Palestine Journal of Mathematics

Vol. 8(1)(2019) , 397–411

A new approach to find a generalized evolute and involute curve in 4-dimensional Minkowski space-time

Muhammad Hanif and Zhong Hua Hou

Communicated by Zafar Ahsan

MSC 2010 Classifications: : Primary 53-XX, 53Axx; Secondary 53A35.

Keywords and phrases: Evolute, Involute, Null Cartan curve, Minkowski space.

This work is supported by the National Natural Science Foundation of China (No.61473059).

Abstract. In this study, we introduce a new approach to find a special kind of generalized evolute and involute curve in Minkowski space-time E_1^4. Necessary and sufficient condition for the curve possessing generalized evolute as well as involute curve were obtained. Furthermore, Cartan null curve is also discussed in detail.

1 Introduction

Many mathematicians did work about the general theory of the curves in Euclidean space. Now, we have much understanding on their local geometry as well as their global geometry. Identification of a regular curve is one of the important and interesting complication in the theory of curves in Euclidean space.

Using two different approaches this complication can be solved, the relation between the frenet vectors of the curve [1], and determining the shape and size of the curve by curvatures κ_1 and κ_2. In differential geometry an evolute is the envelope of the normals of the specific curve. An evolute and its involute are defined in mutual pairs. The evolute of any curve is defined as the locus of the centers of curvatures of the curve. The original curve is then described as the involute of the evolute. Evolutes and involutes (also known as evolvents) were studied by C. Huygens [2]. Later, in [3] the author explore that if evolute occur then the evolute of parallel arc also occur. In [4] the author fixed that evolute Frenet apparatus can be establish by involute apparatus in four dimensional Euclidean space so by this approach another orthonormal of the same space is acquired. In [5] author resolved that the iteration of involutes create a pair of sequences of curves with respect to Minkowski metric and its dual. In (1845), Saint Venant [6] suggested a question either the principle normal of a curve is the principle normal of another on the surface produced by the principle normal of the specific curve. Bertrand answered this question in [7]. He proved that a necessary and sufficient condition for the existence of such a second curve is required. Using this method we define a kind of generalized evolute-involute curve in Minkowski space time. We acquire the necessary and sufficient conditions for the curves with spacelike (1,3)-normal plane to be (1,3)-Evolute curves and we also prove its converse by using the condition of Evolute curve that is spanned by principle normal and the second binormal. In the end we give some examples for these curves. Evolute curves and their identification were studied by some researchers in Minkowski space [8], [9], [10], [11], [12], [13], [14], [15], [16] as well as in Euclidean space. We see that mainly evolute-involute curves have been studied but not so much research has been carried out to find the mate curves of Cartan null curves. In this paper, a kind of generalized evolute and involute curve is considered for Cartan null curve in Minkowski space-time. The necessary and sufficient conditions for a curve possessing generalized evolute as well as involute mate curves is obtained.

2 Preliminaries

Consider the Minkowski space-time (E_1^4, H) where $E_1^4 = \{z = (z_1, z_2, z_3, z_4)|z_i \in R\}$ and $H = -dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2$. For any $U = (x_1, x_2, x_3, x_4)$ and $V = (y_1, y_2, y_3, y_4) \in T_zE$, we

denote
$U \cdot V = H(U,V) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$.
Let I be an open interval in R and $\Gamma : I \to E_1^4$ be a regular curve in E_1^4 parameterized by arc length parameter s and $\{T, N_1, N_2, N_3\}$ be a moving Frenet frame along Γ, consisting of tangent vector T, principal normal vector N_1, the first binormal vector N_2 and the second binormal vector N_3 respectively, so that $T \wedge N_1 \wedge N_2 \wedge N_3$ coincides with the standard orientation of E_1^4. From [17] Frenet seret formula.

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{2.1}$$

where

$$\begin{aligned} &H(T,T) = \epsilon_1, H(N_1,N_1) = \epsilon_2, H(N_2,N_2) = \epsilon_3, \\ &H(N_3,N_3) = \epsilon_4, \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1, \\ &\epsilon_i \in \{1,-1\}, i \in \{1,2,3,4\}. \end{aligned}$$

In specific, the succeeding conditions exist:

$$H(T,N_1) = H(T,N_2) = H(T,N_3) = H(N_1,N_2) = H(N_1,N_3) = H(N_2,N_3) = 0.$$

A curve $\Gamma(s)$ in E_1^4 can be spacelike, timelike, or null if its velocity vectors $\Gamma'(s)$ are commonly spacelike, timelike, or null [18]. A null curve Γ is parametrized by pseudo-arc s if $H(\Gamma''(s), \Gamma''(s)) = 1$ [19]. Further more nonnull curve Γ, we have this condition $H(\Gamma'(s), \Gamma'(s)) = \pm 1$. From [19, 20] if Γ is null Cartan curve, the Cartan Frenet frame is given by

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{2.2}$$

where $\kappa_1(s) = 0$ if $\Gamma(s)$ is a null straight line or $\kappa_1(s) = 1$ in all other cases. In this case
$T \cdot T = N_2 \cdot N_2 = 0, N_1 \cdot N_1 = N_3 \cdot N_3 = 1,$
$T \cdot N_1 = T \cdot N_3 = N_1 \cdot N_2 = N_1 \cdot N_3 = N_2 \cdot N_3 = 0, T \cdot N_2 = 1.$

We established some terminologies in this study. At any point of Γ, the plane spanned by $\{T, N_2\}$ is called *the* $(0,2)$-*tangent plane* of Γ. The plane spanned by $\{N_1, N_3\}$ is called *the* $(1,3)$-*normal plane* of Γ.

Let $\Gamma : I \to E^4$ and $\Gamma^* : I \to E_1^4$ be two regular curves in E_1^4 where s is the arc-length parameter of Γ. Denote $s^* = f(s)$ to be the arc-length parameters of Γ^*. For any $s \in I$, if the $(0,2)$-tangent plane of Γ at $\Gamma(s)$ of coincides with the $(1,3)$-normal plane at $\Gamma^*(s)$ of Γ^*, then Γ^* is called *the* $(0,2)$-*involute curve of* Γ in E_1^4 and Γ is called *the* $(1,3)$-*evolute curve of* Γ^* in E_1^4.

2.1 (1,3)-involute curve of a given curve in E_1^4

In this section, we proceed to study the existence and expression of the $(1,3)$-evolute curve of a given curve in E_1^4.

Let $\Gamma : I \to E_1^4$ be a regular curve with arc-length parameter s so thatκ_1, κ_2 and κ_3 are not zero. Let $\Gamma^* : I \to E_1^4$ be the $(1,3)$-evolute curve of Γ. Denote $\{T^*, N_1^*, N_2^*, N_3^*\}$ to be the Frenet frame along Γ^* and κ_1^*, κ_2^* and κ_3^* to be the curvatures of Γ^*. Then

$$\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}, \qquad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}.$$

3 Theorem

Let $\Gamma : I \to E_1^4$ be a regular curve with arc-length parameter s so that κ_1, κ_2 and κ_3 are not zero. Let $\Gamma^* : I \to E_1^4$ be the $(1,3)$-evolute curve of Γ. Denote $\{T^*, N_1^*, N_2^*, N_3^*\}$ to be the Frenet frame along Γ^* and κ_1^*, κ_2^* and κ_3^* to be the curvatures of Γ^* if and only if there there exists scalar functions Φ, Ψ of arc-length parameter s and real constant numbers $\alpha \neq \pm 1$, β satisfying

$$\Phi'(s) \neq 0, \Psi^{'}(s) \neq 0, \tag{3.1}$$

$$\Phi^{'}(s) = \alpha \Psi^{'}(s), \tag{3.2}$$

$$\beta \alpha k_1(s) = \alpha k_2(s) - k_3(s), \tag{3.3}$$

$$[-\kappa_2\kappa_3(\alpha^2 - 1) + \alpha(\kappa_3^2 - \kappa_1^2 - \kappa_2^2)] \neq 0 \tag{3.4}$$

for all $s \in I$.

Proof. Let $\Gamma : I \to E_1^4$ be a regular curve with arc-length parameter s so that κ_1, κ_2 and κ_3 are not zero. Let $\Gamma^* : I \to E_1^4$ be the $(1,3)$-evolute curve of Γ. Denote $\{T^*, N_1^*, N_2^*, N_3^*\}$ to be the Frenet frame along Γ^* and κ_1^*, κ_2^* and κ_3^* to be the curvatures of Γ^*. Then

$$\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}, \qquad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}.$$

Moreover, we can write the curve Γ^* as follows

$$\Gamma^*(s^*) = \Gamma(s) + \Phi(s)N_1(s) + \Psi(s)N_3, \tag{3.5}$$

for all $s^* \in I^*$, $s \in I$ where $\Phi(s)$ and $\Psi(s)$ are C^∞ functions on I.

Taking derivative of (3.5), using the equation (2.1), we get

$$T^* f' = (1 - \Phi\epsilon_1\kappa_1)T(s) + \Phi^{'}(s)N_1(s) + \Psi^{'}(s)N_3 + \epsilon_3(\Phi(s)\kappa_2 - \Psi(s)\kappa_3)N_2. \tag{3.6}$$

$\{T^*, N_2^*\} \perp \{T, N_2\}$ so $1 - a\epsilon_1\kappa_1 = 0$ and $\epsilon_3(a\kappa_2 - b\kappa_3) = 0$ from these we have $\Phi = \frac{1}{\epsilon_1\kappa_1}$,$\Psi = \frac{\epsilon_1\kappa_2}{\kappa_1\kappa_3}$.

Equation(3.6) gets the form

$$f^{'}T^* = \Phi^{'}(s)N_1 + \Psi^{'}(s)N_3. \tag{3.7}$$

So (3.6) gets the form

$$T^* = \frac{\Phi^{'}}{f^{'}}N_1 + \frac{\Psi^{'}}{f^{'}}N_3. \tag{3.8}$$

Multiplying (3.7) by itself, we get

$$\epsilon_1^*(f^{'})^2 = \epsilon_2(\Phi^{'})^2 + \epsilon_4(\Psi^{'})^2. \tag{3.9}$$

If we denote

$$\eta = \frac{\Phi^{'}}{f'}, \zeta = \frac{\Psi^{'}}{f'}. \tag{3.10}$$

Using equation (3.10) in (3.7), we get

$$T^* = \eta N_1 + \zeta N_3. \tag{3.11}$$

Taking derivative of equation (3.11) using (2.1), we get

$$f'\kappa_1^* N_1^* = \eta' N_1 - \epsilon_1\eta\kappa_1 T + \zeta' N_3 + \epsilon_3(\eta\kappa_2 - \zeta\kappa_3)N_2. \tag{3.12}$$

Taking inner product on both side of (3.12) by N_1 and N_3 respectively, we get

$$\eta' = 0, \zeta' = 0. \tag{3.13}$$

$$f'\kappa_1^* N_1^* = -\epsilon_1 \eta\kappa_1 T + \epsilon_3(\eta\kappa_2 - \zeta\kappa_3)N_2. \tag{3.14}$$

Multiplying (3.14) by itself, we get

$$(f')^2(\kappa_1^*)^2 = \eta^2\kappa_1^2 + (\eta\kappa_2 - \zeta\kappa_3)^2. \tag{3.15}$$

We know that curvatures κ_1^*, κ_2^*, and $\kappa_3^* \neq 0$, so we obtained result (3.1)

$$\Phi' \neq 0, \Psi^{'} \neq 0.$$

From (3.10), we get the result (3.2)

$$\Phi^{'} = \alpha\Psi^{'}. \tag{3.16}$$

Integrating (3.16), we get $\Phi = \alpha\Psi + \eta$ and $\Psi = \frac{\Phi-\eta}{\alpha}$.
Using (3.10) in (3.15), we get

$$(f^{'})^2(\kappa_1^*)^2 = (\frac{\Psi^{'}}{f^{'}})^2[\alpha^2\kappa_1^2 - (\alpha\kappa_2 - \kappa_3)^2] \tag{3.17}$$

Using (3.16) in (3.9), we acquire

$$f'^2 = (\Psi^{'})^2(\alpha^2 + 1). \tag{3.18}$$

Again writing equation (3.17)

$$(f')^2(\kappa_1^*)^2 = (\frac{\Psi^{'}}{f'})^2[\alpha^2\kappa_1^2 - (\alpha\kappa_2 - \kappa_3)^2]. \tag{3.19}$$

Using (3.18) in (3.19), we acquire

$$(f')^2(\kappa_1^*)^2 = \frac{1}{\alpha^2+1}[\alpha^2\kappa_1^2 - (\alpha\kappa_2 - \kappa_3)^2]. \tag{3.20}$$

If we denote

$$\Delta_1 = \frac{\eta\kappa_1}{f'\kappa_1^*} = \frac{\Psi^{'}\alpha}{f'^2\kappa_1^*}[\kappa_1], \tag{3.21}$$

$$\Delta_2 = \frac{\eta\kappa_2 - \zeta\kappa_3}{f'\kappa_1^*} = \frac{\Psi^{'}}{f'^2\kappa_1^*}[\alpha\kappa_2 - \kappa_3]. \tag{3.22}$$

Dividing (3.22) by (3.21), we get the result (3.3)

$$\beta\alpha\kappa_1 = \alpha\kappa_2 - \kappa_3.$$

Putting values of Δ_1, Δ_2 in equation (3.14), we get

$$N_1^* = \Delta_1 T + \Delta_2 B_1. \tag{3.23}$$

Taking derivative of equation (3.23) using equation (2.1), we acquire

$$-\epsilon_1^* f'\kappa_1^* T^* + \epsilon_3^* f'\kappa_2^* N_2^* = -\epsilon_2(\Delta_1\kappa_1 + \Delta_2\kappa_2)N_1 + \epsilon_4\Delta_2\kappa_3 N_3 + \Delta_1^{'}T + \Delta_2^{'}N_2. \tag{3.24}$$

Since $\{T^*, N_2^*\}\perp\{T, N_2\}$, so we acquire

$$\Delta_1^{'} = 0, \Delta_2^{'} = 0. \tag{3.25}$$

Using the (3.7), (3.21), (3.22) and (3.25) in (3.24), we obtain

$$\epsilon_3^* f'\kappa_2^* N_2^* = P(s)N_1 + Q(s)N_3, \tag{3.26}$$

where

$$P(s) = \frac{\Psi^{'}}{f'^2(\alpha^2+1)\kappa_1^*}[-\kappa_2\kappa_3(\alpha^2 - 1) + \alpha(\kappa_3^2 - \kappa_1^2 - \kappa_2^2)], \tag{3.27}$$

$$Q(s) = -\frac{\alpha\Psi'}{f'^2(\alpha^2+1)\kappa_1^*}[-\kappa_2\kappa_3(\alpha^2-1)+\alpha(\kappa_3^2-\kappa_1^2-\kappa_2^2)]. \quad (3.28)$$

Since

$$\epsilon_3^* f' \kappa_2^* N_2^* \neq 0.$$

So we get the result (3.4)

$$[-\kappa_2\kappa_3(\alpha^2-1)+\alpha(\kappa_3^2-\kappa_1^2-\kappa_2^2)] \neq 0$$

Conversely, we suppose that $\Gamma : I \subset R \to E_1^4$ be an evolute curve with arc-length parameter s so that k_1, k_2 and k_3 are not zero. And the relations (3.1), (3.2), (3.3), (3.4) hold for some scalar functions Φ, Ψ and constant real numbers $\alpha \neq 0$, β. Then the curve Γ^* can be expressed

$$\Gamma^*(s^*) = \Gamma(s) + \Phi(s)N_1(s) + \Psi(s)N_3. \quad (3.29)$$

Taking derivative of equation (3.29) using equation (2.1), we acquire

$$\frac{d\Gamma^*}{ds} = \Phi'(s)N_1 + \Psi'(s)N_3. \quad (3.30)$$

From (3.30) and (3.2), we get

$$\frac{d\Gamma^*}{ds} = \Psi'[\alpha N_1 + N_3]. \quad (3.31)$$

From this

$$f' = \frac{ds^*}{ds} = ||\frac{dT^*}{ds}|| = c_1(\Psi')\sqrt{c_2(\alpha^2+1)} > 0, \quad (3.32)$$

such that $c_1(\Psi') > 0$ where $c_1 = \pm 1$ and $c_2 = \pm 1$ such that $\epsilon_2 c_2(\alpha^2+1) > 0$. Again writing the equation (3.31)

$$T^* f' = \Psi'[\alpha N_1 + N_3]. \quad (3.33)$$

Using (3.32) in (3.33), we get

$$T^* = \frac{c_1}{\sqrt{c_2(\alpha^2+1)}}[\alpha N_1 + N_3], \quad (3.34)$$

which indicate that $H(T^*, T^*) = c_2 = \epsilon_1^*$.
Taking derivative of equation (3.34) using (2.1), we acquire

$$\frac{dT^*}{ds^*} = \frac{c_1}{f'\sqrt{c_2(\alpha^2+1)}}[-\epsilon_1\alpha\kappa_1 T + \epsilon_3(\alpha\kappa_2 - \kappa_3)N_2]. \quad (3.35)$$

Using (3.35), we have

$$k_1^* = ||\frac{dT^*}{ds}|| = \frac{\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2-\kappa_3)^2}}{f'\sqrt{c_2(\alpha^2+1)}} > 0 \quad (3.36)$$

From equation (2.37) and (2.38), we acquire

$$N_1^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{c_1}{\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2-\kappa_3)^2}}[-\epsilon_1(\alpha\kappa_1)T + \epsilon_3(\alpha\kappa_2-\kappa_3)N_2], \quad (3.37)$$

which indicate that $H(N_1^*, N_1^*) = 1$.

Let

$$\Delta_3 = \frac{-\epsilon_1(\alpha\kappa_1)}{\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2-\kappa_3)^2}}, \quad \Delta_4 = \frac{\epsilon_3(\alpha\kappa_2-\kappa_3)}{\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2-\kappa_3)^2}}, \quad (3.38)$$

we acquire

$$N_1^* = \Delta_3 T + \Delta_4 N_2. \quad (3.39)$$

Taking derivative of (3.39) using equation (2.1), we acquire

$$f'\frac{dN_1^*}{ds^*} = \Delta_3'T + \Delta_4'N_2 + \epsilon_2(\Delta_3\kappa_1 - \Delta_4\kappa_2)N_1 + \epsilon_4\Delta_4\kappa_3 N_3. \tag{3.40}$$

Taking derivative of (3.3), we acquire

$$(\alpha\kappa_2' - \kappa_3')\alpha\kappa_1 - (\alpha\kappa_2 - \kappa_3)\alpha\kappa_1' = 0. \tag{3.41}$$

Taking derivative of equation (3.38) with respect to s using (3.41), we acquire

$$\Delta_3' = 0, \quad \Delta_4' = 0. \tag{3.42}$$

Substituting the values (3.38) and (3.42) in (3.40), we get

$$\frac{dN_1^*}{ds^*} = \frac{c_1(-\alpha\kappa_1)\kappa_1 + (\alpha\kappa_2 - \kappa_3)\kappa_2}{f'\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}}N_1 + \frac{-c_1(\alpha\kappa_2 - \kappa_3)\kappa_3}{f'\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}}N_3. \tag{3.43}$$

Using equation (3.36) and (3.34), we acquire

$$\epsilon_1^*\kappa_1^*T^* = \frac{c_1\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}}{f'(\alpha^2+1)}[\alpha N_1 + N_3]. \tag{3.44}$$

From (3.43) and (3.44), we acquire

$$\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^*T^* = \frac{c_1(\alpha^2-1)\kappa_2\kappa_3 + \alpha(\kappa_2^2 - \kappa_1^2 - \kappa_3^2)}{f'(\alpha^2+1)\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}}[N_1 - \alpha N_3], \tag{3.45}$$

From (3.45), we have

$$k_2^* = \frac{|(\alpha^2-1)\kappa_2\kappa_3 + \alpha(\kappa_2^2 - \kappa_1^2 - \kappa_3^2)|}{f(\alpha^2+1)\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}} > 0. \tag{3.46}$$

Consider (3.45) and (3.46) together, we acquire

$$N_2^* = \frac{\epsilon_3^*}{\kappa_2^*}[\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^*T^*] = \frac{c_2\epsilon_3^*}{\alpha^2+1}[N_1 + \alpha N_3], \tag{3.47}$$

where $c_2 = \frac{|(\alpha^2-1)\kappa_2\kappa_3+\alpha(\kappa_2^2-\kappa_1^2-\kappa_3^2)|}{|(\alpha^2-1)\kappa_2\kappa_3+\alpha(\kappa_2^2-\kappa_1^2-\kappa_3^2)|} = \pm 1$ and $\epsilon_3^* = \pm 1$.
From (3.47), we acquire $H(N_2^*, N_2^*) = c_1 = \epsilon_3^* = -\epsilon_1^*$, also unit vector N_3^* can be expressed like this $N_3^* = -\Delta_4 T + \Delta_3 N_2$; that is,

$$N_3^* = \frac{c_1\epsilon_3}{\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}}[(\alpha\kappa_2 - \kappa_3)T - \alpha\kappa_1 N_2] \tag{3.48}$$

which indicates that $H(N_3^*, N_3^*) = 1$. In the end we find κ_3^*

$$\kappa_3^* = H(\frac{dN_2^*}{ds^*}, N_3^*) = \frac{c_1c_2\kappa_1\epsilon_3^*(\alpha^2+1)\kappa_3}{f'\sqrt{c_3(\alpha^2+1)}\sqrt{(\alpha\kappa_1)^2 - (\alpha\kappa_2 - \kappa_3)^2}} \neq 0.$$

So we find that Γ^* is (1,3)-Evolute curve of the curve Γ.
Since $\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}$, $\quad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}$.

Case 2 Γ is a Cartan null curve with arc-length parameter s so thatκ_1, κ_2 and κ_3 are not zero and space like vectors N_1 and N_3 and Γ^* is a spacelike or timelike curve with arc-length parameter $s*$ so that curvature functions κ_1^*, κ_2^* and κ_3^* are not equal to zero. For this we have succeeding theorem.

4 Theorem

Let $\Gamma : I \to E_1^4$ be a Cartan null curve with arc-length parameter s so that $\kappa_1 = 1$, and κ_2 and κ_3 are not zero. Then the curve α is a $(1,3)$-Evolute curve and its Evolute mate curve is a spacelike or timelike curve with non zero curvatures if and only if there exists scalar functions Φ, Ψ of arc-length parameter s and constant real numbers β $\alpha \neq \pm 1$, satisfying

$$\Phi'(s) = \alpha \Psi'(s), \tag{4.1}$$

$$\beta\alpha = \alpha\kappa_2(s) - \kappa_3(s), \tag{4.2}$$

$$\alpha^3\kappa_2(s) - 2\alpha^2\kappa_3(s) - \kappa_3(s) \neq 0, \tag{4.3}$$

for all $s \in I$.

Proof We can prove this theorem in same way as we proved theorem 6.

Case 3 Let Γ is spacelike or timelike curve with nonzero curvatures κ_1, κ_2 and κ_3 and space like vectors T and N_2 and Γ^* is also spacelike or timelike curve with κ_1^*, κ_2^* and κ_3^* not equal to zero and vectors T^* and N_2^* are spacelike. For this we have succeeding theorem.

5 Theorem

Let $\Gamma : I \to E_1^4$ be a regular curve with arc-length parameter s so thatκ_1, κ_2 and κ_3 are not zero. Let $\Gamma^* : I \to E_1^4$ be the $(0,2)$-evolute curve of Γ. Denote $\{T^*, N_1^*, N_2^*, N_3^*\}$ to be the Frenet frame along Γ^* and κ_1^*, κ_2^* and κ_3^* to be the curvatures of Γ^* if and only if there exist constant numbers Φ, Ψ, $\alpha \neq \pm 1$, β satisfying

$$\Phi(s) \neq 0, \Psi(s) \neq 0, \tag{5.1}$$

$$\epsilon_2(\Phi(s)\kappa_1(s) - \Psi(s)\kappa_2(s)) = \alpha\epsilon_4\Psi(s)\kappa_3(s), \tag{5.2}$$

$$\beta\alpha\kappa_1(s) = \alpha\kappa_2(s) - \kappa_3(s), \tag{5.3}$$

$$-\kappa_2(s)\kappa_3(s)(\alpha^2 + 1) + \alpha[\kappa_1^2(s) + \kappa_2^2(s) + \kappa_3^2(s)] \neq 0, \tag{5.4}$$

for all $s \in I$.

Proof. Let $\Gamma : I \to E_1^4$ be a regular curve with arc-length parameter s so thatκ_1, κ_2 and κ_3 are not zero. Let $\Gamma^* : I \to E_1^4$ be the $(0,2)$-evolute curve of Γ. Denote $\{T^*, N_1^*, N_2^*, N_3^*\}$ to be the Frenet frame along Γ^* and κ_1^*, κ_2^* and κ_3^* to be the curvatures of Γ^*. Then

$$\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}, \qquad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}.$$

Moreover, we can write the curve Γ^* as follows

$$\Gamma^*(s^*) = \Gamma(s) + \Phi(s)T(s) + \Psi(s)N_2, \tag{5.5}$$

for all $s^* \in I^*$, $s \in I$ where $\Phi(s)$ and $\Psi(s)$ are C^∞ functions on I.
Taking derivative of equation (5.5) using equation (2.1), we acquire;

$$T^* f' = T(s) + \Phi'(s)T(s) + \Psi'(s)N_2 + \epsilon_2(\Phi\kappa_1 - \Psi\kappa_2)N_1 + \epsilon_4\Psi\kappa_3 N_3. \tag{5.6}$$

Taking inner product on both-sides of (5.6) with T and N_2 respectively, we get $1 + \Phi' = 0$ and $\Psi' = 0$, which implies that Ψ is constant and $\Phi = \Phi_0 - s$, where Φ_0 is the integration constant. So (5.6) turns into

$$T^* f' = \epsilon_2(\Phi\kappa_1 - \Psi\kappa_2)N + \epsilon_4\Psi\kappa_3 B_2. \tag{5.7}$$

Multiplying (5.7) by itself, we get

$$\epsilon_1^*(f')^2 = \epsilon_2(\Phi\kappa_1 - \Psi\kappa_2)^2 + \epsilon_4\Psi^2\kappa_3^2. \tag{5.8}$$

If we denote

$$\eta = \frac{\epsilon_2(\Phi\kappa_1 - \Psi\kappa_2)}{f'} and \zeta = \frac{\epsilon_4\Psi\kappa_3}{f'}. \tag{5.9}$$

So (5.7) gets the form

$$T^* = \eta N_1 + \zeta N_3. \tag{5.10}$$

Differentiating equation (5.10) using equation (2.1) , we acquire

$$\epsilon_1^* f'\kappa_1^* N_1^* = \eta' N_1 - \epsilon_1\eta\kappa_1 T + \zeta' N_3 + \epsilon_3(\eta\kappa_2 - \zeta\kappa_3)N_2. \tag{5.11}$$

Multiplying (5.11) by N_1 and N_3 respectively, we get

$$\eta' = 0, \zeta' = 0. \tag{5.12}$$

Using (5.12) in (5.11), we get

$$\epsilon_1^* f'\kappa_1^* N_1^* = -\epsilon_1\eta\kappa_1 T + \epsilon_3(\eta\kappa_2 - \zeta\kappa_3)N_2. \tag{5.13}$$

Multiplying (5.13) by itself, we get

$$\epsilon_2^*(f')^2(\kappa_1^*)^2 = \epsilon_1\eta^2\kappa_1^2 + \epsilon_3(\eta\kappa_2 - \zeta\kappa_3)^2. \tag{5.14}$$

Substituting (5.9) in (5.14), we find

$$(f')^2(\kappa_1^*)^2 = (\frac{\Psi\kappa_3}{f'})^2[\alpha^2\kappa_1^2 + (\alpha\kappa_2 - \kappa_3)^2]. \tag{5.15}$$

Since $\kappa_1, \kappa_2, \kappa_3 \neq 0$, so from (5.9), we get the result (5.1)

$$\Phi \neq 0, \Psi \neq 0.$$

From (5.9), we get

$$\epsilon_2(\Phi\kappa_1 - \Psi\kappa_2)\zeta = \eta\epsilon_4(\Psi\kappa_3). \tag{5.16}$$

From this we get the result (5.2)

$$\epsilon_2(\Phi\kappa_1 - \Psi\kappa_2) = \alpha\epsilon_4\Psi\kappa_3. \tag{5.17}$$

Using (5.17) in (5.7), we get

$$f'^2 = \epsilon_1^*\epsilon_4(\Psi\kappa_3)^2(\alpha^2 + 1). \tag{5.18}$$

Substituting (5.18) in (5.15), we get

$$(f')^2(\kappa_1^*)^2 = \frac{\epsilon_1^*\epsilon_4}{\alpha^2 + 1}[\alpha^2\kappa_1^2 - (\alpha\kappa_2 - \kappa_3)^2]. \tag{5.19}$$

If we denote

$$\Delta_2 = \frac{\eta\kappa_2 - \zeta\kappa_3}{f'\kappa_1^*} = \frac{\epsilon_4\Psi\kappa_3}{f'^2\kappa_1^*}[(\alpha\kappa_2 - \kappa_3)]. \tag{5.20}$$

$$\Delta_1 = \frac{\eta\kappa_1}{f'\kappa_1^*} = \frac{\epsilon_4\Psi\kappa_3}{f'^2\kappa_1^*}\alpha\kappa_1. \tag{5.21}$$

Dividing (5.20) by (5.21) we get result (5.3)

$$\beta\alpha\kappa_1 = \alpha\kappa_2 - \kappa_3.$$

Putting values of Δ_1, Δ_2 in equation (5.13), we get

$$N_1^* = \Delta_1 T + \Delta_2 N_2. \tag{5.22}$$

Taking derivative of the equation (5.22) using equation (2.1), we acquire

$$-\epsilon_1^* f' \kappa_1^* T^* + \epsilon_3^* f' \kappa_2^* N_2^* = \epsilon_2(\Delta_1\kappa_1 - \Delta_2\kappa_2)N_1 + \epsilon_4\Delta_2\kappa_3 N_3 + \Delta_2^{'} N_2 + \Delta_1^{'} T. \tag{5.23}$$

Multiplying equation (5.23) by T and N_2 respectively, we get

$$\Delta_1^{'} = 0, \Delta_2^{'} = 0. \tag{5.24}$$

Using (5.7) and (5.24) in (5.23), we obtain

$$\epsilon_3^* f' k_2^* N_2^* = P(s)N_1 + Q(s)N_3, \tag{5.25}$$

where

$$P(s) = \frac{\epsilon_2\epsilon_4\Psi\kappa_3}{f'^2(\alpha^2+1)\kappa_1^*}[\alpha(\kappa_2^2 - \kappa_1^2 - \kappa_3^2) - \kappa_2\kappa_3(\alpha^2+1)], \tag{5.26}$$

$$Q(s) = \frac{\epsilon_2\epsilon_4\alpha\Psi\kappa_3}{f'^2(\alpha^2+1)\kappa_1^*}[\alpha(\kappa_2^2 - \kappa_1^2 - \kappa_3^2) - \kappa_2\kappa_3(\alpha^2+1)]. \tag{5.27}$$

Since

$$\epsilon_3^* f' k_2^* N_2^* \neq 0.$$

So we get the result (5.4)

$$[\alpha(\kappa_2^2 - \kappa_1^2 - \kappa_3^2) - \kappa_2\kappa_3(\alpha^2+1)] \neq 0. \tag{5.28}$$

Conversely, let $\Gamma : I \subset R \to E_1^4$ be an evolute curve with arc-length parameter s with κ_1, κ_2, κ_3 are not equal to zero and the relations (5.1), (5.2), (5.3), (5.4) exist for constant numbers Φ, Ψ, $\alpha \neq 0$, β. then we can define curve Γ^* like this

$$\Gamma^*(s^*) = \Gamma(s) + \Phi(s)T(s) + \Psi(s)N_2. \tag{5.29}$$

Differentiating (5.29) with respect to s using frenet formula (2.1), we get

$$\frac{d\Gamma^*}{ds} = \epsilon_2(\Phi\kappa_1 - \Psi\kappa_2)N_1 + \epsilon_4(\Psi\kappa_3)N_3. \tag{5.30}$$

From (5.2), we get

$$\frac{d\Gamma^*}{ds} = (\epsilon_4\alpha\Psi\kappa_3)N_1 + \epsilon_4(\Psi\kappa_3)N_3.$$

$$\frac{d\Gamma^*}{ds} = \epsilon_4\Psi\kappa_3[\alpha N_1 + N_3]. \tag{5.31}$$

From (5.31), we get

$$f' = \frac{ds^*}{ds} = ||\frac{dT^*}{ds}|| = c_1(\Psi\kappa_3)\sqrt{\epsilon_2 c_2(\alpha^2+1)} > 0, \tag{5.32}$$

such that $c_1(\Psi k_3) > 0$ where $c_1 = \pm 1$ and $c_2 = \pm 1$ such that $\epsilon_2 c_2(\alpha^2+1) > 0$. Again writing equation (5.31)

$$T^* f' = \epsilon_4\Psi\kappa_3[\alpha N_1 + N_3]. \tag{5.33}$$

Substituting (5.32) in (5.33), we get

$$T^* = \frac{\epsilon_4 c_1}{\sqrt{\epsilon_2 c_2(\alpha^2+1)}}[\alpha N_1 + N_3], \tag{5.34}$$

which indicates that $H(T^*, T^*) = C_2 = \epsilon_1^*$.
Taking derivative of equation (5.34) using equation (2.1), we acquire

$$\frac{dT^*}{ds^*} = \frac{\epsilon_4 c_1}{f'\sqrt{\epsilon_2 c_2(\alpha^2+1)}}[(\alpha\kappa_2 - \kappa_3)T - \alpha\kappa_1 N_2]. \tag{5.35}$$

Using (5.35), we have

$$k_1^* = ||\frac{dT^*}{ds}|| = \frac{\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}{f'\sqrt{\epsilon_2 c_2(\alpha^2+1)}} > 0 \tag{5.36}$$

From (5.35) and (5.36), we get

$$N_1^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{\epsilon_4 c_1}{\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}[(\alpha\kappa_2-\kappa_3)T - \alpha\kappa_1 N_2], \tag{5.37}$$

which leads to $H(N_1^*, N_1^*) = 1$.

If we denote

$$\Delta_3 = \frac{\epsilon_4 c_1(\alpha\kappa_2-\kappa_3)}{\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}, \quad \Delta_4 = -\frac{\epsilon_4\alpha c_1\kappa_1}{\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}, \tag{5.38}$$

Using (5.38) in (5.37), we get

$$N_1^* = \Delta_3 T + \Delta_4 N_2. \tag{5.39}$$

Taking derivative of equation (5.39) using equation (2.1), we acquire

$$f'\frac{dN_1^*}{ds^*} = \Delta_3^{'}T + \Delta_4^{'}N_2 + \epsilon_2(\Delta_3\kappa_1 - \Delta_4\kappa_2)N_1 + \epsilon_4\Delta_4\kappa_3 N_3. \tag{5.40}$$

Differentiating (5.3), we acquire

$$(\alpha\kappa_2^{'} - \kappa_3^{'})\alpha\kappa_1 - (\alpha\kappa_2-\kappa_3)\alpha\kappa_1^{'} = 0. \tag{5.41}$$

Taking derivative (5.38) with respect to s using equation (5.41), we acquire

$$\Delta_3^{'} = 0, \quad \Delta_4^{'} = 0. \tag{5.42}$$

Substituting the values (5.42) and (5.38) in (5.40), we get

$$\frac{dN_1^*}{ds^*} = \frac{c_1(2\alpha\kappa_1\kappa_2 - \kappa_1\kappa_3)}{f'\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}N_1 + \frac{c_1(\alpha\kappa_1\kappa_3)}{f'\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}N_3. \tag{5.43}$$

From (5.34) and (5.36), we get

$$\epsilon_1^*\kappa_1^* T^* = \frac{\epsilon_4 c_1\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}{f'(\alpha^2+1)}[\alpha N_1 + N_3]. \tag{5.44}$$

From (5.43) and (5.44), we get

$$\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^* T^* = \frac{c_1(2\alpha\kappa_2 + \alpha^2\kappa_3 - \kappa_3)}{f'(\alpha^2+1)\sqrt{-2\alpha^2\kappa_2 + 2\alpha\kappa_3}}[N_1 + \frac{1}{\alpha}N_3], \tag{5.45}$$

From (5.45), we have

$$k_2^* = \frac{|2\alpha\kappa_1\kappa_2 + \alpha^2\kappa_1\kappa_3 - \kappa_1\kappa_3|}{f\sqrt{|\alpha|}(\alpha^2+1)\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}} > 0. \tag{5.46}$$

Consider(5.45) and (5.46) together, we obtain

$$N_2^* = \frac{\epsilon_3^*}{\kappa_2^*}[\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^* T^*] = \frac{c_2 c_3\epsilon_3^*\sqrt{|\alpha|}}{2}[N_1 + \frac{1}{\alpha}N_3], \tag{5.47}$$

where $c_3 = \frac{|2\alpha\kappa_1\kappa_2+\alpha^2\kappa_1\kappa_3-\kappa_1\kappa_3|}{|2\alpha\kappa_1\kappa_2+\alpha^2\kappa_1\kappa_3-\kappa_1\kappa_3|} = \pm 1$ and $\epsilon_3^* = \pm$.
From (5.47), we acquire $H(N_2^*, N_2^*) = c_1 = \epsilon_3^* = -\epsilon_1^*$, also unit vector N_3^* can be expressed like this $N_3^* = -\Delta_4 T + \Delta_3 N_2$; that is,

$$N_3^* = \frac{c_2(2\alpha\kappa_1\kappa_2 - \kappa_1\kappa_3)}{\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}T + \frac{c_2\alpha\kappa_1\kappa_3}{\sqrt{2\alpha\kappa_1(\alpha\kappa_2-\kappa_3)}}N_2, \tag{5.48}$$

which indicates that $H(N_3^*, N_3^*) = 1$. In the end we find κ_3^* as,

$$\kappa_3^* = H(\frac{dN_2^*}{ds^*}, N_3^*) = \frac{c_3\epsilon_3^*(2\sqrt{|\alpha|}\kappa_3}{f'\sqrt{2\alpha\kappa_1(\alpha\kappa_2 - \kappa_3)}}.$$

So we find that Γ^* is (0,2)-Evolute curve of the curve Γ since
span$\{T, N_2\}$ = span$\{N_1^*, N_3^*\}$, span$\{N_1, N_3\}$ = span$\{T^*, N_2^*\}$.

Case 4 Γ is a cartan null curve with $\kappa_1 = 1$, κ_2, κ_3 are not equal to zero and $\Gamma*$ is a spacelike or time like curve with κ_1^*, κ_2^* and κ_3^* are not equal to zero and T^* and N_2^* are spacelike vectors. Then we have this theorem

6 Theorem

Let $\Gamma : I \to E_1^4$ be a Cartan null curve with arc-length parameter s so that $\kappa_1 = 1$, κ_2 and κ_3 are not zero. Then the curve Γ is a $(0, 2)$-Evolute curve and its Evolute mate curve is a spacelike or timelike curve with curvatures not equal to zero if and only if there exists constant real numbers Φ, Ψ, $\alpha \neq \pm 1$, β satisfying

$$(\Phi(s)\kappa_1(s) - \Psi(s)\kappa_2(s)) = \alpha\Psi(s)\kappa_3(s), \tag{6.1}$$

$$-\beta\alpha = \alpha\kappa_2(s) - \kappa_3(s), \tag{6.2}$$

$$\kappa_3(s)(\alpha^2 - 1) + 2\alpha k_2(s) \neq 0, \tag{6.3}$$

for all $s \in I$.

Proof. Let $\Gamma : I \to E_1^4$ be a Cartan null curve with arc-length parameter s so thatκ_1, κ_2 and κ_3 are not zero. Let $\Gamma^* : I \to E_1^4$ be the $(0, 2)$-evolute curve of Γ. Denote $\{T^*, N_1^*, N_2^*, N_3^*\}$ to be the Frenet frame along Γ^* and κ_1^*, κ_2^* and κ_3^* to be the curvatures of Γ^*. Then

span$\{T, N_2\}$ = span$\{N_1^*, N_3^*\}$, span$\{N_1, N_3\}$ = span$\{T^*, N_2^*\}$.

Moreover, we can write the curve Γ^* as follows

$$\Gamma^*(s^*) = \Gamma(s) + \Phi(s)T(s) + \Psi(s)N_2 \tag{6.4}$$

for all $s^* \in I^*$, $s \in I$ where $\Phi(s)$ and $\Psi(s)$ are C^∞ functions on I.
Taking derivative (6.4) using equation (2.1), we acquire

$$T^*f' = T(s) + \Phi'(s)T(s) + \Psi'(s)N_2 + (\Phi\kappa_1 - \Psi\kappa_2)N_1 + \Psi\kappa_3 N_3 \tag{6.5}$$

Taking inner product on both-sides of (6.5) with T and N_2 respectively, we get $1 + \Phi' = 0$ and $\Psi' = 0$, which implies that Ψ is constant and $\Phi = \Phi_0 - s$, where Φ_0 is the integration constant. So (6.5) turns into

$$T^*f' = (\Phi\kappa_1 - \Psi\kappa_2)N_1 + \Psi\kappa_3 N_3. \tag{6.6}$$

Multiplying (6.6) by itself, we get

$$\epsilon_1^*(f')^2 = (\Phi\kappa_1 - \Psi\kappa_2)^2 + \Psi^2\kappa_3^2. \tag{6.7}$$

If we denote

$$\eta = \frac{(\Phi\kappa_1 - \Psi\kappa_2)}{f'} and \zeta = \frac{\Psi\kappa_3}{f'}. \tag{6.8}$$

So (6.6) gets the form

$$T^* = \eta N_1 + \zeta N_3. \tag{6.9}$$

Taking derivative of equation (6.9) using equation (2.1) , we acquire

$$\epsilon_1^* f'\kappa_1^* N_1^* = \eta' N_1 + \zeta' N_3 + (\eta\kappa_2 - \zeta\kappa_3)T - \eta\kappa_1 N_2. \tag{6.10}$$

Multiplying (6.10) by N_1 and N_3 respectively, we acquire

$$\eta' = 0, \zeta' = 0. \tag{6.11}$$

Using (6.11) in (6.10), we get

$$\epsilon_2^* f' \kappa_1^* N_1^* = (\eta\kappa_2 - \zeta\kappa_3)T - \eta\kappa_1 N_2. \tag{6.12}$$

Multiplying (6.12) by itself, we acquire

$$\epsilon_2^*(f')^2(\kappa_1^*)^2 = -2\alpha(\frac{\Psi\kappa_3}{f'})^2[\alpha\kappa_2 - \kappa_3]. \tag{6.13}$$

From (6.8), we acquire

$$(\Phi\kappa_1 - \Psi\kappa_2)\zeta = \eta(\Psi\kappa_3). \tag{6.14}$$

From this we acquire the result (6.1)

$$(\Phi\kappa_1 - \Psi\kappa_2) = \alpha\Psi\kappa_3. \tag{6.15}$$

Using (6.15) in (6.7), we acquire

$$f'^2 = \epsilon_1^*(\Psi\kappa_3)^2(\alpha^2 + 1). \tag{6.16}$$

Substituting (6.16) in (6.13), we get

$$(f')^2(\kappa_1^*)^2 = \frac{-2\alpha\epsilon_1^*\epsilon_2^*}{\alpha^2 + 1}[(\alpha\kappa_2 - \kappa_3)]. \tag{6.17}$$

If we denote

$$\Delta_1 = \frac{\eta\kappa_2 - \zeta\kappa_3}{f'\kappa_1^*} = \frac{\Psi\kappa_3}{f'^2\kappa_1^*}[(\alpha\kappa_2 - \kappa_3)]. \tag{6.18}$$

$$\Delta_2 = -\frac{\eta}{f'\kappa_1^*} = -\frac{\Psi\kappa_3}{f'^2\kappa_1^*}\alpha. \tag{6.19}$$

Dividing (6.18) by (6.19), we acquire the result (6.2)

$$-\beta\alpha = \alpha\kappa_2 - \kappa_3.$$

Using (6.18) and (6.19) in (6.12), we get

$$N_1^* = \Delta_1 T + \Delta_2 N_2. \tag{6.20}$$

Taking derivative of (6.20) using equation (2.1), we acquire

$$-\epsilon_1^* f'\kappa_1^* T^* + \epsilon_3^* f'\kappa_2^* N_2^* = (\Delta_1\kappa_1 - \Delta_2\kappa_2)N_1 + \Delta_2\kappa_3 N_3 + \Delta_2^{'} N_2 + \Delta_1^{'} T. \tag{6.21}$$

Multiplying equation (6.21) by T and N_2 respectively, we acquire

$$\Delta_1^{'} = 0, \Delta_2^{'} = 0. \tag{6.22}$$

Using the (6.18), (6.19) and (6.22) in (6.21), we obtain

$$\epsilon_3^* f' k_2^* N_2^* = P(s)N_1 + Q(s)N_3, \tag{6.23}$$

where

$$P(s) = \frac{\Psi\kappa_3}{f'^2(\alpha^2+1)\kappa_1^*}[\kappa_3(\alpha^2 - 1) + 2\alpha k_2], \tag{6.24}$$

$$Q(s) = \frac{\alpha\Psi\kappa_3}{f'^2(\alpha^2+1)\kappa_1^*}[\kappa_3(\alpha^2 - 1) + 2\alpha k_2]. \tag{6.25}$$

Since

$$\epsilon_3^* f' k_2^* N_2^* \neq 0.$$

So we get the result (6.3)

$$\kappa_3(\alpha^2-1)+2\alpha k_2 \neq 0. \tag{6.26}$$

Conversely, we suppose that $\Gamma: I \subset R \to E_1^4$ be a Cartan null curve with arc-length parameter s and κ_1, κ_2, κ_3 not equal to zero and the relations (6.1), (6.2) and (6.3) hold for constant real numbers Φ, Ψ, $\alpha \neq 0$ and β. Then we can define curve Γ^* like this

$$\Gamma^*(s^*)=\Gamma(s)+\Phi(s)T(s)+\Psi(s)N_2. \tag{6.27}$$

Taking derivative of equation (6.27) using equation (2.1), we acquire

$$\frac{d\Gamma^*}{ds}=(\Phi-\Psi\kappa_2)N_1+(\Psi\kappa_3)N_3. \tag{6.28}$$

From (6.1), we gacquire

$$\frac{d\Gamma^*}{ds}=(\alpha\Psi\kappa_3)N_1+(\Psi\kappa_3)N_3.$$

$$\frac{d\Gamma^*}{ds}=\Psi\kappa_3[\alpha N_1+N_3]. \tag{6.29}$$

From this

$$f'=\frac{ds^*}{ds}=||\frac{d\Gamma^*}{ds}||=c_1(\Psi\kappa_3)\sqrt{c_2(\alpha^2+1)}>0, \tag{6.30}$$

such that $c_1(\Psi k_3)>0$ where $c_1=\pm1$ and $c_2=\pm1$ such that $c_2(\alpha^2-1)>0$. Again writing the equation (6.29)

$$T^*f'=\Psi\kappa_3[\alpha N_1+N_3]. \tag{6.31}$$

Substituting (6.30) in (6.31), we get

$$T^*=\frac{c_1}{\sqrt{c_2(\alpha^2+1)}}[\alpha N_1+N_3], \tag{6.32}$$

which indicates that $H(T^*,T^*)=c_2=\epsilon_1^*$.
Taking derivative of the equation (6.32) s using equation (2.1), we acquire

$$\frac{dT^*}{ds^*}=\frac{c_1}{f'\sqrt{c_2(\alpha^2+1)}}[(\alpha\kappa_2-\kappa_3)T-\alpha N_2]. \tag{6.33}$$

Using (6.33), we get

$$k_1^*=||\frac{dT^*}{ds}||=\frac{\sqrt{-2(\alpha^2\kappa_2-\alpha\kappa_3)}}{f'\sqrt{c_2(\alpha^2+1)}}>0 \tag{6.34}$$

From (6.33) and (6.34), we have

$$N_1^*=\frac{1}{\kappa_1^*}\frac{dT^*}{ds^*}=\frac{c_1}{\sqrt{-2(\alpha^2\kappa_2-\alpha\kappa_3)}}[(\alpha\kappa_2-\kappa_3)T-\alpha N_2], \tag{6.35}$$

which indicates that $H(N_1^*,N_1^*)=1$.
If we denote

$$\Delta_3=\frac{c_1(\alpha\kappa_2-\kappa_3)}{\sqrt{-2(\alpha^2\kappa_2-\alpha\kappa_3)}},\quad \Delta_4=-\frac{c_1\alpha}{\sqrt{-2(\alpha^2\kappa_2-\alpha\kappa_3)}}. \tag{6.36}$$

Using (6.36) in (6.35), we get

$$N_1^*=\Delta_3T+\Delta_4N_2. \tag{6.37}$$

Taking derivative of (6.37) using equation (2.1), we acquire

$$f'\frac{dN_1^*}{ds^*}=\Delta_3'T+\Delta_4'N_2+(\Delta_3-\Delta_4\kappa_2)N_1+\Delta_4\kappa_3N_3. \tag{6.38}$$

Differentiating (6.2), we get

$$(\alpha\kappa_2'-\kappa_3')=0. \tag{6.39}$$

Taking derivative of equation (6.36) with respect to s using (6.39), we acquire

$$\Delta_3^{'} = 0, \quad \Delta_4^{'} = 0. \tag{6.40}$$

Substituting (6.40) and (6.36) in (6.38), we get

$$\frac{dN_1^*}{ds^*} = \frac{c_1(2\alpha\kappa_2 - \kappa_3)}{f'\sqrt{-2(\alpha^2\kappa_2 - \alpha\kappa_3)}}N + \frac{c_1\alpha\kappa_3}{f'\sqrt{-2(\alpha^2\kappa_2 - \alpha\kappa_3)}}N_3. \tag{6.41}$$

From (6.32) and (6.34), we get

$$\epsilon_1^*\kappa_1^*T^* = \frac{c_1\sqrt{-2(\alpha^2\kappa_2 - \alpha\kappa_3)}}{f'(\alpha^2+1)}[\alpha N_1 + N_3]. \tag{6.42}$$

From (6.41) and (6.42), we get

$$\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^*T^* = \frac{c_1(2\alpha\kappa_2 + \alpha^2\kappa_3 - \kappa_3)}{f'(\alpha^2+1)\sqrt{-2\alpha^2\kappa_2 + 2\alpha\kappa_3}}[N_1 + \frac{1}{\alpha}N_3], \tag{6.43}$$

From (6.43), we have

$$k_2^* = \frac{|2\alpha\kappa_2 + \alpha^2\kappa_3 - \kappa_3|}{f\sqrt{|\alpha|}(\alpha^2+1)\sqrt{-2\alpha^2\kappa_2 + 2\alpha\kappa_3}} > 0. \tag{6.44}$$

Considering (6.43) and (6.44) together, we obtain

$$N_2^* = \frac{\epsilon_3^*}{\kappa_2^*}[\frac{dN_1^*}{ds^*} + \epsilon_1^*\kappa_1^*T^*] = \frac{c_2c_3\epsilon_3^*\sqrt{|\alpha|}}{2}[N_1 + \frac{1}{\alpha}N_3], \tag{6.45}$$

where $c_3 = \frac{|2\alpha\kappa_2+\alpha^2\kappa_3-\kappa_3|}{|2\alpha\kappa_2+\alpha^2\kappa_3-\kappa_3|} = \pm 1$ and $\epsilon_3^* = \pm$. From (2.52) $H(N_2^*, N_2^*) = c_1 = \epsilon_3^* = -\epsilon_1^*$, also unit vector N_3^* can be expressed like this $N_3^* = -\Delta_4 T + \Delta_3 N_2$; that is,

$$N_3^* = \frac{c_2(\alpha\kappa_2 - \kappa_3)}{\sqrt{-2(\alpha^2\kappa_2 - \alpha\kappa_3)}}T + \frac{c_2\alpha}{\sqrt{-2(\alpha^2\kappa_2 - \alpha\kappa_3)}}N_2, \tag{6.46}$$

which indicates that $H(N_3^*, N_3^*) = 1$. In the end we find κ_3^* as,

$$\kappa_3^* = H(\frac{dN_2^*}{ds^*}, N_3^*) = \frac{c_3\epsilon_3^*(2\sqrt{|\alpha|}\kappa_3}{f'\sqrt{-2(\alpha^2\kappa_2 - \alpha\kappa_3)}} \neq 0.$$

So we find that Γ^* is spacelike or timelike curve and a (0,2)-Evolute curve of the curve Γ considering $\text{span}\{T, N_2\} = \text{span}\{N_1^*, N_3^*\}, \qquad \text{span}\{N_1, N_3\} = \text{span}\{T^*, N_2^*\}$.

Conclusion: In present study, we established a new kind of generalized evolutes and involutes curve in 4-Dimensional Minkowski space. We obtain necessary and sufficient condition for the curve possessing generalized Evolute as well as an Involute curve. Many researchers have developed extensive significant research contribution in the field of general theory of the curves in Euclidean space as well as in Minkowski space. However the special characters of the curve are not considered which is a research gap in this technique. In this article, we described a new type of (1,3)-Evolute curve in 4-Dimensional Minkowski space. We introduced several theorems with necessary and sufficient conditions and obtained interesting results. The understanding of Evolute curves with this type of Evolute and Involute curve, researchers will do more research in 4-dimensional Minkowski space. Evolute curves are used in mathematics and different branches of engineering this work maybe help full for researchers for future studies. In the future, we plan to improve our proposed framework for involutes of order K of a null Cartan curve in Minkowski spaces.

References

[1] R. S. Millman and G. D. Parker, Elements of Differential Geometry, Prentice-Hall, Englewood Cliffs, NJ, USA, 1977.

[2] C. Boyer, A History of Mathematics, New York: Wiley, 1968.

[3] Suleyman Senyurt, Seyda Kihcoglu. On the Differential Geometric Elements of the Involute D-Scroll in E^3, Adv. Appl. Clifford Algebras **25**, 977–988 (2015).

[4] Ozyilmaz, Emin and Suha Yilmaz,Involute-evolute curve couples in the euclidean 4-Space, Int. J. Open Problems Compt. Math **2**, 168–174 (2009).

[5] Marcos Craizer, Iteration of involutes of constant width curves in the Minkowski plane, Beitr Algebra Geom **55**, 479–496 (2014).

[6] Saint Venant B. Memoire sur les lignes courbes non planes, Journal de lEcole Polytechnique (in French) **18**, 1–76 (1845).

[7] Bertrand JM, Memoire sur la theorie des courbes a double courbure, Comptes Rendus (in French) **15**, 332–350 (1850).

[8] Hanif, M, Hou, Z. H, Generalized involute and evolute curve-couple in Euclidean space, Int. J. Open Problems Compt. Math **15**, (2018)

[9] Bukcu. B, Murat K Karacan, On the involute and evolute curves of spacelike curves with spacelike principal normal in Minkowski 3-space, Int. J. of mathematical combinatorics **1**, 27–37 (2009).

[10] Bukcu. B, Murat K Karacan, On the Involute and Evolute Curves of the Spacelike Curve with a Spacelike Binormal in Minkowski 3-Space, Int. J. Contemp. Math. Sciences **5**, 221–232 (2007).

[11] Dae Won Yoon, On the evolute offsets of ruled surfaces in Minkowski 3-space, Turk J Math **40**, 594–604 (2016).

[12] Engin As and Ayhan Sarioglugil, On the Bishop curvatures of Involute-Evolute curve couple in E^3 **9**, 140–145 (2014).

[13] Hanif, M, Hou, Z, Nisar, K, On Special Kinds of Involute and Evolute Curves in 4-Dimensional Minkowski Space, Symmetry **10**, (2018).

[14] Haiou Yu, Donghe Pei, Xiupeng Cui, Evolutes of fronts on Euclidean 2-sphere, J. Nonlinear Sci **8**, 678–686 (2015).

[15] M. Turgut, S. Yilmaz, On the frenet frame and a characterization of spacelike involute-evolute curve couple in Minkowski space-time, Int. Math. forum **16**, 2889–2905 (2010).

[16] T. Fukunaga, Masatomo Takahashi, Evolutes and Involutes of fronts in Euclidean plane, Journal of singularities **10**, 92–107 (2014).

[17] Ilarslan K, Nesovic E. Spacelike and timelike normal curves in Minkowski space-time. Publications de Linstitut Math **85**, 111-118 (2009).

[18] O'Neill B, Semi-Riemannian Geometry with applications to relativity. New York, NY, USA:Academic Press, 1983.

[19] Bonnor WB, Null curves in minkowski space-time, Tensor **20**, 229-242 (1969).

[20] Bonnor WB, Curves with null normals in minkowski space-time, Dadhich NJ, editor, A Random Walk in relativity and Cosmology, New York, NY, USA: Wiley 33-47, (1950).

Author information

Muhammad Hanif and Zhong Hua Hou, School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China.
E-mail: `hanifmayo84@gmail.com, zhhou@dlut.edu.cn`

Received: July 18, 2018.
Accepted: August 29, 2018