilon, \tag{2.19}$$

for all $x, y, u, v \in X$. Then F has a coupled fixed point.

Theorem 2.5. *Let (X, d) be a metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be two mappings satisfying (2.1) and $\{F, g\}$ satisfies the (EA) property. Then F and g have a coupled coincidence point. Furthermore, if one of the conditions (a) to (d) holds. Then F and g have a common coupled fixed point.*

Proof. Since $\{F, g\}$ satisfies the (EA) property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some s, t in X and A, B in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n &= s \in A = \lim_{n\to\infty} F(x_n,\ y_n), \\ \lim_{n\to\infty} gy_n &= t \in B = \lim_{n\to\infty} F(y_n,\ x_n). \end{aligned} \tag{2.20}$$

Since $g(X)$ is a closed subset of X, then there exist $x, y \in X$, we have

$$s = gx \text{ and } t = gy. \tag{2.21}$$

Now, by using condition (2.3), we get

$$\begin{aligned} &\frac{H(F(x_n,\ y_n),\ F(x,\ y)) + H(F(y_n,\ x_n),\ F(y,\ x))}{2} \\ < \quad &\frac{d(gx_n,\ gx) + d(gy_n,\ gy)}{2}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (2.20) and (2.21), we get

$$H(A,\ F(x,\ y)) = 0 \text{ and } H(B,\ F(y,\ x)) = 0.$$

Since $gx \in A$ and $gy \in B$, therefore

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, $(x,\,y)$ is a coupled coincidence point of F and g. Hence $C(F,\,g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x,\,y) \in C(F,\,g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{2.22}$$

where $u,\ v \in X$. Since g is continuous at u and v. We have, by (2.22), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{2.23}$$

As F and g are $w-$compatible, so

$$(g^n x,\ g^n y) \in C(F,\ g),\ \text{ for all } n \geq 1, \tag{2.24}$$

that is, for all $n \geq 1$,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x). \tag{2.25}$$

Now, by using (2.3) and (2.25), we obtain

$$\begin{aligned}
&\frac{D(g^n x,\ F(u,\ v)) + D(g^n y,\ F(v,\ u))}{2} \\
\leq\ &\frac{H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v)) + H(F(g^{n-1}y,\ g^{n-1}x),\ F(v,\ u))}{2} \\
<\ &\frac{d(g^n x,\ gu) + d(g^n y,\ gv)}{2}.
\end{aligned}$$

On taking limit as $n \to \infty$ in above inequality, by using (2.22) and (2.23), we get

$$D(gu,\ F(u,\ v)) = 0 \text{ and } D(gv,\ F(v,\ u)) = 0,$$

it follows that

$$gu \in F(u,\ v) \text{ and } gv \in F(v,\ u), \tag{2.26}$$

Now, from (2.23) and (2.26), we have

$$u =\ gu \in F(u,\ v) \text{ and } v = gv \in F(v,\ u),$$

that is, $(u,\,v)$ is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,\,y) \in C(F,\,g)$, g is $F-$weakly commuting, that is $g^2x \in F(gx,\,gy)$ and $g^2y \in F(gy,\,gx)$ and $g^2x = gx$ and $g^2y = gy$. Thus $gx = g^2x \in F(gx,\,gy)$ and $gy = g^2y \in F(gy,\,gx)$, that is, $(gx,\,gy)$ is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x,\,y) \in C(F,\,g)$ and for some $u,\,v \in X$,

$$\lim_{n\to\infty} g^n u = x \text{ and } \lim_{n\to\infty} g^n v = y.$$

Since g is continuous at x and y. Therefore x and y are fixed points of g, that is,

$$gx = x \text{ and } gy = y. \tag{2.27}$$

Since $(x,\,y) \in C(F,\,g)$, Therefore, by using (2.27), we obtain

$$x = gx \in F(x,\ y) \text{ and } y = gy \in F(y,\ x),$$

that is, $(x,\,y)$ is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F,\,g)) = \{(x,\,x)\}$. Then $\{x\} = \{gx\} = F(x,\,x)$. Hence $(x,\,x)$ is a common coupled fixed point of F and g. □

Theorem 2.6. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (2.1) and $\{F, g\}$ is occasionally $w-$compatible. Then F and g have a common coupled fixed point.*

Proof. Since the pairs $\{F, g\}$ is occasionally $w-$compatible, therefore there exists some point $(x, y) \in X \times X$ such that

$$gx \in F(x,\ y),\ gy \in F(y,\ x) \text{ and } gF(x,\ y) \subseteq F(gx,\ gy). \tag{2.28}$$

It follows that

$$g^2x \in F(gx,\ gy) \text{ and } g^2y \in F(gy,\ gx). \tag{2.29}$$

Now, suppose $u = gx$ and $v = gy$, then by (2.29), we get

$$gu \in F(u,\ v) \text{ and } gv \in F(v,\ u). \tag{2.30}$$

Now, we claim that $u = gx = gu$ and $v = gy = gv$. If not, then by condition (2.3) and by triangle inequality, we have

$$\begin{aligned} & \frac{d(gx,\ gu) + d(gy,\ gv)}{2} \\ \leq\ & \frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2} \\ <\ & \frac{d(gx,\ gu) + d(gy,\ gv)}{2}, \end{aligned}$$

which is a contradiction. Hence we must have

$$u = gx = gu \text{ and } v = gy = gv. \tag{2.31}$$

Thus, by (2.30) and (2.31), we get

$$u = gu \in F(u,\ v) \text{ and } v = gv \in F(v,\ u),$$

that is, (u, v) is a common coupled fixed point of F and g. $\square$

Example 2.7. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X \times X \to K(X)$ be defined as

$$F(x,\ y) = \begin{cases} \{0\}, \text{ for } x,\ y = 1, \\ \left[0,\ \frac{x^2+y^2}{3}\right], \text{ for } x,\ y \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X.$$

Suppose for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \frac{d(gx,\ gu) + d(gy,\ gv)}{2} \leq \varepsilon + \delta(\varepsilon).$$

Now, for all $x, y, u, v \in X$ with $x, y, u, v \in [0, 1)$, we have

Case (a). If $x^2 + y^2 = u^2 + v^2$, then

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ =\ & \frac{u^2 + v^2}{3} \\ \leq\ & \frac{1}{3}\max\{x^2,\ u^2\} + \frac{1}{3}\max\{y^2,\ v^2\} \\ \leq\ & \frac{1}{3}d(gx,\ gu) + \frac{1}{3}d(gy,\ gv) \\ \leq\ & \frac{2}{3}\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) \\ \leq\ & \frac{2}{3}(\varepsilon + \delta(\varepsilon)) < \varepsilon. \end{aligned}$$

Case (b). If $x^2+y^2 \neq u^2+v^2$ with $x^2+y^2 < u^2+v^2$, then

$$\begin{aligned}
& H(F(x,\ y),\ F(u,\ v)) \\
= \ & \frac{u^2+v^2}{3} \\
\leq \ & \frac{1}{3}\max\{x^2,\ u^2\} + \frac{1}{3}\max\{y^2,\ v^2\} \\
\leq \ & \frac{1}{3}d(gx,\ gu) + \frac{1}{3}d(gy,\ gv) \\
\leq \ & \frac{2}{3}\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right) \\
\leq \ & \frac{2}{3}(\varepsilon+\delta(\varepsilon)) < \varepsilon.
\end{aligned}$$

Similarly, we obtain the same result for $u^2+v^2 < x^2+y^2$. Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$. Again, for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$, we have

$$\begin{aligned}
& H(F(x,\ y),\ F(u,\ v)) \\
= \ & \frac{x^2+y^2}{3} \\
\leq \ & \frac{1}{3}\max\{x^2,\ u^2\} + \frac{1}{3}\max\{y^2,\ v^2\} \\
\leq \ & \frac{1}{3}d(gx,\ gu) + \frac{1}{3}d(gy,\ gv) \\
\leq \ & \frac{2}{3}\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right) \\
\leq \ & \frac{2}{3}(\varepsilon+\delta(\varepsilon)) < \varepsilon.
\end{aligned}$$

Thus the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (2.1), for all $x,\ y,\ u,\ v \in X$. In addition, all the other conditions of Theorem 2.1, Theorem 2.5 and Theorem 2.6 are satisfied and $z = (0,\ 0)$ is a common coupled fixed point of hybrid pair $\{F,\ g\}$. The function $F : X \times X \to K(X)$ involved in this example is not continuous at the point $(1,\ 1) \in X \times X$.

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