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k–Quotient cordial labeling of graphs

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Abstract. In this paper we introduce k–Quotient cordial labeling of graphs. Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 \leq k \leq |V(G)|$. For each edge uv assign the label $\left[\frac{f(u)}{f(v)}\right]$ (or) $\left[\frac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. f is called a k–Quotient cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a k–Quotient cordial labeling is called a k–Quotient cordial graph. We investigate the quotient cordial labeling behavior of path, cycle, wheel, complete graph, star, bistar and some more graphs.

1 Introduction

Graphs considered here are finite and simple. Graph labeling is used in several areas of science and technology like coding theory, astronomy, circuit design etc. For more details refer Gallian [2]. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$. The graph $W_n = C_n + K_1$ is called a wheel. In a wheel, a vertex of degree 3 is called a rim vertex. A vertex which is adjacent to all the rim vertices is called the central vertex. The edges with one end incident with the rim and the other incident with the central vertex are called spokes. Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. The bistar $B_{m,n}$ is the graph obtained by making adjacent the two central vertices of $K_{1,m}$ and $K_{1,n}$. The square of a path P_n^2 is obtained from the path P_n by adding edges that joins all vertices u and v with $d(u, v) = 2$. Cahit [1], introduced the concept of cordial labeling of graphs. Recently Ponraj et al. [4], introduced Quotient cordial labeling of graphs and investigated the Quotient cordial labeling behavior of path, cycle, complete graph, star, bistar. In [5], Ponraj et al. investigate the Quotient cordial labeling behavior of subdivided star $S(K_{1,n})$, subdivided bistar $S(B_{n,n})$ and union of some star related graphs. Motivated by this labeling we introduce k–Quotient cordial labeling of graphs. Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 \leq k \leq |V(G)|$. For each edge uv assign the label $\left[\frac{f(u)}{f(v)}\right]$ (or) $\left[\frac{f(v)}{f(u)}\right]$ according as $f(u) \geq f(v)$ or $f(v) > f(u)$. f is called a k–Quotient cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with a k–Quotient cordial labeling is called a k–Quotient cordial graph. In this paper we investigate the k–Quotient cordial labeling behavior of path, cycle, wheel, complete graph, star, bistar etc. $[x]$ denote the smallest integer less than or equal to x. Terms are not defined here follows from Harary [3].

2 3–Quotient cordial labeling

Here we investigate the 3–Quotient cordial labeling behavior of some graphs. First we prove the following theorem.

Theorem 2.1. *Every graph is a subgraph of a connected* 3*–Quotient cordial graph.*

Proof. Let G be a (p, q) graph. Consider three copies of the complete graph K_p. Let $u_1^i, u_2^i, \ldots, u_p^i$, $(1 \le i \le 3)$ be the vertices of the i^{th} copy of K_p. Let $m = \binom{p}{2}$. Next consider three copies of the star $K_{1,m}$ with $v_1^i, v_2^i, \ldots, v_m^i$, $(1 \le i \le 3)$ is the pendent vertices of the i^{th} star. The connected super graph G^* of G is obtained as follows: Let $V(G^*) = \bigcup_{i=1}^{3} \{u_j^i : 1 \le j \le p\} \cup \{u_1^1 v_j^1 : 1 \le j \le m\} \cup \{u_1^2 v_j^1 : 1 \le j \le m\} \cup \{u_1^3 v_j^1 : 1 \le j \le m\} \cup \{u_1^1 u_1^2, u_1^2 u_1^3\}$. Clearly G^* has $3p + 3m$ vertices and $4m + 2$ edges. We now assign the label to the vertices of G^*. Assign the label 1 to all the vertices of the first copy of K_p and 2 to all the vertices of the second copy and 3 to the thir copy. Next assign the label 2 to the vertices v_j^1 $(1 \le j \le m)$, 3 to the vertices v_j^2 $(1 \le j \le m)$ and 3 to the vertices v_j^3 $(1 \le j \le m)$. This vertex labeling f is a 3–Quotient cordial labeling of G^* . Since $v_f(1) = v_f(2) = v_f(3) = p + m$ and $e_f(0) = e_f(1) = 2m + 1$. □

Now we investigate the 3–Quotient cordial labeling behavior of K_n.

Theorem 2.2. *The complete graph K_n is* 3*–Quotient cordial iff $n \le 4$.*

Proof. Suppose f is 3–Quotient cordial labeling of K_n. The proof is divided into three cases.
Case 1. $n \equiv 0 \pmod 3$.
Let $n = 3t$, $t \in \mathbb{N}$ and $t \ge 2$. Then $v_f(1) = v_f(2) = v_f(3) = t$. This implies $e_f(1) = \binom{t}{2} + \binom{t}{2} + \binom{t}{2} + t^2 + t^2$ and $e_f(0) = t^2$. Therefore $e_f(1) - e_f(0) = 3\binom{t}{2} + t^2 = \frac{5t^2-3t}{2} \ge 7$ as $t \ge 2$, a contradiction.
Case 2. $n \equiv 1 \pmod 3$.
Let $n = 3t + 1$, $t \in \mathbb{N}$ and $t \ge 2$. In this case any one of the following arises.
Type A: $v_f(1) = t + 1$, $v_f(2) = v_f(3) = t$.
Type B: $v_f(1) = t$, $v_f(2) = t + 1$, $v_f(3) = t$.
Type C: $v_f(1) = v_f(2) = t$, $v_f(3) = t + 1$.
Now we examine the above three types.
Type A and Type B:
In this type $e_f(1) - e_f(0) = 3\binom{t+1}{2} + t^2 + 2\binom{t}{2} = \frac{5t^2-t}{2} \ge 9$ as $t \ge 2$, a contradiction.
Type C:
In this type $e_f(1) - e_f(0) = 2\binom{t}{2} + \binom{t+1}{2} + t^2 + 2t = \frac{5t^2+3t}{2} \ge 13$ as $t \ge 2$, a contradiction.
Case 3. $n \equiv 2 \pmod 3$.
Let $n = 3t + 2$, $t \in \mathbb{N}$ and $t \ge 1$. In this case any one of the following type arises.
Type A: $v_f(1) = t$, $v_f(2) = v_f(3) = t + 1$.
Type B: $v_f(1) = v_f(3) = t + 1$, $v_f(2) = t$.
Type C: $v_f(1) = v_f(2) = t + 1$, $v_f(3) = t$.
Now we examine the above three types.
Type A and Type B:
In this type $e_f(1) - e_f(0) = \binom{t}{2} + \binom{t+1}{2} + \binom{t+1}{2} + t(t + 1) + (t + 1)^2 - t(t + 1) = \frac{5t^2+5t+2}{2} \ge 6$ as $t \ge 1$, a contradiction.
Type C:
In this type $e_f(1) - e_f(0) = \binom{t+1}{2} + \binom{t+1}{2} + \binom{t}{2} + 2t(t + 1) - (t + 1)^2 = \frac{5t^2+t-2}{2} \ge 2$ as $t \ge 1$, a contradiction.
Thus K_n is not 3–Quotient cordial for all $n \ge 5$ and 3–Quotient cordial labeling of K_1, K_2, K_3 and K_4 are given in Figure 1.

□

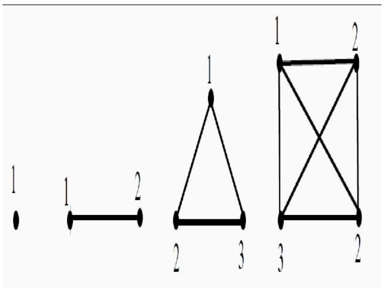

Figure 1.

The next investigation is about the 3–Quotient cordiality of paths and cycles.

Theorem 2.3. *Any path P_n is 3–Quotient cordial.*

Proof. Let P_n be the path $u_1u_2 \ldots u_n$.
Case 1. n is even.
Assign the label 1 to the vertices $u_1, u_3, u_5, \ldots, u_{\frac{n}{2}-1}$ and 2 to the vertices u_2, u_4, u_6, $\ldots$, $u_{\frac{n}{2}}$. Next assign the label 3 to the vertex $u_{\frac{n}{2}+1}$. Then assign the label 1 to the vertices $u_{\frac{n}{2}+2}, u_{\frac{n}{2}+3}, \ldots, u_{\frac{n}{2}+t+1}$ where $t = \lceil \frac{n}{3} \rceil - \lceil \frac{n}{4} \rceil$. Next we move to the other pendent vertex u_n. Assign the label 2 to the vertices $u_n, u_{n-1}, \ldots, u_{n-t}$. Finally assign the label 3 to the non labeled vertices.
Case 2. n is odd.
Assign the label to the vertices of the path $P_{n-1} : u_1u_2 \ldots u_{n-1}$ as in case 1. Next assign 1 to the vertex u_n. Finally relabel the vertex $u_{\frac{n+3}{2}}$ by 3. Clearly this vertex labeling is 3–Quotient cordial labeling. □

Corollary 2.4. *All cycles are 3–Quotient cordial.*

Proof. The 3–Quotient cordial labeling of path P_n given in Theorem 2.3 is also a 3–Quotient cordial labeling of cycle C_n. □

In the next theorem we give a necessary and sufficient condition for a wheel which admits a 3–Quotient cordial labeling.

Theorem 2.5. *The wheel W_n is 3–Quotient cordial iff $n \equiv 1 \pmod 3$.*

Proof. Let $W_n = C_n + K_1$ where C_n is the cycle $u_1u_2 \ldots u_nu_1$ and $V(K_1) = \{u\}$. Suppose f is a 3–Quotient cordial labeling of W_n.
Case 1. $n \equiv 2 \pmod 3$.
Subcase 1a. $f(u) = 3$.
Clearly all the n spokes received the label 1. Also atleast $\lfloor \frac{n}{3} \rfloor - 1$ rims get the label 1. This forces $e_f(1) \geq n + \lfloor \frac{n}{3} \rfloor - 1$, a contradiction.
Subcase 1b. $f(u) = 2$.
$n = 3t+2$. Minimum possible rims with label 1 is $t+1+1$. That is $t+2$. Also possible spokes with label 1 is $2t+1$. Thus $e_f(1) \geq 3t+3$, a contradiction.
Subcase 1c. $f(u) = 1$.
Similar to subcase 1a, we get a contradiction.
Case 2. $n \equiv 0 \pmod 3$.
Let $n = 3t$.
Subcase 2a. $f(u) = 3$.
Similar to subcase 1a.
Subcase 2b. $f(u) = 2$.
In this case $e_f(1) \geq 3t+2$, a contradiction.
Subcase 2c. $f(u) = 1$.
Similar to subcase 2b.
Case 3. $n \equiv 1 \pmod 3$.
Let $n = 3t+1$. We assign the label to the vertices as follows: Assign the label 2 to the central vertex. Next assign the labels 3 to the t vertices $u_1, u_2, \ldots, u_t$. Finally assign the labels 1 and

2 alternatively to the non labeled vertices. It is easy to verify that this vertex labeling is a 3–Quotient cordial labeling of W_n. □

The following theorem establish that all combs are 3–Quotient cordial.

Theorem 2.6. *The comb $P_n \odot K_1$ is 3–Quotient cordial.*

Proof. Let P_n be the path $u_1u_2\ldots u_n$. and v_i be the vertices adjacent to u_i $(1 \le i \le n)$. Clearly $P_n \odot K_1$ has $2n$ vertices and $2n-1$ edges.
Case 1. $n \equiv 0 \pmod 3$.
Let $n = 3t$. Assign the label 1, 2 alternatively to the vertices $u_1, u_2, \ldots, u_{n-1}$. Note that in this process the last vertex u_n of the path is received the label 1 or 2 according as $n \equiv 3 \pmod 6$ or $n \equiv 0 \pmod 6$.
Subcase 1a. $n \equiv 0 \pmod 6$.
Assign the label 3 to the vertices $v_1, v_2, \ldots, v_{2t}$. Next assign the labels 1, 2 alternatively to the remaining t vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{3t}$. Clearly in this pattern the last vertex v_n received the label 2.
Subcase 1b. $n \equiv 3 \pmod 6$.
In this case assign the label 3 to the vertices $v_1, v_2, \ldots, v_{2t-1}$ and assign the labels 2 and 3 respectively to the vertices v_{2t} and v_{3t}. Finally assign the labels 1, 2 alternatively to the remaining pendent vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{3t-1}$.
Case 2. $n \equiv 1 \pmod 3$.
As in case 1, assign the label to the vertices u_i, v_i $(1 \le i \le n-1)$. Then assign the labels 1 and 3 respectively to the vertices u_n and v_n.
Case 3. $n \equiv 2 \pmod 3$.
Assign the label to the vertices u_i, v_i $(1 \le i \le n-1)$ as in case 2. Then assign the labels 2 and 3 to the vertices u_n and v_n respectively. □

Finally we investigate the 3–Quotient cordial behavior of star, bistar and square of a path.

Theorem 2.7. *The star $K_{1,n}$ is 3–Quotient cordial iff $n \in \{1,2,3,4,5,6,7,9\}$.*

Proof. Let u be the centre of the star and u_i $(1 \le i \le n)$ be the vertices adjacent to u. The 3–Quotient cordial labeling of the star $n \in \{1,2,3,4,5,6,7,9\}$ is given in Table 1.

n	u	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	1	2								
2	1	2	3							
3	1	2	2	3						
4	1	2	2	3	3					
5	1	2	2	3	3	1				
6	1	2	2	2	3	3	1			
7	1	2	2	2	3	3	3	1		
9	1	2	2	2	2	3	3	3	1	1

Table 1.

Assume $n \notin \{1,2,3,4,5,6,7,9\}$. Suppose f is a 3–Quotient cordial labeling.
Case 1. $n \equiv 0 \pmod 3$.
Let $n = 3t$. The any one of the following types occur.
Type 1: $v_f(1) = t+1$, $v_f(2) = t$, $v_f(3) = t$.
Type 2: $v_f(1) = t$, $v_f(2) = t+1$, $v_f(3) = t$.
Type 3: $v_f(1) = t$, $v_f(2) = t$, $v_f(3) = t+1$.
Subcase 1a. $f(u) = 1$ or 2.
In this case $e_f(0) \le t+1$, a contradiction.
Subcase 1b. $f(u) = 3$.
In this case $e_f(0) = 0$ and $e_f(1) = 3t$, again a contradiction.

Case 2. $n \equiv 1 \pmod 3$.
Let $n = 3t + 1$.
Subcase 2a. $f(u) = 1$ or 2.
As in subcase 1a, $e_f(0) \leq t + 1$, a contradiction.
Subcase 2b. $f(u) = 3$.
In this case $e_f(0) = 0$, a contradiction.
Case 3. $n \equiv 2 \pmod 3$.
Let $n = 3t + 2$. As in case 2, $e_f(0) \leq t + 1$ or $e_f(0) = 0$ according as $f(u) \in \{1, 2\}$ or 3.
Thus f is not a 3–Quotient cordial labeling for all $n \notin \{1, 2, 3, 4, 5, 6, 7, 9\}$. □

Theorem 2.8. *The bistar $B_{n,n}$ is 3–Quotient cordial.*

Proof. Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$.
Assign the label 2 and 1 respectively to the vertices u, v.
Case 1. $n \equiv 0 \pmod 3$.
Let $n = 3t$. Assign the labels 1 to $u_1, u_2, \ldots, u_{2t}$ and 3 to the vertices $v_1, v_2, \ldots, v_{2t}$. Finally assign the labels 2 to the remaining non labeled vertices $u_{2t+1}, u_{2t+2}, \ldots, u_{3t}$, $v_{2t+1}, v_{2t+2}, \ldots, v_{3t}$.
Case 2. $n \equiv 1 \pmod 3$.
As in case 1, assign the label to the vertices u_i, v_i $(1 \leq i \leq n - 1)$. Then assign the labels 1, 3 to the vertices u_n and v_n respectively.
Case 3. $n \equiv 2 \pmod 3$.
Assign the label to the vertices u_i, v_i $(1 \leq i \leq n - 1)$ as in case 2. Finally assign the labels 3, 2 respectively to the vertices u_n and v_n. This vertex labeling is a 3–Quotient cordial labeling follows from Table 2.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$e_f(0)$	$e_f(1)$
$n = 3t$	$2t + 1$	$2t + 1$	$2t$	$3t + 1$	$3t$
$n = 3t + 1$	$2t + 2$	$2t + 1$	$2t + 1$	$3t + 1$	$3t$
$n = 3t + 2$	$2t + 2$	$2t + 2$	$2t + 2$	$3t + 1$	$3t$

Table 2.

□

Theorem 2.9. *The square of a path, P_n^2 is 3–Quotient cordial if and only if $n \equiv 1, 2 \pmod 3$ and $n = 3, 6$.*

Proof. Let P_n be the path $u_1 u_2 \ldots u_n$.
Case 1. $n \equiv 1 \pmod 3$.
Let $n = 3t + 1$. Assign the label 3 to the first consecutive vertices $u_1, u_2, \ldots, u_t$. Next assign the labels 1 and 2 alternatively to the remaining nonlabeled vertices $u_{t+1}, \ldots, u_{3t-2}$. Finally assign the labels 2, 2 and 1 respectively to the vertices u_{3t-1}, u_{3t}, and u_{3t+1}.
Case 2. $n \equiv 2 \pmod 3$.
Let $n = 3t + 2$. As in case 1, assign the label 3 to the vertices $u_1, u_2, \ldots, u_t$ and assign the labels 1 and 2 alternatively to the vertices $u_{t+1}, \ldots, u_{3t-1}$. Next assign the labels 2, 2, 1 to the remaining nonlabeled vertices u_{3t}, u_{3t+1}, and u_{3t+2} respectively.
Case 3. $n \equiv 0 \pmod 3$, $n \neq 3, 6$.
The maximum possible edges with label 2 occur only when 1 and 2 should be labeled alternatively to the vertices. But in this case $e_f(0) \leq e_f(1) - 2$, a contradiction.
The table 3 establish that, the labeling in case 1,2 is 3-quotient cordial labeling.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$e_f(0)$	$e_f(1)$
$n = 3t + 1$	$t + 1$	$t + 1$	t	$3t - 1$	$3t$
$n = 3t + 2$	$t + 1$	$t + 1$	t	$3t$	$3t + 1$

Table 3.

□

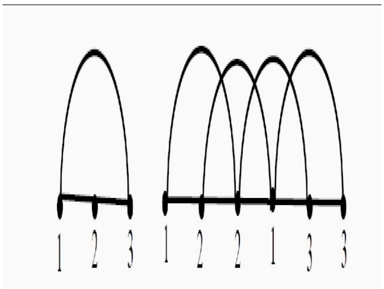

Figure 2.

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