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GROWTH PROPERTIES OF COMPOSITE ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES IN UNIT POLYDISC FROM THE VIEW POINT OF THEIR NEVANLINNA L^{*}-ORDERS

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Abstract. In this paper we introduce the idea of Nevanlinna n variables based L^{*}-order and Nevanlinna n variables based L^{*}-lower order in the unit polydisc. Hence we study some growth properties of Nevanlinna's Characteristic function relating to the composition of two analytic function in the unit polydisc on the basis of Nevanlinna n variables based L^{*} -order and Nevanlinna n variables based L^{*}-lower order as compared to the growth of their corresponding left and right factors.

1 Introduction, Definitions and Notations

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order [4] if there exist a number μ such that Nevanlinna characteristic function

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} log^{+} \left| f\left(re^{i\theta}\right) \right| d\theta$$

satisfies $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such number μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r \to 1} \frac{\log T_f(r)}{-\log(1-r)} .$$

Similarly, Nevanlinna lower order λ_f of f is given by

$$\lambda_f = \liminf_{r \to 1} \frac{\log T_f(r)}{-\log(1-r)} .$$

Somasundaram and Thamizharasi [6] introduced the notions of L-order (L-lower order) for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. In the line of Somasundaram and Thamizharasi [6] one may introduce the notion of Nevanlinna L-order for an analytic function f in the unit disc $U = \{z : |z| < 1\}$ where $L \equiv L\left(\frac{1}{1-r}\right)$ is a positive continuous function in the unit disc U increasing slowly i.e., $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$, for every positive constant 'a', in the following manner:

Definition 1.1. If f be analytic in U, then the Nevanlinna L-order ρ_f^L of f is defined as

$$\rho_f^L = \inf \left\{ \mu > 0 : T_f(r) < \left[\frac{L\left(\frac{1}{1-r}\right)}{(1-r)} \right]^{\mu} \text{ for all } 0 < r_0(\mu) < r < 1 \right\} .$$

Similarly one may define λ_f^L, the Nevanlinna L-lower order of f in the following way:

$$\lambda_f = \liminf_{r \to 1} \frac{\log T_f(r)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} .$$

The more generalised concept of Nevanlinna L-order and the Nevanlinna L-lower order of an analytic function f in the unit disc U are the Nevanlinna L^*-order and the Nevanlinna L^*-lower order. Their definitions are as follows:

Definition 1.2. [2] The Nevanlinna L^*-order $\rho_f^{L^*}$ and Nevanlinna L^*-lower order $\lambda_f^{L^*}$ of an analytic function f in the unit disc U are defined as

$$\rho_f^{L^*} = \limsup_{r \to 1} \frac{\log T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} \text{ and}$$

$$\lambda_f^{L^*} = \liminf_{r \to 1} \frac{\log T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} \text{ respectively.}$$

Extending the notion of single variable to several variable, let $f(z_1, z_2, \cdots, z_n)$ be a non-constant analytic function of n complex variables $z_1, z_2, \cdots, z_{n-1}$ and z_n in the unit polydisc

$$U = \{(z_1, z_2, \cdots, z_n) : |z_j| \le 1, j = 1, 2, \cdots, n; r_1 > 0, r_2 > 0, \cdots, r_n > 0\} .$$

Now in the line of Nevanlinna L^*-order and Nevanlinna L^*-lower order, in this paper we introduce the Nevanlinna n variables based L^*-order and the Nevanlinna n variables based L^*-lower order for functions of n complex variables analytic in a unit polydisc as follows :

$$_{v_n}\rho_f^{L^*} = \limsup_{r_1, r_2, \cdots, r_n \to 1} \frac{\log T_f(r_1, r_2, \cdots, r_n)}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}$$

and

$$_{v_n}\lambda_f^{L^*} = \liminf_{r_1, r_2, \cdots, r_n \to 1} \frac{\log T_f(r_1, r_2, \cdots, r_n)}{\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}$$

where $L \equiv L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)$ is a positive continuous function in the unit polydisc U increasing slowly i.e., $L\left(\frac{a}{1-r_1}, \frac{a}{1-r_2}, \cdots, \frac{a}{1-r_n}\right) \sim L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)$ as $r \to 1$, for every positive constant 'a'.

In this paper we study some growth properties of Nevanlinna's Characteristic function relating to the composition of two analytic function in the unit polydisc on the basis of Nevanlinna n variables based L^*-order and Nevanlinna n variables based L^*-lower order as compared to the growth of their corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [1], [3] and [5].

2 Theorems

In this section we present the main results of the paper.

Theorem 2.1. *If f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\lambda_{f \circ g}^{L^*} \le {}_{v_n}\rho_{f \circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\lambda_f^{L^*} \le {}_{v_n}\rho_f^{L^*} < \infty$ then*

$$\frac{_{v_n}\lambda_{f \circ g}^{L^*}}{_{v_n}\rho_f^{L^*}} \le \liminf_{r_1, r_2, \cdots, r_n \to 1} \frac{\log T_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n)} \le \frac{_{v_n}\lambda_{f \circ g}^{L^*}}{_{v_n}\lambda_f^{L^*}}$$

$$\le \limsup_{r_1, r_2, \cdots, r_n \to 1} \frac{\log T_{f \circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n)} \le \frac{_{v_n}\rho_{f \circ g}^{L^*}}{_{v_n}\lambda_f^{L^*}} .$$

Proof. From the definition of ${}_{v_n}\rho_f^{L^*}$ and ${}_{v_n}\lambda_{f\circ g}^{L^*}$, we have for arbitrary positive ε and for all large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\log T_{f\circ g}(r_1, r_2, \cdots, r_n) \geqslant$$

$$\left({}_{v_n}\lambda_{f\circ g}^{L^*} - \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right] \tag{2.1}$$

and

$$\log T_f(r_1, r_2, \cdots, r_n) \leq$$

$$\left({}_{v_n}\rho_f^{L^*} + \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]. \tag{2.2}$$

Now from (2.1), (2.2) it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log T_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n)} \geqslant \frac{\left({}_{v_n}\lambda_{f\circ g}^{L^*} - \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\left({}_{v_n}\rho_f^{L^*} + \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\log T_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n)} \geqslant \frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}. \tag{2.3}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log T_{f\circ g}(r_1, r_2, \cdots, r_n) \leq$$

$$\left({}_{v_n}\lambda_{f\circ g}^{L^*} + \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right] \tag{2.4}$$

and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log T_f(r_1, r_2, \cdots, r_n) \geqslant$$

$$\left({}_{v_n}\lambda_f^{L^*} - \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]. \tag{2.5}$$

Combining (2.4) and (2.5) we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log T_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log T_f(r_1, r_2, \cdots, r_n)} \leq \frac{\left({}_{v_n}\lambda_{f\circ g}^{L^*} + \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\left({}_{v_n}\lambda_f^{L^*} - \varepsilon\right) \log \left[\frac{\exp\left\{L\left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \cdots, \frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}{\log T_f\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}\le\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}}\,. \tag{2.6}$$

Also for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdot\cdot\cdot$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_f\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)\le$$

$$\left({}_{v_n}\lambda_f^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot\cdot,\frac{1}{1-r_n}\right)\right\}}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\cdot\left(1-r_n\right)}\right]. \tag{2.7}$$

Now from (2.1) and (2.7), we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdot\cdot\cdot$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log T_{f\circ g}\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}{\log T_f\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}\ge\frac{\left({}_{v_n}\lambda_{f\circ g}^{L^*}-\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\left({}_{v_n}\lambda_f^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}\,.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}{\log T_f\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}\ge\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}}\,. \tag{2.8}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdot\cdot\cdot$ and $\left(\frac{1}{1-r_n}\right)$,

$$\log T_{f\circ g}\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)\le$$

$$\left({}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdot\cdot\cdot,\frac{1}{1-r_n}\right)\right\}}{\left(1-r_1\right)\left(1-r_2\right)\cdot\cdot\cdot\left(1-r_n\right)}\right]. \tag{2.9}$$

Now it follows from (2.5) and (2.9) for all sufficiently large values of $\left(\frac{1}{1-r_1}\right),\left(\frac{1}{1-r_2}\right),\cdot\cdot\cdot$ and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log T_{f\circ g}\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}{\log T_f\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}\le\frac{\left({}_{v_n}\rho_{f\circ g}^{L^*}+\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\left({}_{v_n}\lambda_f^{L^*}-\varepsilon\right)\log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}{\log T_f\left(r_1,r_2,\cdot\cdot\cdot,r_n\right)}\le\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}}\,. \tag{2.10}$$

Thus the theorem follows from (2.3), (2.6), (2.8) and (2.10). □

The following theorem can be proved in the line of Theorem 2.1 and so the proof is omitted.

Theorem 2.2. *If f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\lambda_{f\circ g}^{L^*} \le {}_{v_n}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\lambda_{g}^{L^*} \le {}_{v_n}\rho_{g}^{L^*} < \infty$ then*

$$\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\rho_{g}^{L^*}} \le \liminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)} \le \frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_{g}^{L^*}}$$
$$\le \limsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_g(r_1,r_2,\cdots,r_n)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\lambda_{g}^{L^*}} .$$

Theorem 2.3. *If f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < {}_{v_n}\rho_{f\circ g}^{L^*} < \infty$ and $0 < {}_{v_n}\rho_{f}^{L^*} < \infty$ then*

$$\liminf_{r_1,r_2,\cdots,r_n\to 1} \frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{f}^{L^*}} \le \limsup_{r_1,r_2,\cdots,r_n\to 1} \frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)} .$$

Proof. From the definition of ${}_{v_n}\rho_{f}^{L^*}$, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log T_f(r_1,r_2,\cdots,r_n) \geqslant$$
$$\left({}_{v_n}\rho_{f}^{L^*} - \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right] . \tag{2.11}$$

Now from (2.9) and (2.11), it follows for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)} \le \frac{\left({}_{v_n}\rho_{f\circ g}^{L^*} + \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\left({}_{v_n}\rho_{f}^{L^*} - \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]} .$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_f(r_1,r_2,\cdots,r_n)} \le \frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_{f}^{L^*}} . \tag{2.12}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log T_{f\circ g}(r_1,r_2,\cdots,r_n) \geqslant$$
$$\left({}_{v_n}\rho_{f\circ g}^{L^*} - \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right] . \tag{2.13}$$

So combining (2.2) and (2.13), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right), \left(\frac{1}{1-r_2}\right), \cdots$ and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log T_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log T_h^{-1} T_f(r) \log T_f(r_1,r_2,\cdots,r_n)} \geqslant \frac{\left({}_{v_n}\rho_{f\circ g}^{L^*} - \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]}{\left({}_{v_n}\rho_{f}^{L^*} + \varepsilon\right) \log\left[\frac{\exp\left\{L\left(\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots,\frac{1}{1-r_n}\right)\right\}}{(1-r_1)(1-r_2)\cdots(1-r_n)}\right]} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_f\left(r_1,r_2,\cdots,r_n\right)}\geqslant\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}\ . \tag{2.14}$$

Thus the theorem follows from (2.12) and (2.14) . □

The following theorem can be carried out in the line of Theorem 2.3 and therefore we omit its proof.

Theorem 2.4. *If f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0<{}_{v_n}\rho_{f\circ g}^{L^*}<\infty$ and $0<{}_{v_n}\rho_g^{L^*}<\infty$ then*

$$\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_g\left(r_1,r_2,\cdots,r_n\right)}\leq\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_g^{L^*}}\leq\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_g\left(r_1,r_2,\cdots,r_n\right)}\ .$$

The following theorem is a natural consequence of Theorem 2.1 and Theorem 2.3.

Theorem 2.5. *If f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0<{}_{v_n}\lambda_{f\circ g}^{L^*}\leq{}_{v_n}\rho_{f\circ g}^{L^*}<\infty$ and $0<{}_{v_n}\lambda_f^{L^*}\leq{}_{v_n}\rho_f^{L^*}<\infty$ then*

$$\begin{aligned}\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_f\left(r_1,r_2,\cdots,r_n\right)}&\leq\min\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}\right\}\\ &\leq\max\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_f^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_f^{L^*}}\right\}\leq\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_f\left(r_1,r_2,\cdots,r_n\right)}\ .\end{aligned}$$

The proof is omitted.

Analogously, one may state the following theorem without its proof.

Theorem 2.6. *If f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0<{}_{v_n}\lambda_{f\circ g}^{L^*}\leq{}_{v_n}\rho_{f\circ g}^{L^*}<\infty$ and $0<{}_{v_n}\lambda_g^{L^*}\leq{}_{v_n}\rho_g^{L^*}<\infty$ then*

$$\begin{aligned}\liminf_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_g\left(r_1,r_2,\cdots,r_n\right)}&\leq\min\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_g^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_g^{L^*}}\right\}\\ &\leq\max\left\{\frac{{}_{v_n}\lambda_{f\circ g}^{L^*}}{{}_{v_n}\lambda_g^{L^*}},\frac{{}_{v_n}\rho_{f\circ g}^{L^*}}{{}_{v_n}\rho_g^{L^*}}\right\}\leq\limsup_{r_1,r_2,\cdots,r_n\to 1}\frac{\log T_{f\circ g}\left(r_1,r_2,\cdots,r_n\right)}{\log T_g\left(r_1,r_2,\cdots,r_n\right)}\ .\end{aligned}$$

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