Palestine Journal of Mathematics

Vol. 8(1)(2019) , 35–43

FINITE RANK COMPRESSION OF SLANT HANKEL OPERATORS

Gopal Datt and Anshika Mittal

Communicated by Fuad Kittaneh

MSC 2010 Classifications: Primary 47B37; Secondary 47B35.

Keywords and phrases: Hankel operator, weighted Hankel Operator, weighted slant Hankel operator.

Abstract. Aim of the paper is to study the structure of the compression of slant Hankel operators that are of finite rank. Slant Hankel operators become a particular case of the notion of weighted slant Hankel operators and paper describes symbols so that the compression of the k^{th}-order slant Hankel operators for integer $k \geq 2$ is a finite rank operator.

1 Introduction

The Hardy space H^2 of analytic functions in the open unit disk $\mathbb{D}$ is defined as

$$H^2 = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

Let μ denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$ (the boundary of $\mathbb{D}$) and L^2 the Hilbert space of all complex-valued measurable functions f defined on $\mathbb{T}$ satisfying

$$\int |f|^2 d\mu \ < \infty.$$

It is customary to identify the functions of H^2 with the space of their boundary functions (see [8]). The boundary functions correspond to those functions in L^2 whose negative Fourier coefficients vanish. With this identification, H^2 is a closed subspace of L^2.

Hankel operators, which first appeared in the work of Hankel, arise in many applications, constitute one of the important classes of non isometric operators. In terms of operator equation, a Hankel operator is seen as an operator H satisfying the equation $U^*H = HU$ (see [2, 3, 9]), where U denotes the unilateral shift operator on H^2. A symbolic representation of Hankel operators is obtained by Nehari Theorem [3], by which, a Hankel operator H on H^2 is defined as $H = PJM_\phi$ for some $\phi \in L^\infty$, where P is the orthogonal projection of L^2 to H^2, J is the operator on L^2 given by $Jf(z) = f(\overline{z})$ and M_ϕ is the multiplication operator defined as $M_\phi f(z) = \phi(z)f(z)$. In this terminology H is said to be induced by the symbol $\phi \in L^\infty$ and is denoted as H_ϕ. For the details and applications of Hankel operators, we refer [3, 6, 7, 9, 10].

The study of Hankel operators becomes more demanding with the inception of the notion of slant Hankel operators [1], having the property that their matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Hankel operators. The study in this direction is further enhanced over various function spaces which led to different generalizations of the original concept, like, k^{th}-order slant Hankel operators, compression of slant Hankel operators, $\lambda-$Hankel operators and $(\lambda, \mu)-$Hankel operators (see [2],[4],[12] and the references therein).

Around the year 1974, Shields [11] brought forth the attention of mathematicians towards the study of the weighted multiplication operator $M_\phi^\beta(f \mapsto \phi f)$ on $L^2(\beta)$ with the symbol $\phi \in L^\infty(\beta)$. However, weighted Hardy spaces appeared in the work of Zorboska [13], where he discussed the notion of composition operators on these spaces. In the year 2005, Lauric [8] discussed the notion of weighted Toeplitz operator $T_\phi^\beta = P^\beta M_\phi^\beta$ on $H^2(\beta)$. The study is further extended with the introduction of the notions of weighted Hankel operators and weighted slant Hankel operators in [5], where the authors also discuss the compression of k^{th}-order weighted slant Hankel operators. It is also shown that an operator A on $H^2(\beta)$ is compression of a k^{th}-order weighted slant Hankel operator only if it satisfies the equation $T^{\beta}_{z^{-1}}A = AT^{\beta}_{z^k}$.

The spaces $L^2(\beta)$, $H^2(\beta)$ and $L^\infty(\beta)$ considered in the paper are under the assumption that $\{\beta_n\}$ is a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$. Throughout the paper, an additional condition of semi-duality on the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ (that is $\beta_n = \beta_{-n}$ for each n) is assumed.

The space $L^2(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ (whether or not the series converges for any values of z) for which $\|f\|_\beta < \infty$. The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product $\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n\, \bar{b}_n {\beta_n}^2$, for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|^2_\beta = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

The space $L^\infty(\beta)$ is complete with respect to $\|\cdot\|_\infty$. The space $H^\infty(\beta)$ denotes the set of formal power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. In the present paper, first section comprises of notational familiarities needed in the paper. In the second section, we provide the structure of rank one compression of k^{th}-order weighted slant Hankel operators for some specific bounded sequences $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ along with a characterization for rank one compression of k^{th}-order slant Hankel operators. Partial isometries of these operators are also discussed. One of the important results in the theory of Hankel operators is the Kronecker theorem, which characterizes the symbols inducing finite rank Hankel operators [3]. Our third section explores the Kronecker theorem for the finite rank compressions of k^{th}-order slant Hankel operators which provides a necessary condition for the symbols inducing these operators.

2 Rank One Compression

We begin with the following definitions, the detailed study of which can be seen in [1], [3] and [5].

Definition 2.1. [5] For fixed integer $k \geq 2$ and $\phi \in L^\infty(\beta)$, a k^{th}-order weighted slant Hankel operator $D^\beta_{k,\phi}$ on $L^2(\beta)$ is given by $D^\beta_{k,\phi} = J^\beta W_k M^\beta_\phi$, where J^β is the reflection operator given by $J^\beta e_n = e_{-n}$ for each $n \in \mathbb{Z}$ and W_k is given by

$$W_k e_n(z) = \begin{cases} \frac{\beta_m}{\beta_{km}} e_m(z) & \text{if } n = km \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

It is clear from the definition that if $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, then for each integer j,

$$D^\beta_{k,\phi} e_j = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{-nk-j}\beta_{-n} e_n.$$

The 2^{nd}-order weighted slant Hankel operators are simply called weighted slant Hankel operators and denoted by D^β_ϕ, $\phi \in L^\infty(\beta)$. If we take the particular case of the sequence β with $\beta_n = 1$ for each n, then $D^\beta_{k,\phi}$, $\phi \in L^\infty(\beta)$ on $L^2(\beta)$ is nothing but a k^{th}-order slant Hankel operator on L^2.

Definition 2.2. [1] A k^{th}-order slant Hankel operator $D_{k,\phi}$ on L^2 is given by

$$D_{k,\phi}e_j = \sum_{n=-\infty}^{\infty} a_{-nk-j}e_n.$$

for each integer j.

Definition 2.3. [3] The reproducing kernel function of H^2 at $\omega \in \mathbb{D}$ is denoted by K_ω, that is,

$$K_\omega(z) = \sum_{n=0}^{\infty} \bar{\omega}^n z^n.$$

We list here some known facts about rank one operator $f \otimes g$ on a Hilbert space, defined as $(f \otimes g)h = \langle h, g\rangle f$ (see [3]).

(i) $\|f \otimes g\| = \|f\|_2\|g\|_2$.

(ii) $(f \otimes g)^* = g \otimes f$.

(iii) For operators S and T, $S(f \otimes g)T = Sf \otimes T^*g$.

(iv) Two non-zero rank one operators $f_1 \otimes g_1$ and $f_2 \otimes g_2$ are equal if and only if there exists a non-zero complex number c such that $f_1 = cf_2$ and $g_2 = \bar{c}g_1$.

These elementary properties are used to study the structure of the compression of k^{th}-order weighted slant Hankel operators on $H^2(\beta)$ that are of rank one.

Theorem 2.4. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a bounded sequence such that $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. A rank one compression on $H^2(\beta)$ of a k^{th}-order weighted slant Hankel operator is always written as a linear combination of rank one operators of the form $K_{\bar{\alpha}} \otimes T^{\beta}_{z^i}V^{\beta}_{k,i+1}K_\alpha$, $i = 0, 1, \cdots, k-1$, for some $|\alpha| < 1$, where $T^{\beta}_{z^i}$ stands for the weighted Toeplitz operator induced by z^i ($T^{\beta}_{z^0}$ is the identity operator) and each $V^{\beta}_{k,i+1}$ is a bounded operator on $L^2(\beta)$ defined as $V^{\beta}_{k,i+1}e_n = \frac{\beta_{kn}}{\beta_n\beta^2_{kn+i}}e_{kn}$ for each $n \in \mathbb{Z}$.*

Proof. Let $A = f \otimes g$ be the compression of a k^{th}-order weighted slant Hankel operator for some non-zero elements $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ of $H^2(\beta)$. Then using [1,4], $T^{\beta}_{z^{-1}}(f \otimes g) = (f \otimes g)T^{\beta}_{z^k}$. Hence, for each $n\geq 0$, $\langle e_n, g\rangle T^{\beta}_{z^{-1}}f = \langle e_n, T^{\beta *}_{z^k}g\rangle f$. This yields that for $m, n\geq 0$

$$a_{m+1}\bar{b}_n\beta^2_n = \bar{b}_{n+k}\beta^2_{n+k}a_m. \tag{2.1}$$

Pick non-negative integer m_0 such that $a_{m_0} \neq 0$ (such m_0 exists as f is non-zero). Now we complete the proof by considering the two cases.

$\underline{Case(i)}$: Let $a_{m_0+1} = 0$. Then equation (2.1) provides $\bar{b}_n = 0$ for each $n \geq k$ and $a_m = 0$ for each $m \geq 1$. This means that $m_0 = 0$ and as a consequence, we have

$$\begin{aligned} A &= a_0\beta_0e_0 \otimes (b_0\beta_0e_0 + b_1\beta_1e_1 + \cdots + b_{k-1}\beta_{k-1}e_{k-1}) \\ &= a_0\bar{b}_0(e_0 \otimes \beta_0e_0) + a_0\overline{b_1}(e_0 \otimes \beta_1e_1) + \cdots + a_0\overline{b_{k-1}}(e_0 \otimes \beta_{k-1}e_{k-1}). \end{aligned}$$

Now for each $0 \leq i \leq k-1$, $(e_0 \otimes \beta_ie_i) = (e_0 \otimes e_0)T^{\beta *}_{z^i} = c_iK_{\bar{0}} \otimes T^{\beta}_{z^i}V^{\beta}_{k,i+1}K_0$ for some constant $c_i = a_0\bar{b}_i\beta^2_i$, where $V^{\beta}_{k,i+1}$ and K_0 are same as defined in the statement of the theorem. This provides the required form for A with $\alpha = 0$.

$\underline{Case(ii)}$: Suppose $a_{m_0+1} \neq 0$. Each element $g = \sum_{n=0}^{\infty} b_n z^n$ in $H^2(\beta)$ can be expressed as $g = \sum_{i=0}^{k-1} g_i$, where each $g_i = \sum_{n=0}^{\infty} b_{kn+i}z^{kn+i} \in H^2(\beta)$. Using (2.1), we find that if $b_n = 0$ for some positive integer n and n_0 is the least positive integer satisfying $b_{n_0} = 0$ then $n_0 < k$ and $b_{n_0+kn} = 0$ for each $n\geq 0$ so that $g_{n_0} = 0$. Let $S = \{i : 0 \leq i \leq k-1 \text{ and } b_i \neq 0\}$. Then $g = \sum_{i\in S} g_i$. The following facts can be gathered using (2.1):

(i) S is non-empty and for each $i \in S$, $b_{kn+i} \neq 0$ for every $n \geq 0$.

(ii) $a_m \neq 0$ for each $m \geq 0$ and $\frac{a_{m+1}}{a_m}$ is independent of the choice of integer $m \geq 0$.

(iii) For $i \in S$, $b_{k(n+1)+i} = \bar{\alpha}^{n+1} \frac{\beta_i^2}{\beta_{k(n+1)+i}^2} b_i$ for each $n \geq 0$, where $\alpha = \frac{a_{m+1}}{a_m}$ satisfies $|\alpha| < 1$.

These facts give that for $i \in S$, $g_i = b_i \beta_i^2 \sum_{n=0}^{\infty} \frac{\bar{\alpha}^n}{\beta_{kn+i}^2} z^{kn+i}$ and $a_n = \alpha^n a_0$ for each $n \geq 0$. A simple computation shows that for $i \in S$, $f \otimes g_i = d_i(K_{\bar{\alpha}} \otimes T_{z^i}^{\beta} V_{k,i+1}^{\beta} K_{\alpha})$, where $d_i = a_0 \bar{b}_i \beta_i^2$. This provides the required form as $A = f \otimes g = \sum_{i=0}^{k-1} c_i(K_{\bar{\alpha}} \otimes T_{z^i}^{\beta} V_{k,i+1}^{\beta} K_{\alpha})$, where

$$c_i = \begin{cases} d_i & if\ i \in S \\ 0 & otherwise \end{cases}.$$

The structure of rank one compression of the weighted slant Hankel operators is given by the following result, which is nothing but the case $k = 2$ of Theorem 2.4.

Corollary 2.5. *The rank one compression of a weighted slant Hankel operator is always of the form* $C_0(K_{\bar{\alpha}} \otimes V_{2,1}^{\beta} K_{\alpha}) + C_1(K_{\bar{\alpha}} \otimes T_z^{\beta} V_{2,2}^{\beta} K_{\alpha})$*, for constants* C_0, C_1 *and some* $|\alpha| < 1$*, where* $V_{2,1}^{\beta}$*,* $V_{2,2}^{\beta}$ *and* $\{\beta_n\}$ *are same as defined in Theorem 2.4.*

The existence of rank one compression of k^{th}-order weighted slant Hankel operators can be justified with the following example.

Example 2.6. Let $k \geq 6$ be a fixed integer. Consider a bounded sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ defined as

$$\beta_n = \begin{cases} 1 & if\ n = 0 \\ 2^{|n|} & if\ 0 \neq |n| < 5 \\ 64 & otherwise \end{cases}.$$

Then $\{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence of positive numbers with $\beta_0 = 1$, $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded and $\frac{1}{2} \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$. Let $\phi(z) = z^{-k+1}$. Clearly $\phi \in L^{\infty}(\beta)$. Consider the k^{th}-order weighted slant Hankel operator $D_{k,\phi}^{\beta}$ induced by ϕ given as $D_{k,\phi}^{\beta} = J^{\beta} W_k M_{\phi}^{\beta}$. Then

$$D_{k,\phi}^{\beta}(e_j) = \begin{cases} \frac{\beta_n}{\beta_{kn+k-1}} e_{-n} & if\ j = kn + k - 1\ for\ n \in \mathbb{Z} \\ 0 & otherwise \end{cases}.$$

Taking $L_{k,\phi}^{\beta}$ to be the compression of $D_{k,\phi}^{\beta}$ on $H^2(\beta)$, we have

$$L_{k,\phi}^{\beta}(e_j) = \begin{cases} \frac{1}{64} e_0 & if\ j = k - 1 \\ 0 & otherwise \end{cases}.$$

Thus $L_{k,\phi}^{\beta}$ is a rank one operator. Further, we can see that $L_{k,\phi}^{\beta} = f \otimes g$, where $f(z) = \frac{1}{8192} e_0$ and $g(z) = 2z^{k-1}$. We can further prove that $L_{k,\phi}^{\beta}$ can be expressed as $L_{k,\phi}^{\beta} = c_{k-1}(K_{\bar{\alpha}} \otimes T_{z^{k-1}}^{\beta} V_{k,k}^{\beta} K_{\alpha})$ with $c_{k-1} = 1$ and $\alpha = 0$, which verifies Theorem 2.4 for rank one compression $L_{k,\phi}^{\beta}$.

However, we show with the help of next example that each rank one operator of the form $c_0(K_{\bar{\alpha}} \otimes V_{k,1}^{\beta} K_{\alpha})$ need not be a compression of k^{th}-order weighted slant Hankel operator.

Example 2.7. Let $\{\beta_n\}_{n\in\mathbb{Z}}$ be the same sequence as defined in the last example. Let $k \geq 6$ be a fixed integer. Consider a rank one operator $A = i(\sum_{n=0}^{\infty} \alpha^n z^n \otimes \sum_{n=0}^{\infty} \frac{\bar{\alpha}^n}{\beta_{kn}^2} z^{kn})$, then we have

$A = c_0(K_{\bar{\alpha}} \otimes V^{\beta}_{k,1} K_{\alpha})$ where $c_0 = i$ and $\alpha = \frac{1}{2}$. Further we see that $A(e_{kn+i}) = 0$ for each $n \geq 0, i = 1, 2, \cdots, k-1$ and

$$A(e_{kn}) = \begin{cases} f_0 & n = 0 \\ \frac{1}{2^{n+6}} f_0 & n > 0 \end{cases},$$

where f_0 is an element of $H^2(\beta)$ given by $f_0(z) = i(1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots)$. We find that this operator A doesn't satisfy the equation $T^{\beta}_{z^{-1}} A = A T^{\beta}_{z^k}$ as $T^{\beta}_{z^{-1}} A(e_{kn}) = \frac{1}{2^{n+6}}(f_0 - i)$ and $A T^{\beta}_{z^k}(e_{kn}) = \frac{1}{2^{n+1}} f_0$ for $n > 0$. Thus, A is not a compression of any k^{th}-order weighted slant Hankel operator.

Remark 2.8. We have proved Theorem 2.4 under the assumption of the boundedness of sequence $\{\beta_n\}_{n\in\mathbb{Z}}$. However, the existence of rank one compression of the form suggested in Theorem 2.4 can also be seen when the sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ is unbounded. For, consider the sequence $\{\beta_n\}$ defined as

$$\beta_n = \begin{cases} 1 & n = 0 \\ |n| & otherwise \end{cases}.$$

Then $\{\beta_n\}_{n\in\mathbb{Z}}$ is an unbounded semi-dual sequence of positive numbers with $\beta_0 = 1$, $\{\frac{\beta_{kn}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded and $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, where $k \geq 6$ is a fixed integer and $r = 1/2$. Now the compression $L^{\beta}_{k,\phi}$ of k^{th}-order weighted slant Hankel operator $D^{\beta}_{k,\phi} = J^{\beta} W_k M^{\beta}_{\phi}$ with $\phi(z) = z^{-k+1}$ is a rank one operator given as

$$L^{\beta}_{k,\phi}(e_j) = \begin{cases} \frac{1}{k-1} e_0 & if\ j = k-1 \\ 0 & otherwise \end{cases}.$$

Also, we find that it can be expressed as $L^{\beta}_{k,\phi} = c_{k-1}(K_{\bar{\alpha}} \otimes T^{\beta}_{z^{k-1}} V^{\beta}_{k,k} K_{\alpha})$ with $c_{k-1} = 1$ and $\alpha = 0$.

Now on considering the sequence β under $\beta_n = 1$ for each n, Theorem 2.4 provides the structure for the rank one compression on H^2 of k^{th}-order slant Hankel operators as $\sum_{i=0}^{k-1} c_i(K_{\bar{\alpha}} \otimes T_{z^i} V_k K_{\alpha})$, where $c_i, \alpha \in \mathbb{C}$ with $|\alpha| < 1$ and V_k is an operator on L^2 given by $V_k(e_n) = e_{kn}$ for each $n \in \mathbb{N}$. In this case, it is interesting to observe that the converse of Theorem 2.4 also holds. A simple computation shows that every operator A of the form $A = \sum_{i=0}^{k-1} c_i(K_{\bar{\alpha}} \otimes T_{z^i} V_k K_{\alpha})$, where $c_i, \alpha \in \mathbb{C}$ with $|\alpha| < 1$, satisfies $U^* A = A U^k$. Hence A is compression of a k^{th}-order slant Hankel operator (using [1], where it is shown that a necessary and sufficient condition for an operator A to be compression of a k^{th}-order slant Hankel operator is $U^* A = A U^k$). Pick $a_0 \neq 0$ and $b_i = \overline{(\frac{c_i}{a_0})}$ for $0 \leq i \leq k-1$. Clearly $b_i \neq 0$ for some i. Define $a_n = \alpha^n a_0$ for $n \geq 1$ and $b_{nk+k+i} = \bar{\alpha}^{(n+1)} b_i$ for $n \geq 0$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then $f, g \in H^2$ and it is evident to see that A is a rank one operator satisfying $A = f \otimes g$. Thus we have the following.

Theorem 2.9. *Let $k \geq 2$ be fixed. The set consisting of all rank one compressions on H^2 of k^{th}-order slant Hankel operators is*

$$\{\sum_{i=0}^{k-1} c_i(K_{\bar{\alpha}} \otimes T_{z^i} V_k K_{\alpha}) : c_i, \alpha \in \mathbb{C} \textit{ and } |\alpha| < 1\},$$

where V_k is the operator on L^2 given by $V_k(e_n) = e_{kn}$.

Theorem 2.10. *A rank one compression A on H^2 of a k^{th}-order slant Hankel operator given by $A = \sum_{i=0}^{k-1} c_i(K_{\bar{\alpha}} \otimes T_{z^i} V_k K_{\alpha})$ with $|\alpha| < 1$, is partial isometry if and only if $(\frac{1}{1-|\alpha|^2})^2(c_0\overline{c_0} + c_1\overline{c_1} + \cdots + c_{k-1}\overline{c_{k-1}}) = 1$.*

Proof. Let $A = \sum_{i=0}^{k-1} c_i(K_{\bar{\alpha}} \otimes T_{z^i} V_k K_\alpha)$. Then $A^* = \sum_{i=0}^{k-1} T_{z^i} V_k K_\alpha \otimes c_i K_{\bar{\alpha}}$ so that for each $0 \leq i \leq k-1$,

$$A^* e_{kn+i} = \overline{c_0}\bar{\alpha}^{kn+i} \sum_{m=0}^{\infty} \bar{\alpha}^m z^{km} + \cdots + \overline{c_{k-1}}\bar{\alpha}^{kn+i} \sum_{m=0}^{\infty} \bar{\alpha}^m z^{km+k-1}.$$

With a routine computation, it can be easily seen that

$$AA^* A e_{kn+i} = c_i \alpha^n (\frac{1}{1-|\alpha|^2})^2 (c_0\overline{c_0} + c_1\overline{c_1} + \cdots + c_{k-1}\overline{c_{k-1}}) \sum_{m=0}^{\infty} \alpha^m z^m.$$

Now result follows using the fact that A be a partial isometry if and only if $AA^*A = A$. □

Example 2.11. Let $k \geq 2$ be fixed. If we consider the operator $T : H^2 \to H^2$ defined as

$$Te_j = \begin{cases} \frac{2}{3^n} g_0 & if\ j = kn,\ n \geq 0 \\ \frac{i}{3^n} g_0 & if\ j = kn+1,\ n \geq 0 \\ 0 & otherwise \end{cases},$$

where g_0 is given by $g_0(z) = 1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots$. It can be easily verified that $T = c_0(K_{\bar{\alpha}} \otimes V_k K_\alpha) + c_1(K_{\bar{\alpha}} \otimes T_z V_k K_\alpha)$, where $c_0 = 2$, $c_1 = i$ and $\alpha = \frac{1}{3}$. Therefore by Theorem 2.9, operator T is a rank one compression of a k^{th}-order slant Hankel operator. Further, we find that $(\frac{1}{1-|\alpha|^2})^2(c_0\overline{c_0} + c_1\overline{c_1} + \cdots + c_{k-1}\overline{c_{k-1}}) \neq 1$, which shows that T is not a partial isometry using Theorem 2.10.

Corollary 2.12. *An operator A on H^2 is a rank one compression of a slant Hankel operator if and only if it is of the form $A = c_0(K_{\bar{\alpha}} \otimes VK_\alpha) + c_1(K_{\bar{\alpha}} \otimes T_z VK_\alpha)$ for constants c_0, c_1 and $|\alpha| < 1$. Further, A is partial isometry if and only if $(\frac{1}{1-|\alpha|^2})^2(c_0\overline{c_0} + c_1\overline{c_1}) = 1$.*

Example 2.13. Let $k \geq 2$ be fixed. Define an operator A on H^2 given by

$$Ae_j = \begin{cases} \frac{i}{4}\left(\frac{1}{2}\right)^n g_0 & if\ j = kn,\ n \geq 0 \\ \frac{i}{\sqrt{2}}\left(\frac{1}{2}\right)^n g_0 & if\ j = kn+1,\ n \geq 0 \\ 0 & otherwise \end{cases},$$

where g_0 is given by $g_0(z) = (1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots)$. Here A is of the form $c_0(K_{\bar{\alpha}} \otimes V_k K_\alpha) + c_1(K_{\bar{\alpha}} \otimes T_z V_k K_\alpha)$ for $c_0 = \frac{i}{4}$, $c_1 = \frac{i}{\sqrt{2}}$ and $\alpha = \frac{1}{2}$. Thus A is a rank one compression of a k^{th}-order slant Hankel operator which is partial isometry as we have $(\frac{1}{1-|\alpha|^2})^2(c_0\overline{c_0} + c_1\overline{c_1}) = 1$.

3 Main Results

Throughout this section, we use the symbol H_ϕ, $\phi \in L^\infty$ to denote a non-zero Hankel operator on H^2. For a fixed integer $k \geq 2$ and $\psi \in L^\infty$, the operators $E_{k,\psi}$ on H^2 represents a compression of the k^{th}-order slant Hankel operator induced by ψ and L_ψ denotes the compression of slant Hankel operator induced by ψ. Then $E_{k,\psi} = W_k|_{H^2} H_\psi$ and $L_\psi = W|_{H^2} H_\psi$. If no confusion arises, we simply write $E_{k,\psi} = W_k H_\phi$ and $L_\psi = WH_\phi$. We use the symbol U to denote the unilateral shift operator.

A simple observation is that if the product $H_\phi E_{k,\psi}$ is a Hankel operator then $(U^* H_\phi E_{k,\psi} U^k)U = U^*(U^* H_\phi E_{k,\psi})U^k = U^*(U^* H_\phi E_{k,\psi} U^k)$ so that $U^* H_\phi E_{k,\psi} U^k$ is also a Hankel operator. For a given symbol $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, the notations $\widetilde{\phi}$ and $\overline{\phi}$ respectively mean the expressions $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$ and $\overline{\phi}(z) = \sum_{n=-\infty}^{\infty} \overline{a_{-n}} z^n$. We now attain the following.

Theorem 3.1. *A necessary and sufficient condition for the product $H_\phi E_{k,\psi}$ to be a Hankel operator is that H_ϕ is a constant multiple of rank one Hankel operator $K_{\bar{\omega}} \otimes K_\omega$ and $E_{k,\psi}$ is a constant multiple of rank one operator $K_{\bar{\omega}^k} \otimes K_\omega$.*

Proof. Let $H_\phi E_{k,\psi}$ be a Hankel operator. Then $H_\phi E_{k,\psi} - U^* H_\phi E_{k,\psi} U^k$ is also a Hankel operator. Moreover, we find that it is non-zero as

$$\begin{aligned} H_\phi E_{k,\psi} - U^* H_\phi E_{k,\psi} U^k &= H_\phi (e_0 \otimes e_0) E_{k,\psi} \\ &= H_\phi e_0 \otimes (W_k H_\psi)^* e_0 \\ &= PJ\phi e_0 \otimes PJ\psi^* e_0 \\ &= P\widetilde{\phi} \otimes P\bar{\psi}, \end{aligned}$$

and $P\widetilde{\phi} \otimes P\bar{\psi} = 0$ provides that either $H_\phi = 0$ or $E_{k,\psi} = 0$. Being $P\widetilde{\phi} \otimes P\bar{\psi}$ a rank one Hankel operator, we have $P\widetilde{\phi} \otimes P\bar{\psi} = cK_{\bar{\omega}} \otimes K_\omega$ for some $|\omega| < 1$ and scalar c. As a consequence, we get that $P\widetilde{\phi} = aK_{\bar{\omega}}$ and $P\bar{\psi} = bK_\omega$ for some scalars a and b. Then we can easily see that $P\overline{\psi} = \bar{b}K_{\bar{\omega}}$. Hence the co-analytic parts of ϕ and ψ are multiples of $K_{\bar{\omega}}$. As a consequence of this, $H_\phi = aK_{\bar{\omega}} \otimes K_\omega$ and $E_{k,\psi} = W_k H_\psi = \bar{b} W_k K_{\bar{\omega}} \otimes K_\omega = \bar{b} K_{\bar{\omega}^k} \otimes K_\omega$.

Conversely, let $H_\phi = aK_{\bar{\omega}} \otimes K_\omega$ and $E_{k,\psi} = bK_{\bar{\omega}^k} \otimes K_\omega$. Then for each $f \in H^2$,

$$\begin{aligned} H_\phi E_{k,\psi} f &= ab(K_{\bar{\omega}} \otimes K_\omega)(K_{\bar{\omega}^k} \otimes K_\omega) f \\ &= ab(K_{\bar{\omega}} \otimes K_\omega)\langle f, K_\omega\rangle W_k K_{\bar{\omega}} \\ &= ab\langle W_k K_{\bar{\omega}}, K_\omega\rangle (K_{\bar{\omega}} \otimes K_\omega) f. \end{aligned}$$

Hence, $H_\phi E_{k,\psi}$ $(= ab\langle W_k K_{\bar{\omega}}, K_\omega\rangle (K_{\bar{\omega}} \otimes K_\omega))$ is a Hankel operator. This completes the proof. □

An immediate consequence of Theorem 3.1 is the following.

Corollary 3.2. *The product $H_\phi L_\psi$ of the Hankel operator H_ϕ and the compression L_ψ of a slant Hankel operator is a Hankel operator if and only if H_ϕ is a constant multiple of rank one Hankel operator $K_{\bar{\omega}} \otimes K_\omega$ and L_ψ is a constant multiple of rank one compression of slant Hankel operator $K_{\bar{\omega}^2} \otimes K_\omega$.*

It is easy to observe that if $H_\phi E_{k,\psi}$ on H^2 is compression of a k^{th}-order slant Hankel operator i.e. satisfies the equation $H_\phi E_{k,\psi} U^k = U^* H_\phi E_{k,\psi}$ then $(U^* H_\phi E_{k,\psi} U^k) U^k = U^*(U^* H_\phi E_{k,\psi}) U^k$ $= U^*(U^* H_\phi E_{k,\psi} U^k)$. Hence, $U^* H_\phi E_{k,\psi} U^k$ is also a compression of a k^{th}-order slant Hankel operator. Along the lines of arguments applied in Theorem 3.1, we can prove the following.

Theorem 3.3. *The product $H_\phi E_{k,\psi}$ of operators H_ϕ and $E_{k,\psi}$ on H^2 is compression of a k^{th}-order slant Hankel operator if and only if H_ϕ is a constant multiple of rank one Hankel operator $K_{\bar{\omega}} \otimes K_\omega$ and $E_{k,\psi}$ is a linear combination of operators $K_{\bar{\omega}} \otimes V_k K_\omega, K_{\bar{\omega}} \otimes T_z V_k K_\omega, \cdots, K_{\bar{\omega}} \otimes T_{z^{k-1}} V_k K_\omega$.*

Proof. If $H_\phi E_{k,\psi}$ satisfies $H_\phi E_{k,\psi} U^k = U^* H_\phi E_{k,\psi}$ then $P\widetilde{\phi} \otimes P\bar{\psi} = H_\phi E_{k,\psi} - U^* H_\phi E_{k,\psi} U^k$ is a rank one operator satisfying $(P\widetilde{\phi} \otimes P\bar{\psi}) U^k = U^*(P\widetilde{\phi} \otimes P\bar{\psi})$ and is of the form

$$\begin{aligned} P\widetilde{\phi} \otimes P\bar{\psi} &= c_0 K_{\bar{\omega}} \otimes V_k K_\omega + \cdots + c_{k-1} K_{\bar{\omega}} \otimes T_{z^{k-1}} V_k K_\omega \\ &= K_{\bar{\omega}} \otimes (c_0 V_k K_\omega + c_1 T_z V_k K_\omega + \cdots + c_{k-1} T_{z^{k-1}} V_k K_\omega). \end{aligned}$$

Thus, $P\widetilde{\phi} = aK_{\bar{\omega}}$ and $P\bar{\psi} = b(c_0 V_k K_\omega + \cdots + c_{k-1} T_{z^{k-1}} V_k K_\omega)$ for some constants a and b. Suppose $\psi = \sum_{n=-\infty}^{\infty} b_n z^n$. Then $P\bar{\psi} = \sum_{n=0}^{\infty} \bar{b}_{-n} z^n$ and hence

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{b}_{-n} z^n &= b(c_0 I + c_1 T_z + \cdots + c_{k-1} T_{z^{k-1}}) V_k K_\omega \\ &= bc_0 \sum_{n=0}^{\infty} \bar{\omega}^n z^{kn} + \cdots + bc_{k-1} \sum_{n=0}^{\infty} \bar{\omega}^n z^{kn+k-1}. \end{aligned}$$

On comparing the coefficients both sides, we get that for each $0 \leq i \leq k-1$, $\bar{b}_{-kn-i} = bc_i\bar{\omega}^n$ for all $n \geq 0$. As a consequence

$$\begin{aligned} P\bar{\psi} = \sum_{n=0}^{\infty} b_{-n} z^{-n} &= \sum_{n=0}^{\infty} b_{-kn} z^{-kn} + \cdots + \sum_{n=0}^{\infty} b_{-kn-k+1} z^{-kn-k+1} \\ &= \overline{bc_0} \sum_{n=0}^{\infty} \omega^n z^{-kn} + \cdots + \overline{bc_{k-1}} \sum_{n=0}^{\infty} \omega^n z^{-kn-k+1} \\ &= d_0 V_k \widetilde{K_{\bar{\omega}}} + \cdots + d_{k-1} \bar{z}^{k-1} V_k \widetilde{K_{\bar{\omega}}}. \end{aligned}$$

Thus the co-analytic part of ϕ is a constant multiple of $K_{\bar{\omega}}$ and co-analytic part of ψ is a linear combination of vectors $V_k\widetilde{K_{\bar{\omega}}},\ \bar{z}V_k\widetilde{K_{\bar{\omega}}},\ \cdots,\ \bar{z}^{k-1}V_k\widetilde{K_{\bar{\omega}}}$. This provides the desired structures of H_ϕ and $E_{k,\psi}$.

Converse part follows with a straight forward computation. □

It is easy to obtain the following from here.

Corollary 3.4. *The product $H_\phi L_\psi$ is a compression of slant Hankel operator if and only if H_ϕ is a constant multiple of rank one Hankel operator $K_{\bar{\omega}} \otimes K_\omega$ and L_ψ is a linear combination of $K_{\bar{\omega}} \otimes VK_\omega$ and $K_{\bar{\omega}} \otimes T_z VK_\omega$.*

In the next result, we find a necessary condition for the symbol inducing finite rank compression of a k^{th}-order slant Hankel operator.

Theorem 3.5. *Let $\phi \in L^\infty$. A necessary condition for $E_{k,\phi}$ to be of finite rank is that $\phi \in e^{i\theta}H^\infty + \mathbb{R}(z^k) + \frac{1}{z}\mathbb{R}(z^k) + \cdots + \frac{1}{z^{k-1}}\mathbb{R}(z^k)$, where $\mathbb{R}$ is set of rational functions with poles inside $\mathbb{D}$.*

Proof. Let rank of $E_{k,\phi}$ be r. Then each of the set $\{E_{k,\phi}e_i,\ E_{k,\phi}e_{k+i},\ E_{k,\phi}e_{2k+i},\ E_{k,\phi}e_{3k+i},\cdots\cdot\cdot,\ E_{k,\phi}e_{kr+i}\}$, $0 \leq i \leq k-1$, is linearly dependent. Hence, there exist constants $c_0^i, c_1^i, \cdots, c_r^i$, not all zero, such that

$$c_0^i E_{k,\phi}e_i + c_1^i E_{k,\phi}e_{k+i} + \cdots + c_r^i E_{k,\phi}e_{kr+i} = 0.$$

This provides that for each $0 \leq i \leq k-1$, $\sum\limits_{j=0}^{r} c_j^i a_{-kj-i-kl} = 0$ for all $l \geq 0$.

For each $0 \leq i \leq k-1$, define $\psi_i(z) = c_0^i + c_1^i z + \cdots + c_r^i z^r$ and $\phi_i(z) = \frac{a_{-i}}{z^i} + \frac{a_{-k-i}}{z^{k+i}} + \frac{a_{-2k-i}}{z^{2k+i}} + \cdots$. Then, we have

$$\begin{aligned} \psi_i(z^k)\phi_i(z) &= (\sum_{j=0}^{r} c_j^i z^{kj})(\sum_{n=0}^{\infty} a_{-kn-i} z^{-kn-i}) \\ &= \sum_{j=0}^{r}\sum_{n=0}^{j} c_j^i a_{-kn-i} z^{kj-kn-i} + \sum_{j=0}^{r}\sum_{n=j+1}^{\infty} c_j^i a_{-kn-i} z^{kj-kn-i} \\ &= \sum_{j=0}^{r}\sum_{n=0}^{j} c_j^i a_{-kn-i} z^{kj-kn-i} + \sum_{l=1}^{\infty}(\sum_{j=0}^{r} c_j^i a_{-kj-kl-i)}) z^{-kl-i} \\ &= \frac{1}{z^i}\sum_{j=0}^{r}\sum_{n=0}^{j} c_j^i a_{-kn-i} z^{kj-kn}. \end{aligned}$$

Now, if we define $\gamma_i(z) = \sum\limits_{j=0}^{r}\sum\limits_{n=0}^{j} c_j^i a_{-kn-i} z^{j-n}$ and $\delta_i = \frac{\gamma_i}{\psi_i}$, then we have $\phi_i(z) = \frac{1}{z^i}\delta_i(z^k)$.

It can be seen here that each δ_i is a rational function with poles inside $\mathbb{D}$. For, if δ_i has a pole outside $\mathbb{D}$,then $\delta_i(z^k)$ has a pole outside $\mathbb{D}$. Now, we set $\delta_i^1 = \delta_i(\frac{1}{z})$ then clearly $\delta_i^1(z^k) \in H^2(\mathbb{D})$ for each i which is not possible because $\delta_i(z^k)$ has a pole outside $\mathbb{D}$. This completes the result as $\phi(z) = \sum\limits_{n=1}^{\infty} a_n z^n + \sum\limits_{n=-\infty}^{0} a_n z^n$ and $\sum\limits_{n=-\infty}^{0} a_n z^n = \delta_0(z^k) + \frac{1}{z}\delta_1(z^k) + \cdots + \frac{1}{z^{k-1}}\delta_{k-1}(z^k)$. □

With this theorem, we can prove the following.

Corollary 3.6. *For $\phi \in L^\infty$, if L_ϕ is finite rank compression of a slant Hankel operator then $\phi \in e^{i\theta}H^\infty + \mathbb{R}(z^2) + \frac{1}{z}\mathbb{R}(z^2)$, where $\mathbb{R}$ is set of rational functions with poles inside $\mathbb{D}$.*

References

[1] S. C. Arora and J. Bhola, The compression of a k^{th}-order slant Hankel operator, *Ganita* **59 (1)**, 1–11 (2008).

[2] R. Martìnez-Avendaño, A generalization of Hankel operators, *J. Func. Anal.* **190**, 418–446 (2002).

[3] Ruben A. Martìnez-Avendaño and Peter Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Springer, New York (2007).

[4] G. Datt and R. Aggarwal, On a Generalization of Hankel Operators via Operator Equations, *Extracta Mathematicae* **28 (2)**, 197–211 (2013).

[5] G. Datt and D. K. Porwal, Generalization of weighted slant Hankel operators, *Mathematica Slovaca* **66 (5)**, 1–14 (2016).

[6] T. Goodman, C. Micchelli and J. Ward, *Spectral radius formula for subdivision operators*, Recent advances in Wavelet Analysis, Edited by L. Schumaker and G. Webb, Acedemic Press, 335–360 (1994).

[7] H. Hankel, *Über eine besondere Classe der symmetrischen Determinanten(Leipzger) dissertation*, *Göttigen* (1861).

[8] V. Lauric, On a Weighted Toeplitz operator and its commutant, *International Jr. Math. and Math. Sc.* **6**, 823–835 (2005).

[9] V. Peller, *Hankel operators and applications*, Springer-Verlag, New York (2003).

[10] V. Peller and S. V. Khrushchev, *Hankel operators, best approximations and stationary Gaussian processes*, Russian Math. Surveys **37(1)**, 61–144 (1982).

[11] A. L. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys, No.13, American Mathematical Society, Rhode Ireland (1974).

[12] Taddesse Zegeye and S. C. Arora, The compression of a slant Hankel operator to H^2, *Publications De L Institut Mathematique Ât' Nouvelle sÂt'erie Tome* **74(88)**, 129–136 (2003).

[13] N. Zorboska, Compact composition operators on some weighted Hardy spaces, *Journal of operator theory* **22**, 233–241 (1989).

Author information

Gopal Datt and Anshika Mittal, Department of Mathematics, PGDAV College, University of Delhi, Delhi - 110065, INDIA.
E-mail: gopal.d.sati@gmail.com

Received: September 9, 2017.
Accepted: April 12, 2018.