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GENERALIZED LEGENDRE WAVELET METHOD AND ITS APPLICATIONS IN APPROXIMATION OF FUNCTIONS OF BOUNDED DERIVATIVES

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Abstract. In this paper, five new estimates $E^{(1)}_{\mu^k,0}, E^{(2)}_{\mu^k,M}, E^{(3)}_{\mu^k,0}, E^{(4)}_{\mu^k,1}\ and\ E^{(5)}_{\mu^k,M}$ of any function f on $[0,1)$ having bounded derivative are obtained by extended Legendre wavelet method. These new estimators are sharper and best possible in wavelet analysis.

1 Introduction

Approximation of function by n^{th} partial sums of its Fourier series is at common place of analysis. It is known that wavelet approximation is a new and better tool than that of the Fourier approximation. The wavelet approximations of certain functions by Haar wavelet method have been discussed by number of researcher like Devore[4], Debnath[3], Meyer[6], Morlet[2], and Lal and Kumar[5]. In this paper, an extended Legendre wavelet method is introduced. This method is a generalization of Legendre wavelet method. But till now no work seems to have been done to obtain the extended Legendre approximation of a functions having bounded first and second derivative. i.e $0 < |f^{'}(x)| < \infty$ and $0 < |f^{''}(x)| < \infty$. In an attempt to make advance study in this direction, in this paper, the wavelet approximations of the function f with $0 < |f^{'}(x)| < \infty$ as well as $0 < |f^{''}(x)| < \infty$ have been established by extended Legendre Wavelet Method. Our approximations obtain in this paper, are better and sharper in wavelet analysis to the best of our knowledge. It is observed that the estimate of a function f having bounded second derivative i.e $0 < |f^{''}(x)| < \infty$ is better and sharper than the estimate of the function f with first bounded derivative $0 < |f^{'}(x)| < \infty$.

2 Definitions and Preliminaries :

2.1 Legendre Wavelet and Extended Legendre Wavelet

Wavelets constitute a family of functions generated from translation and dilation of a single function $\psi \in L^2(\mathbb{R})$called mother wavelet. When the dilation parameter a and translation parameter b vary continuously, following family of continuous wavelets are obtained.

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi(\tfrac{x-b}{a})\ ;\ a,b \in (\mathbb{R}) \text{ and } a \neq 0.$$

If we can restrict the value of dilation and translation parameter to $a = a_0^{-n}$, $b = mb_0a_0^{-n}$, $a_0 > 1$, $b_0 > 0$ respectively. We have following family of discrete wavelets.

$$\psi_{n,m}(x) = |a_0|^{\frac{n}{2}}\psi(a_0^n x - mb_0).$$

One dimensional Legendre wavelet over the interval $[0,1)$ is defined as :

$$\psi_{n,m}(x) = \begin{cases} (m+\frac{1}{2})^{\frac{1}{2}}2^{\frac{k}{2}}L_m(2^k x - \hat{n}) \text{ for } \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k} \\ 0, \qquad\qquad\qquad\qquad otherwise, \end{cases}$$

where $n = 1, 2, \ldots, 2^{k-1}$, $\hat{n} = 2n - 1$ and $m = 0, 1, 2, \ldots$, k is a positive integer, x is normalized time and $L_m(x)$ is the Legendre polynomial of degree m over the interval [-1,1]. which is defined as follows :

$$L_0(x) = 1,\ L_1(x) = x\ ,\ L_2(x) = \frac{3x^2-1}{2}$$

and the recurrence formula for Legendre polynomial is given by

$$L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), m = 1, 2, \ldots .$$

The set $\{L_m(x)\ ;\ m = 1, 2, \ldots\}$in the Hilbert space $L^2[-1, 1]$ is a complete orthogonal set. Orthogonality of Legendre polynomial on the interval [-1,1] implies that

$$< L_m(x), L_n(x) >= \int_{-1}^{1} L_m(x)\bar{L_n}(x)\mathrm{dx} = \begin{cases} \frac{2}{2m+1}, & \text{for } m = n \\ 0, & otherwise. \end{cases}$$

Furthermore, the set of wavelets $\psi_{n,m}(x)$ makes an orthonormal basis in $L^2[0, 1)$.
i.e

$$\int_0^1 \psi_{n,m}(x)\psi_{n',m'}(x)dx = \delta_{n,n'}\delta_{m,m'},$$

in which δ denotes Kronecker delta function defined by

$$\delta_{n,n'} = \begin{cases} 1 \text{ for } n = n' \\ 0, \quad otherwise. \end{cases}$$

The function $f \in L^2[0, 1)$ is expressed in the Legendre wavelet series as

$$f(\mathrm{x}) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x),$$

where $c_{n,m} =< f(x), \psi_{n,m}(x) >$. The $(2^{k-1}, M)^{th}$ partial sum of the above series is given by

$$(S_{2^{k-1},M}f)(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m}\psi_{n,m}(x).$$

2.2 Extended Legendre Wavelet

The extended Legendre wavelet over $[0, 1)$, denoted by $\psi^{(\mu)}_{n,m}$, is defined by:

$$\psi^{(\mu)}_{n,m}(x) = \begin{cases} (m + \frac{1}{2})^{\frac{1}{2}} \mu^{\frac{k}{2}} L_m(\mu^k x - \hat{n}) \text{ for } \frac{\hat{n}-1}{\mu^k} \le x < \frac{\hat{n}+1}{\mu^k} \\ 0, \qquad otherwise. \end{cases} \tag{2.1}$$

If we take $\mu = 2$ in extended Legendre wavelet then it reduces to standard Legendre wavelet.

3 Extended Legendre Wavelet Expansion

Any function $f(x)$ defined over $[0, 1)$ can be expanded in terms of the extended Legendre wavelet as

$$f(x) \approx \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\mu)}_{n,m}(x), \tag{3.1}$$

where $c_{n,m} = < f(x), \psi^{(\mu)}_{n,m}(x) >$ and $< .,. >$ denotes the inner product on $L^2[0, 1)$. If the above infinite series is truncated, then it can be written as

$$f(x) = \sum_{n=1}^{\mu^k} \sum_{m=0}^{M} c_{n,0}\psi^{(\mu)}_{n,0}(x) = C^T\Psi(x),$$

where C and $\Psi(x)$are $\hat{m} = \mu^k(M+1)$ columns vector given by

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M} | c_{2,0}, c_{2,1}, \ldots c_{2,M} | \ldots | c_{\mu^k,0}, c_{\mu^k,1}, \ldots, c_{\mu^k,M}].$$

$$\Psi(x) = [\psi^{(\mu)}_{1,0}, \psi^{(\mu)}_{1,1}, \ldots \psi^{(\mu)}_{1,M} | \psi^{(\mu)}_{2,0}, \psi^{(\mu)}_{2,1}, \ldots, \psi^{(\mu)}_{2,M} | \ldots | \psi^{(\mu)}_{\mu^k,0}, \psi^{(\mu)}_{\mu^k,1}, \ldots \psi^{(\mu)}_{\mu^k,M}].$$

4 The Extended Legendre Wavelet Approximation

The extended Legendre wavelet approximation $E_{\mu^k,M}(f)$ of a function $f \in L^2[0,1)$ is given by

$$E_{\mu^k,M}(f) = min||f - S_{\mu^k,M}(f)||_2,$$

where $(S_{\mu^k,M}f)(x) = \sum\limits_{n=1}^{\mu^k} \sum\limits_{m=0}^{M} c_{n,m}\psi^{(\mu)}_{n,m}(x)$ the $(\mu^k, M)^{th}$ partial sum of

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\mu)}_{n,m}(x),$$

$$\text{and } ||f||_2 = \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}}.$$

If $E_{\mu^k,M}(f) \to 0$ as $k \to \infty$, $M \to \infty$ then $E_{\mu^k,M}(f)$ is called the best approximation of f of order (μ^k, M), Zygmund[1].

5 Theorems

In this paper, we prove the following Theorems

Theorem 5.1. *Let*

$$\psi^{(\mu)}_{n,0}(x) = \begin{cases} \frac{\mu^{\frac{k}{2}}}{\sqrt{2}} \; for \; \frac{\hat{n}-1}{\mu^k} \le x < \frac{\hat{n}+1}{\mu^k} \\ 0, \qquad otherwise, \end{cases}$$

$n = 1, 2, \_\_\_, (\mu^k - \mu^{k-1})$; *k is a positive integer. If a function* $f \in L^2[0,1)$ *such that its first derivative is bounded i.e* $0 < |f'(x)| < \infty \; \forall \; x \in [0,1)$ *and its extended Legendre wavelet expansion for m = 0 is written as*

$$f(x) = \sum_{n=1}^{\infty} c_{n,0}\psi^{(\mu)}_{n,0}(x). \tag{5.1}$$

Then the extended Legendre wavelet approximation of f by $(\mu^k, 0)^{th}$ partial sums $(S_{\mu^k,0}f)(x) = \sum\limits_{n=1}^{\mu^k} c_{n,0}\psi^{(\mu)}_{n,0}(x)$ *of the extended Legendre wavelet series (5.1) is given by*

$$E^{(1)}_{\mu^k,0}(f) = min||f - S_{\mu^k,0}(f)||_2 = O\left(\frac{1}{\mu^k}\right).$$

Proof. By defining error between f(x) and its expansion over a subinterval $\left[\frac{\hat{n}-1}{\mu^k}, \frac{\hat{n}+1}{\mu^k}\right)$ as:

$$e_n(x) = c_{n,0}\psi^{(\mu)}_{n,0}(x) - f(x); \; x \in \left[\frac{\hat{n}-1}{\mu^k}, \frac{\hat{n}+1}{\mu^k}\right).$$

We obtain,

$$\begin{aligned}
||e_n||^2 &= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} e_n^2(x)dx \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(c_{n,0}\psi_{n,0}^{(\mu)}(x) - f(x)\right)^2 dx \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(c_{n,0}^2\left(\psi_{n,0}^{(\mu)}(x)\right)^2 + f^2(x) - 2c_{n,0}\psi_{n,0}^{(\mu)}(x)f(x)\right) dx \\
&= c_{n,0}^2 \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(\psi_{n,0}^{(\mu)}(x)\right)^2 dx + \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx - 2c_{n,0}\int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \psi_{n,0}^{(\mu)}(x)f(x)dx \\
&= c_{n,0}^2 + \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx - 2c_{n,0}^2 \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx - c_{n,0}^2.
\end{aligned}$$

By Taylor's expansion,

$$f(x) = f\left(\frac{\hat{n}-1}{\mu^k} + h\right) = f\left(\frac{\hat{n}-1}{\mu^k}\right) + hf'\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right); 0 \le h < \frac{2}{\mu^k}, 0 < \theta < 1.$$

Then, $$||e_n||^2 = \int_0^{\frac{2}{\mu^k}} \left(f\left(\frac{\hat{n}-1}{\mu^k}\right) + hf'\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right)\right)^2 dh - c_{n,0}^2 = I_1 - c_{n,0}^2. \tag{5.2}$$

Next,

$$\begin{aligned}
c_{n,0} &= < f(x), \psi_{n,0}^{(\mu)}(x) > \\
&= \int_0^1 f(x)\psi_{n,0}^{(\mu)}(x)dx \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f(x)\frac{\mu^{\frac{k}{2}}}{\sqrt{2}}dx \\
&= \frac{\mu^{\frac{k}{2}}}{\sqrt{2}} \int_o^{\frac{2}{\mu^k}} \left(f\left(\frac{\hat{n}-1}{\mu^k}\right) + hf'\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right)\right) dh \\
&= \frac{\mu^{\frac{k}{2}}}{\sqrt{2}} \left[\frac{2}{\mu^k} f\left(\frac{\hat{n}-1}{\mu^k}\right) + \int_0^{\frac{2}{\mu^k}} hf'\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right) dh\right].
\end{aligned}$$

Now,

$$\begin{aligned} c_{n,0}^2 &= \frac{\mu^k}{2}\left[\frac{4}{\mu^{2k}} f^2\left(\frac{\hat{n}-1}{\mu^k}\right) + \left(\int_0^{\frac{2}{\mu^k}} h f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh\right)^2\right] \\ &\quad + 2f\left(\frac{\hat{n}-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} h f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh \\ &= \frac{2}{\mu^k} f^2\left(\frac{\hat{n}-1}{\mu^k}\right) + \frac{\mu^k}{2}\left(\int_0^{\frac{2}{\mu^k}} h f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh\right)^2 \\ &\quad + 2f\left(\frac{\hat{n}-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} h f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh. \end{aligned} \tag{5.3}$$

$$\begin{aligned} I_1 &= \int_0^{\frac{2}{\mu^k}} \left[f^2\left(\frac{\hat{n}-1}{\mu^k}\right) + h^2\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2 + 2hf\left(\frac{\hat{n}-1}{\mu^k}\right) f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right] dh \\ &= \left[\frac{2}{\mu^k} f^2\left(\frac{\hat{n}-1}{\mu^k}\right)\right] + \int_0^{\frac{2}{\mu^k}} h^2\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2 dh \\ &\quad + 2f\left(\frac{\hat{n}-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} h f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh. \end{aligned} \tag{5.4}$$

By equations (5.2) to (5.4), we have

$$\begin{aligned} ||e_n||_2^2 &= \int_0^{\frac{2}{\mu^k}} h^2\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2 dh - \frac{\mu^k}{2}\left(\int_0^{\frac{2}{\mu^k}} h f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh\right)^2 \\ &\le \int_0^{\frac{2}{\mu^k}} h^2\left|\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2\right| dh + \frac{\mu^k}{2}\left(\int_0^{\frac{2}{\mu^k}} h\left|f^{'}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right| dh\right)^2 \\ &\le M_1^2\int_0^{\frac{2}{\mu^k}} h^2 dh + M_1^2\frac{\mu^k}{2}\left(\int_0^{\frac{2}{\mu^k}} h dh\right)^2 \quad ; \text{ since } f^{'} \text{ bounded} \\ &= M_1^2\left(\frac{8}{3\mu^{3k}} + \frac{4\mu^k}{2\mu^{4k}}\right) \\ &= \frac{M_1^2}{\mu^{3k}}\left(\frac{8}{3}+2\right) = \frac{14M_1^2}{3\mu^{3k}}. \end{aligned} \tag{5.5}$$

Lastly,

$$\begin{aligned}(E^{(1)}_{\mu^k,0}(f))^2 &= \int_0^1 \left(f(x)-(S_{\mu^k,0}f)(x)\right)^2 dx\\ &= \int_0^1 \left(\sum_{n=1}^{\mu^k}(c_{n,0}\psi^{(\mu)}_{n,0}(x)-f(x)\right)^2 dx\\ &= \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n(x)\right)^2 dx\\ &= \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n^2(x)\right) dx + 2\sum\sum_{n<\hat{n}}\int_0^1 e_n(x)e_{\hat{n}}(x)dx.\end{aligned}$$

Now, due to disjointness of the supports of these basis functions we have:

$$\begin{aligned}(E^{(1)}_{\mu^k,0}(f))^2 &= \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n^2(x)\right) dx\\ &= \sum_{n=1}^{\mu^k} ||e_n(x)||_2^2. \qquad (5.6)\end{aligned}$$

Substituting (5.5) into (5.6),we obtain:

$$\begin{aligned}(E^{(1)}_{\mu^k,0}(f))^2 &\leq \sum_{n=1}^{\mu^k}\frac{14M_1^2}{3\mu^{3k}} = \frac{14M_1^2}{3\mu^{2k}}.\\ \text{Therefore, } E^{(1)}_{\mu^k,0}(f) &\leq \sqrt{\frac{14}{3}}\frac{M_1}{\mu^k}.\\ \text{Hence, } E^{(1)}_{\mu^k,0}(f) &= O\left(\frac{1}{\mu^k}\right).\end{aligned}$$

Theorem 5.2. *If $f \in L^2[0,1)$, $0 < |f^{'}(x)| < \infty$ and its extended Legendre wavelet expansion is $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\mu)}_{n,m}(x)$. Then extended Legendre wavelet approximation of f by $(\mu^k, M)^{th}$ partial sums $(S_{\mu^k,M}f)(x) = \sum_{n=1}^{\mu^k}\sum_{m=0}^{M} c_{n,m}\psi^{(\mu)}_{n,m}(x)$ under the norm $||\ ||_2$ satisfies*

$$E^{(2)}_{\mu^k,M}(f) = min||f - S_{\mu^k,M}(f)||_2 = O\left(\frac{1}{\mu^k(2M+1)^{\frac{1}{2}}}\right) \text{; for } M \geq 0.$$

Proof.
$$\begin{aligned}c_{n,m} &= \int_0^1 f(x)\psi^{(\mu)}_{n,m}(x)dx\\ &= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f(x)\sqrt{\frac{2m+1}{2}}\mu^{\frac{k}{2}}L_m(\mu^k x-2n+1)dx, \quad \text{taking } t=\mu^k x-2n+1\\ &= \int_{-1}^{1} f\left(\frac{t+2n-1}{\mu^k}\right)\sqrt{\frac{2m+1}{2}}\mu^{\frac{k}{2}}L_m(t)\frac{dt}{\mu^k}\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{2m+1}{2\mu^k}} \int_{-1}^{1} f\left(\frac{t+2n-1}{\mu^k}\right) L_m(t)dt, \text{ since } L_m(t) = \frac{L'_{m+1}(t) - L'_{m-1}(t)}{2m+1},\ m \geq 1 \\
&= \frac{1}{\sqrt{2(2m+1)\mu^k}} \int_{-1}^{1} f\left(\frac{t+2n-1}{\mu^k}\right) d(L_{m+1}(t) - L_{m-1}(t))dt \\
&= \frac{1}{\sqrt{2(2m+1)\mu^k}} \left[f\left(\frac{t+2n-1}{\mu^k}\right)(L_{m+1}(t) - L_{m-1}(t))\right]_{-1}^{1} \\
&\quad - \frac{1}{\sqrt{2(2m+1)\mu^k}} \left[\int_{-1}^{1} f'\left(\frac{t+2n-1}{\mu^k}\right)\frac{1}{\mu^k}(L_{m+1}(t) - L_{m-1}(t))dt\right], \\
&\qquad \text{integrating by parts} \\
&= \frac{1}{\sqrt{2(2m+1)\mu^k}} \left[f\left(\frac{2n}{\mu^k}\right)(1-1) - f\left(\frac{2(n-1)}{\mu^k}\right)((-1)^{m+1} - (-1)^{m-1})\right] \\
&\quad - \frac{1}{\sqrt{2(2m+1)\mu^k}} \left[\frac{1}{\mu^k}\int_{-1}^{1} f'\left(\frac{t+2n-1}{\mu^k}\right)(L_{m+1}(t) - L_{m-1}(t))dt\right] \\
&= \frac{-1}{\sqrt{2(2m+1)\mu^{3k}}} \int_{-1}^{1} f'\left(\frac{t+2n-1}{\mu^k}\right)(L_{m+1}(t) - L_{m-1}(t))\,dt.
\end{aligned}
$$

$$
\begin{aligned}
Then,\ |c_{n,m}| &\leq \frac{1}{\sqrt{2(2m+1)\mu^{3k}}} \\
&\quad \times \int_{-1}^{1} \left|f'\left(\frac{t+2n-1}{\mu^k}\right)\right| |(L_{m+1}(t) - L_{m-1}(t))|\,dt\ , 0 < \left|f'\left(\frac{t+2n-1}{\mu^k}\right)\right| \leq M_1 \\
&\leq \frac{M_1}{\sqrt{2(2m+1)\mu^{3k}}} \int_{-1}^{1} |L_{m+1}(t) - L_{m-1}(t)|\,dt \\
&\leq \frac{M_1}{\sqrt{2(2m+1)\mu^{3k}}} \left(\int_{-1}^{1} (1)^2 dt\right)^{\frac{1}{2}} \left(\int_{-1}^{1} |L_{m+1}(t) - L_{m-1}(t)|^2\,dt\right)^{\frac{1}{2}}, \\
&\qquad \text{, by Holders inequality} \\
&\leq \frac{\sqrt{2}M_1}{\sqrt{2(2m+1)\mu^{3k}}} \left(\int_{-1}^{1} (L^2_{m+1}(t) + L^2_{m-1}(t) - 2L_{m+1}(t)L_{m-1}(t))dt\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{2}M_1}{\sqrt{2(2m+1)\mu^{3k}}} \left(\int_{-1}^{1} L^2_{m+1}(t)dt + \int_{-1}^{1} L^2_{m-1}(t)dt - 2\int_{-1}^{1} L_{m+1}(t)L_{m-1}(t)dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

By orthogonal property of Legendre polynomials,

$$
\begin{aligned}
|c_{n,m}| &\leq \frac{\sqrt{2}M_1}{\sqrt{2(2m+1)\mu^{3k}}} \left(\frac{2}{2m+3} + \frac{2}{2m-1}\right)^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}M_1}{\sqrt{2(2m+1)\mu^{3k}}} \left(\frac{2}{2m-1} + \frac{2}{2m-1}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{2}M_1}{\sqrt{2(2m+1)\mu^{3k}}}\left(\frac{4}{2m-1}\right)^{\frac{1}{2}} \\
&= \frac{2\sqrt{2}M_1}{\sqrt{2(2m+1)(2m-1)\mu^{3k}}} \\
&\leq \frac{2\sqrt{2}M_1}{\sqrt{2(2m-1)(2m-1)\mu^{3k}}} \\
&= \frac{2M_1}{(2m-1)\mu^{\frac{3k}{2}}}.
\end{aligned}
$$

$$
\text{Thus,}\quad |c_{n,m}|^2 \leq \frac{4M_1^2}{(2m-1)^2\mu^{3k}},\quad m\geq 1. \tag{5.7}
$$

$$
\begin{aligned}
f(x)-(S_{\mu^k,M}f)(x) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\mu)}(x) - \sum_{n=1}^{\mu^k}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}^{(\mu)}(x) \\
&= \sum_{n=1}^{\mu^k}\left(\sum_{m=0}^{M} + \sum_{m=M+1}^{\infty}\right) c_{n,m}\psi_{n,m}^{(\mu)}(x) \\
&\quad -\sum_{n=1}^{\mu^k}\sum_{m=0}^{M} c_{n,m}\psi_{n,m}^{(\mu)}(x), \quad \text{by definition of } \psi_{n,m}^{(\mu)}(x) \\
&= \sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}^{(\mu)}(x). \\
\left(f(x)-(S_{\mu^k,M}f)(x)\right)^2 &= \left(\sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}\psi_{n,m}^{(\mu)}(x)\right)^2 \\
&= \sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}^2\left(\psi_{n,m}^{(\mu)}(x)\right)^2 + \sum_{n=1}^{\mu^k}\sum_{M+1\leq m\neq}\sum_{\hat{m}\leq\infty} c_{n,m}c_{n,\hat{m}}\psi_{n,m}^{(\mu)}(x)\psi_{n,\hat{m}}^{(\mu)}(x) \\
&\quad + \sum_{1\leq n\neq}\sum_{\hat{n}\leq\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}c_{\hat{n},m}\psi_{n,m}^{(\mu)}(x)\psi_{\hat{n},m}^{(\mu)}(x) \\
&\quad + \sum_{1\leq n\neq}\sum_{\leq\mu^k}\sum_{M+1\leq m}\sum_{\neq m\leq\infty} c_{n,m}c_{\hat{n},\hat{m}}\psi_{n,m}^{(\mu)}(x)\psi_{\hat{n},\hat{m}}^{(\mu)}(x). \\
\left(E_{\mu^k,M}^{(2)}(f)\right)^2 &= \int_0^1 \left(f(x)-(S_{\mu^k,M}f)(x)\right)^2 dx \\
&= \sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}^2\left(\int_0^1 \psi_{n,m}^{(\mu)}(x)dx\right)^2 \\
&\quad +\sum_{n=1}^{\mu^k}\sum_{M+1\leq m}\sum_{\neq\hat{m}\leq\infty} c_{n,m}c_{n,\hat{m}}\int_0^1\left(\psi_{n,m}^{(\mu)}(x)\psi_{n,\hat{m}}^{(\mu)}(x)\right)dx \\
&\quad + \sum_{1\leq n\neq}\sum_{\hat{n}\leq\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}c_{\hat{n},m}\int_0^1\left(\psi_{n,m}^{(\mu)}(x)\psi_{\hat{n},m}^{(\mu)}(x)\right)dx
\end{aligned}
$$

$$+\sum_{1\leq n\neq}\sum_{\hat{n}\leq\mu^k}\sum_{M+1\leq m}\sum_{\neq m\leq\infty} c_{n,m}c_{\hat{n},\hat{m}}\int_0^1\left(\psi^{(\mu)}_{n,m}(x)\psi^{(\mu)}_{\hat{n},\hat{m}}(x)\right)dx.$$

Since $||\psi^{(\mu)}_{n,m}||_2^2 = 1$ and other terms vanish by orthogonality of $\psi^{(\mu)}_{n,m}$

$$\begin{aligned}
(E^{(2)}_{\mu^k,M})^2 &= (f(x)-(S_{\mu^k,M}f)(x))^2\\
&= \left(\sum_{n=1}^{\mu^k}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\mu)}_{n,m}(x)-\sum_{n=1}^{\mu^k}\sum_{m=0}^{M} c_{n,m}\psi^{(\mu)}_{n,m}(x)\right)^2\\
&= \left(\sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty} c_{n,m}\psi^{(\mu)}_{n,m}(x)\right)^2\\
||E^{(2)}_{\mu^k,M}(f)||^2 &= \sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty}|c_{n,m}|^2. \qquad (5.8)
\end{aligned}$$

From equation (5.7) and (5.8) we have,

$$\begin{aligned}
||E^{(2)}_{\mu^k,M}(f)||_2^2 &\leq \sum_{n=1}^{\mu^k}\sum_{m=M+1}^{\infty}\left(\frac{4M_1^2}{(2m-1)^2\mu^{3k}}\right)\\
&= \sum_{n=1}^{\mu^k}\left(\frac{4M_1^2}{\mu^{3k}}\right)\sum_{m=M+1}^{\infty}(2m-1)^{-2}\\
&= \frac{4M_1^2}{\mu^{2k}}\int_{M+1}^{\infty}(2m-1)^{-2}dm\\
&= \frac{4M_1^2}{\mu^{2k}}\left[\frac{(2m-1)^{-1}}{-2}\right]_{M+1}^{\infty}\\
&= \frac{4M_1^2}{2(2M+1)\mu^{2k}}\\
&= \frac{2M_1^2}{\mu^{2k}(2M+1)}\\
||E^{(2)}_{\mu^k,M}(f)|| &\leq \frac{\sqrt{2}M_1}{\mu^k(2M+1)^{\frac{1}{2}}}.\\
\text{Hence,}\quad ||E^{(2)}_{\mu^k,M}(f)|| &= O\left(\frac{1}{\mu^k(2M+1)^{\frac{1}{2}}}\right);\ \text{ for }\ M\geq 0.
\end{aligned}$$

Theorem 5.3. *Let $\mathbb{R}$ be the set of all real number. If $f:[0,1)\to\mathbb{R}$ is a real - valued function such that $0<|f''(x)|<\infty$ and $f(x)=\sum_{n=1}^{\infty}c_{n,0}\psi^{(\mu)}_{n,0}(x)$, for m=0. Then the extended Legendre wavelet approximation $E^{(3)}_{\mu^k,0}(f)$ of f by $(\mu^k,0)^{th}$ partial sums $(S_{\mu^k,0}f)(x)=\sum_{n=1}^{\mu^k}c_{n,0}\psi^{(\mu)}_{n,0}(x)$ is estimated as*

$$E^{(3)}_{\mu^k,0}(f)=min||f-S_{\mu^k,0}(f)||_2=O\left(\tfrac{1}{\mu^k}(1+\tfrac{1}{\mu^k})\right).$$

Proof. Following the proof of Theorem 5.1, we have

$$\begin{aligned}
||e_n||_2^2 &= \int\limits_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx - c_{n,0}^2. \qquad (5.9)\\
c_{n,0} &= < f(x), \psi_{n,0}^{(\mu)}(x) >\\
&= \int\limits_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f(x)\frac{\mu^{\frac{k}{2}}}{\sqrt{2}}dx\\
&= \frac{\mu^{\frac{k}{2}}}{\sqrt{2}}\int\limits_{0}^{\frac{2}{\mu^k}}\left(f\left(\frac{\hat{n}-1}{\mu^k}\right)+hf^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)+\frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)dh \ \ ,x=\left(\frac{\hat{n}-1}{\mu^k}\right)+h\\
&= \frac{\mu^{\frac{k}{2}}}{\sqrt{2}}\left[\left[\frac{2}{\mu^k}f\left(\frac{\hat{n}-1}{\mu^k}\right)+\frac{2}{\mu^{2k}}f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right]+\int\limits_{o}^{\frac{2}{\mu^k}}\frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{\mu}+\theta h\right)dh\right]\\
c_{n,0}^2 &= \frac{2}{\mu^k}f^2\left(\frac{\hat{n}-1}{\mu^k}\right)+\frac{2}{\mu^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right)^2+\frac{4}{\mu^{2k}}f\left(\frac{\hat{n}-1}{\mu^k}\right)f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\\
&\quad+\frac{\mu^k}{2}\left(\int\limits_{0}^{\frac{2}{\mu^k}}\frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{\mu^k}\right)dh\right)^2\\
&\quad+\left[f\left(\frac{\hat{n}-1}{\mu^k}\right)+\frac{1}{\mu^k}f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right]\int\limits_{0}^{\frac{2}{\mu^k}}h^2f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh. \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
\int\limits_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx &= \int\limits_{0}^{\frac{2}{\mu^k}}\left(f^2\left(\frac{\hat{n}-1}{\mu^k}\right)+h^2\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right)^2+\frac{h^4}{4}\left(f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2\right)dh\\
&\quad+\int\limits_{0}^{\frac{2}{\mu^k}}2hf\left(\frac{\hat{n}-1}{\mu^k}\right)f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)dh\\
&\quad+\int\limits_{0}^{\frac{2}{\mu^k}}\left(h^3f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)+h^2f\left(\frac{\hat{n}-1}{\mu^k}\right)f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)dh\\
&= \left[\frac{2}{\mu^k}f^2\left(\frac{\hat{n}-1}{\mu^k}\right)+\frac{8}{3\mu^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right)^2+\frac{4}{\mu^{2k}}f\left(\frac{\hat{n}-1}{\mu^k}\right)f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right]\\
&\quad+\int\limits_{0}^{\frac{2}{\mu^k}}\frac{h^4}{4}\left(f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2dh+\int\limits_{0}^{\frac{2}{\mu^k}}h^3f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh\\
&\quad+f\left(\frac{\hat{n}-1}{\mu^k}\right)\int\limits_{0}^{\frac{2}{\mu^k}}h^2f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh. \qquad (5.11)
\end{aligned}$$

Using equation (5.9), (5.10) and (5.11),we have

$$\begin{aligned}
||e_n||_2^2 &= \frac{2}{3\mu^{3k}}\left(f'\left(\frac{\hat{n}-1}{\mu^k}\right)\right)^2 + \int_0^{\frac{2}{\mu^k}} \frac{h^4}{4}\left(f''\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2 dh \\
&\quad - \frac{\mu^k}{2}\left(\int_0^{\frac{2}{\mu^k}} \frac{h^2}{2} f''\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh\right)^2 \\
&\quad + f'\left(\frac{\hat{n}-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} \left(h^3 - \frac{h^2}{\mu^k}\right) f''\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh \\
&\leq \frac{2M_1^2}{3\mu^{3k}} + \frac{32M_2^2}{20\mu^{5k}} + \frac{64M_2^2}{72\mu^{5k}} + M_1M_2\left[\frac{4}{\mu^{4k}} - \frac{8}{3\mu^{4k}}\right] \\
&= \frac{2}{3\mu^{3k}}\left(M_1^2 + \frac{56}{15}\frac{M_2^2}{\mu^{2k}} + \frac{2M_1M_2}{\mu^k}\right) \\
&\leq \frac{2}{3\mu^{3k}}\left(M_1^2 + \frac{4M_2^2}{\mu^{2k}} + \frac{4M_1M_2}{\mu^k}\right) \\
&= \frac{2}{3\mu^{3k}}\left(M_1 + \frac{2M_2}{\mu^k}\right)^2 .
\end{aligned}$$

$$\begin{aligned}
\text{Next, } \left(E^{(3)}_{\mu^k,0}(f)\right)^2 &= \sum_{n=1}^{\mu^k} ||e_n(x)||^2. \\
&\leq \sum_{n=1}^{\mu^k}\left(\frac{2}{3\mu^{3k}}\left(M_1 + \frac{2M_2}{\mu^k}\right)^2\right) \\
&= \frac{2}{3\mu^{2k}}\left(M_1 + \frac{2M_2}{\mu^k}\right)^2 \\
&\leq \frac{2}{3\mu^{2k}}(M_1+2M_2)^2\left(1+\left(\frac{1}{\mu^k}\right)\right)^2 . \\
\text{Hence, } E^{(3)}_{\mu^k,0}(f) &= O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^k}\right)\right).
\end{aligned}$$

Theorem 5.4. *If the second derivative of f is bounded i.e $0 < |f''(x)| < \infty$ and $f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{1} c_{n,m}\psi^{(\mu)}_{n,m}(x)$, then extended Legendre wavelet approximation $E^{(4)}_{\mu^k,1}$ of f by $(\mu^k,1)^{th}$ partial sums $(S_{\mu^k,1}f)(x) = \sum_{n=1}^{\mu^k}\sum_{m=0}^{1} c_{n,m}\psi^{(\mu)}_{n,m}(x)$ is obtained as*

$$E^{(4)}_{\mu^k,1}(f) = min||f - S_{\mu^k,1}(f)||_2 = O\left(\tfrac{1}{\mu^{2k}}\right).$$

Proof. By defining error between f(x) and its expansion over any subinterval as:

$$e_n(x) = c_{n,0}\psi^{(\mu)}_{n,0}(x) + c_{n,1}\psi^{(\mu)}_{n,1}(x) - f(x); \;\; x \in \left[\frac{\hat{n}-1}{\mu^k}, \frac{\hat{n}+1}{\mu^k}\right).$$

We obtain,

$$
\begin{aligned}
||e_n||_2^2 &= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} e_n^2(x)dx \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(c_{n,0}\psi_{n,0}^{(\mu)}(x) + c_{n,1}\psi_{n,1}^{(\mu)}(x) - f(x)\right)^2 dx \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(c_{n,0}^2\left(\psi_{n,0}^{(\mu)}(x)\right)^2 + c_{n,1}^2\left(\psi_{n,1}^{(\mu)}(x)\right)^2 + f^2(x) + 2c_{n,0}\psi_{n,0}^{(\mu)}(x)c_{n,1}\psi_{n,1}^{(\mu)}(x)\right) dx \\
&\quad - \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(2c_{n,0}\psi_{n,0}^{(\mu)}(x)f(x) + 2c_{n,1}\psi_{n,1}^{(\mu)}(x)f(x)\right) dx \\
&= c_{n,0}^2 \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(\psi_{n,0}^{(\mu)}(x)\right)^2 dx + c_{n,1}^2 \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \left(\psi_{n,1}^{(\mu)}(x)\right)^2 dx + \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx \\
&\quad + 2c_{n,0}c_{n,1} \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \psi_{n,0}^{(\mu)}(x)\psi_{n,1}^{(\mu)}(x)dx \\
&\quad - 2c_{n,0} \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} \psi_{n,0}^{(\mu)}(x)f(x)dx - 2c_{n,1} \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}-1}{\mu^k}} \psi_{n,1}^{(\mu)}(x)f(x)dx \\
&= c_{n,0}^2 + c_{n,1}^2 + \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx - 2c_{n,0}^2 - 2c_{n,1}^2 \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f^2(x)dx - c_{n,0}^2 - c_{n,1}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
||e_n||_2^2 &= \int_0^{\frac{2}{\mu^k}} \left(f\left(\frac{\hat{n}-1}{\mu^k}\right) + hf^{'}\left(\frac{\hat{n}-1}{\mu^k}\right) + \frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{\mu^k} + \theta h\right)\right)^2 dh \\
&\quad - c_{n,0}^2 - c_{n,1}^2. \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
c_{n,1} &= < f(x), \psi_{n,1}^{(\mu)}(x) > \\
&= \int_0^1 f(x)\psi_{n,1}^{(\mu)}(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f(x)\sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} L_1(\mu^k x - 2n + 1)dx \\
&= \int_{\frac{\hat{n}-1}{\mu^k}}^{\frac{\hat{n}+1}{\mu^k}} f(x)\sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} (\mu^k x - 2n + 1)dx \\
&= \int_0^{\frac{2}{\mu^k}} \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \left(\mu^k\left(\frac{\hat{n}-1}{\mu^k}+h\right) - 2n + 1\right)\left(f\left(\frac{\hat{n}-1}{\mu^k}\right) + hf^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right)dh \\
&\quad + \int_0^{\frac{2}{\mu^k}} \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \left(\mu^k\left(\frac{\hat{n}-1}{\mu^k}+h\right) - 2n + 1\right)\frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh \\
&= \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \int_0^{\frac{2}{\mu^k}} (\mu^k h - 1)\left(f\left(\frac{\hat{n}-1}{\mu^k}\right) + hf^{'}\left(\frac{\hat{n}-1}{\mu^k}\right) + \frac{h^2}{2}f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)dh \\
&= \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \left[\left(\frac{\mu^k h^2}{2} - h\right)f\left(\frac{\hat{n}-1}{\mu^k}\right) + \left(\frac{\mu^k h^3}{3} - \frac{h^2}{2}\right)f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right]_0^{\frac{2}{\mu^k}} \\
&\quad + \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \int_0^{\frac{2}{\mu^k}} \frac{(\mu^k h^3 - h^2)}{2} f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh \\
&= \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \left[\left(\frac{2}{\mu^k} - \frac{2}{\mu^k}\right)f\left(\frac{\hat{n}-1}{\mu^k}\right) + \left(\frac{8}{3\mu^{2k}} - \frac{4}{2\mu^{2k}}\right)f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right] \\
&\quad + \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \int_0^{\frac{2}{\mu^k}} \frac{(\mu^k h^3 - h^2)}{2} f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh \\
&= \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \left(\frac{2}{3\mu^{2k}} f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right) + \sqrt{\frac{3}{2}}\mu^{\frac{k}{2}} \int_0^{\frac{2}{\mu^k}} \frac{(\mu^k h^3 - h^2)}{2} f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh
\end{aligned}
$$

$$
\begin{aligned}
c_{n,1}^2 &= \frac{2}{3\mu^{3k}}\left(f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\right)^2 + \frac{3\mu^k}{8}\left(\int_0^{\frac{2}{\mu^k}} (\mu^k h^3 - h^2) f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh\right)^2 \\
&\quad + \frac{1}{\mu^k}f^{'}\left(\frac{\hat{n}-1}{\mu^k}\right)\int_0^{\frac{2}{\mu^k}} (\mu^k h^3 - h^2) f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)dh, \qquad (5.13)
\end{aligned}
$$

and $c_{n,0}^2$ is given in (5.10),we have

$$
||e_n||_2^2 = \int_0^{\frac{2}{\mu^k}} \frac{h^4}{4}\left(f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right)\right)^2 dh
$$

$$
\begin{aligned}
& - \frac{\mu^k}{8}\left(\int_0^{\frac{2}{\mu^k}} h^2 f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh\right)^2 \\
& - \frac{3\mu^k}{8}\left(\int_0^{\frac{2}{\mu^k}} \left(\mu^k h^3 - h^2\right) f^{''}\left(\frac{\hat{n}-1}{\mu^k}+\theta h\right) dh\right)^2 \\
\leq \;& M_2^2\left(\frac{32}{20\mu_{5k}}+\frac{64}{72\mu^{5k}}+\frac{3\mu^k}{8}\left[\frac{4}{\mu^{3k}}-\frac{8}{3\mu^{3k}}\right]^2\right) \\
= \;& \frac{142M_2^2}{45\mu^{5k}}.
\end{aligned}
$$

$$
\begin{aligned}
\left(E^{(4)}_{\mu^k,1}(f)\right)^2 &= \sum_{n=1}^{\mu^k} ||e_n(x)||_2^2 \\
&\leq \sum_{n=1}^{\mu^k} \frac{142M_2^2}{45\mu^{5k}} \\
&= \frac{142M_2^2}{45\mu^{4k}}. \\
Hence,\;\; E^{(4)}_{\mu^k,1}(f) &= O\left(\frac{1}{\mu^{2k}}\right).
\end{aligned}
$$

Theorem 5.5. *If* $0<|f^{''}(x)|<\infty\ \forall x\in[0,1),\ f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi^{(\mu)}_{n,m}(x)$ *and* $(S_{\mu^k,M}f)(x)=\sum_{n=1}^{\mu^k}\sum_{m=0}^{M} c_{n,m}\psi^{(\mu)}_{n,m}(x)$, *then extended Legendre wavelet approximation* $E^{(5)}_{\mu^k,M}(f)$ *of* f *by* $(S_{\mu^k,M}f)(x)$ *is calculated as*

$$
E^{(5)}_{\mu^k,M}(f) = min||f-S_{\mu^k,M}(f)||_2 = O\left(\frac{1}{\mu^{2k}(2M-1)^{\frac{3}{2}}}\right); \textit{for } M\geq 1.
$$

Proof. Following the proof of Theorem 5.2, we have

$$
\begin{aligned}
c_{n,m} &= \frac{-1}{\sqrt{2(2m+1)\mu^{3k}}}\int_{-1}^{1} f^{'}\left(\frac{t+2n-1}{\mu^k}\right)\left(L_{m+1}(t)-L_{m-1}(t)\right)dt,\ m\geq 1 \\
&= \frac{-1}{\sqrt{2(2m+1)\mu^{3k}}} \\
&\quad \times \int_{-1}^{1} f^{'}\left(\frac{t+2n-1}{\mu^k}\right)\left(\left(\frac{L^{'}_{m+2}(t)-L^{'}_{m}(t)}{2m+3}\right)-\left(\frac{L^{'}_{m}(t)-L^{'}_{m-2}(t)}{2m-1}\right)\right)dt,\ for\ m\geq 2 \\
&= \frac{-1}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{3k}}} \\
&\quad \times \left[\int_{-1}^{1} f^{'}\left(\frac{t+2n-1}{\mu^k}\right)\left((2m-1)L^{'}_{m+2}(t)-2(2m+1)L^{'}_{m}(t)+(2m+3)L^{'}_{m-2}(t)\right)dt\right], \\
&= \frac{1}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}}
\end{aligned}
$$

$$\times \left[\int_{-1}^{1} f'' \left(\frac{t+2n-1}{\mu^k} \right) ((2m-1)L_{m+2}(t) - 2(2m+1)L_m(t) + (2m+3)L_{m-2}(t)) \, dt \right]$$

integrating by parts.

$$
\begin{aligned}
\text{Then, } |c_{n,m}| &\leq \frac{1}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} \\
&\quad \times \int_{-1}^{1} \left| f'' \left(\frac{t+2n-1}{\mu^k} \right) \right| |(2m-1)L_{m+2}(t) - 2(2m+1)L_m(t) + (2m+3)L_{m-2}(t)| \, dt \\
&\leq \frac{M_2}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} \\
&\quad \times \int_{-1}^{1} |(2m-1)L_{m+2}(t) - 2(2m+1)L_m(t) + (2m+3)L_{m-2}(t)| \, dt \\
&\leq \frac{M_2}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} \\
&\quad \times \left(\int_{-1}^{1} (1)^2 dt \right)^{\frac{1}{2}} \left(\int_{-1}^{1} ((2m-1)L_{m+2}(t) - 2(2m+1)L_m(t) + (2m+3)L_{m-2}(t))^2 dt \right)^{\frac{1}{2}} \\
&= \frac{M_2}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} \\
&\quad \times \sqrt{2} \left(\int_{-1}^{1} \left((2m-1)^2 L_{m+2}^2(t) + 4(2m+1)^2 L_m^2(t) + (2m+3)^2 L_{m-2}^2(t) \right) dt \right)^{\frac{1}{2}} \\
&= \frac{M_2}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} \\
&\quad \times \sqrt{2} \left((2m-1)^2 \frac{2}{2m+5} + 4(2m+1)^2 \frac{2}{2m+1} + (2m+3)^2 \frac{2}{2m-3} \right)^{\frac{1}{2}}, \\
&\qquad \text{by orthogonality property} \\
&\leq \frac{M_2}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} \\
&\quad \times \sqrt{2} \left((2m+3)^2 \frac{2}{2m-3} + 4(2m+3)^2 \frac{2}{2m-3} + (2m+3)^2 \frac{2}{2m-3} \right)^{\frac{1}{2}} \\
&= \frac{M_2}{(2m+3)(2m-1)\sqrt{2(2m+1)\mu^{5k}}} (2m+3) \sqrt{\frac{24}{(2m-3)^{\frac{1}{2}}}} \\
&\leq \frac{\sqrt{12} M_2}{(2m-3)^2 \mu^{\frac{5k}{2}}} \\
|c_{n,m}|^2 &\leq \frac{12 M_2^2}{\mu^{5k}(2m-3)^4}, \quad m \geq 2. \\
(E^{(5)}_{\mu^k,M}(f))^2 &= \sum_{n=1}^{\mu^k} \sum_{m=M+1}^{\infty} |c_{n,m}|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\mu^k} \sum_{m=M+1}^{\infty} \frac{12M_2^2}{\mu^{5k}(2m-3)^4} \\
&= \frac{12M_2^2}{\mu^{4k}} \sum_{m=M+1}^{\infty} (2m-3)^{-4} \\
&= \frac{12M_2^2}{\mu^{4k}} \int_{M+1}^{\infty} (2m-3)^{-4} dm \\
&= \frac{2M_2^2}{\mu^{4k}(2M-1)^3} \\
E^{(5)}_{\mu^k,M}(f) &\leq \frac{\sqrt{2}M_2}{\mu^{2k}(2M-1)^{\frac{3}{2}}}; for \;\; M \geq 1 \\
E^{(5)}_{\mu^k,M}(f) &= O\left(\frac{1}{\mu^{2k}(2M-1)^{\frac{3}{2}}}\right); \;\; for M \geq 1.
\end{aligned}
$$

Following corollary is deduced from our Theorems (5.1)

Corollary 5.6. *Let*

$$
\psi_{n,0}(x) = \begin{cases} \frac{2^{\frac{k}{2}}}{\sqrt{2}} \; for \;\; \frac{\hat{n}-1}{2^k} \leq x < \frac{\hat{n}+1}{2^k} \\ 0, \qquad otherwise, \end{cases}
$$

$n = 1, 2, \_\_\_, 2^{(k-1)}$; *k is a positive integer. If a function* $f \in L^2[0,1)$ *such that its first derivative is bounded i.e* $0 < |f^{'}(x)| < \infty \; \forall \; x \in [0,1)$ *and its the Legendre wavelet expansion for m = 0 is written as*

$$
f(x) = \sum_{n=1}^{\infty} c_{n,0}\psi_{n,0}(x). \tag{5.14}
$$

Then the Legendre wavelet approximation of f by $(2^k, 0)^{th}$ *partial sums*

$$
(S_{2^k,0}f)(x) = \sum_{n=1}^{2^k} c_{n,0}\psi_{n,0}
$$

of the Legendre wavelet series (5.14)) is given by

$$
E_{2^k,0}(f) = min||f - S_{2^k,0}(f)||_2 = O\left(\tfrac{1}{2^k}\right).
$$

Proof. Proof of corollary 5.6 can be developed on the same line of proofs of the Theorem (5.1) by taking $\mu = 2$.

6 Conclusions

(1) The estimates of Theorem (5.1), (5.5), (5.3), (5.4) and (5.5) are obtained as:

$$
\begin{aligned}
E^{(1)}_{\mu^k,0}(f) &= O\left(\frac{1}{\mu^k}\right), \\
E^{(2)}_{\mu^k,M}(f) &= O\left(\frac{1}{\mu^k(2M+1)^{\frac{1}{2}}}\right); \text{for } \; M \geq 0,
\end{aligned}
$$

$$E^{(3)}_{\mu^k,0}(f) = O\left(\frac{1}{\mu^k}\left(1+\frac{1}{\mu^k}\right)\right),$$
$$E^{(4)}_{\mu^k,1}(f) = O\left(\frac{1}{\mu^{2k}}\right),$$
$$E^{(5)}_{\mu^k,M}(f) = O\left(\frac{1}{\mu^{2k}(2M-1)^{\frac{3}{2}}}\right); \text{for } M \geq 1.$$

Since
$E^{(1)}_{\mu^k,0}(f) \to 0, E^{(2)}_{\mu^k,M}(f) \to 0, E^{(3)}_{\mu^k,0}(f) \to 0\ E^{(4)}_{\mu^k,1}(f) \to 0\ \ and\ \ E^{(5)}_{\mu^k,M}(f) \to 0$ as $k \to \infty$. Therefore these estimates are best approximation in wavelet analysis.Zygmund[1].
(2) It is observed that estimates of f having higher order derivatives are more sharper than those function f having less derivatives.

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