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GENERALIZATION OF PASCUE-TYPE p-VALENT FUNCTIONS ASSOCIATED WITH THE CONVOLUTION STRUCTURE

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Abstract. In the present paper, we introduce a new class of p-valent functions by making use of convolution structure and study some of their interesting properties such as coefficient bounds, inclusion relation, distortion inequalities, extreme points and integral means inequalities.

1 Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=2p+1}^{\infty} a_k z^k, \qquad (p \in N = \{1, 2, 3, ..., \}) . \tag{1.1}$$

which are analytic and p-valent in the open unite disk $U = \{z : z \in C \text{ set of all complex numbers and } |z| < 1\}$. A function $f \in A_p$ is β-pascue convex of order α if

$$\frac{1}{p} \operatorname{Re} \left[\frac{(1-\beta) z f'(z) + \frac{\beta}{p} z \left(z f'(z) \right)'}{(1-\beta) f(z) + \frac{\beta}{p} f'(z)} \right] > \alpha \qquad (0 \le \beta \le 1, 0 \le \alpha < 1) .$$

In other words $(1-\beta) f(z) + \frac{\beta}{p} f'(z)$ is in $f \in S_p^*$ the class of p-valent starlike functions.(for details [6],see also [1],[5]).

Given two functions $f, g \in A_p$,where f is given by (1.1) and g is given by

$$g(z) = z^p + \sum_{k=2p+1}^{\infty} b_k z^k \qquad (p \in N) ,$$

the Hardmard product (or convolution) $f * g$ is defined by

$$(f * g)(z) = z^p + \sum_{k=2p+1}^{\infty} a_k b_k z^k = (g * f)(z) , \qquad z \in U. \tag{1.2}$$

For functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U written as

$f(z) \prec g(z) \qquad (z \in U)$,

if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that

$f(z) = g(w(z)) \qquad (z \in U)$.

In particular, if the function g is univalent in U, the above sobordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

See also Duren [2].

Salagean [7] introduced the following operator which is popularly known as the Salagean derivative operator :

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$

....

$$D^{n} f(z) = D\left(D^{n-1} f(z)\right) \qquad (n \in N_0 = N \cup \{0\}).$$

We easily find from (1.1) that

$$D^{n} f(z) = p^{n} z^{p} + \sum_{k=2p+1}^{\infty} k^{n} a_k z^{k} \qquad (f \in A_p; n \in N_0).$$

We denote by T_p the subclass of A_p consisting of functions of the form

$$f(z) = z^{p} - \sum_{k=2p+1}^{\infty} a_k z^{k}, \qquad (a_k \geq 0, p \in N). \tag{1.3}$$

which are p-valent in U.

For a given function $g \in A_p$ defined by

$$g(z) = z^{p} + \sum_{k=2p+1}^{\infty} b_k z^{k}, \qquad (b_k \geq 0, p \in N). \tag{1.4}$$

We introduce here a new class $AS_g^*(m,n,p,\alpha,\beta)$ of functions belonging to the class T_p which consists of functions $f(z)$ of the form (1.3) satisfying the following inequality :

$$\frac{1}{p}\operatorname{Re}\left[\frac{(1-\beta) D^{n+m}(f*g)(z) + \frac{\beta}{p} D^{n+m+1}(f*g)(z)}{(1-\beta) D^{n+m-1}(f*g)(z) + \frac{\beta}{p} D^{n+m}(f*g)(z)}\right] > \alpha \tag{1.5}$$

where $(0 \leq \beta \leq 1, 0 \leq \alpha < 1, m, n, p \in N)$.

We note that for $m = 1$, this class was introduced and studied by Birgul Oner and Sevtap Sumer Eker [4].

In this paper, we determine the coefficient inequalities, distortion theorem as well as integral means inequalities for functions in the class $AS_g^*(m,n,p,\alpha,\beta)$.

2 Coefficient inequalities and some inclusion relations

We first prove a necessary and sufficient condition for functions to be in the class $AS_g^*(m,n,p,\alpha,\beta)$ as following:

Theorem 2.1. *A function $f(z)$ given by (1.3) is in $AS_g^*(m,n,p,\alpha,\beta)$ if and only if for* $0 \leq \alpha < 1, 0 \leq \beta \leq 1, m, n, p \in N$,

$$\sum_{k=2p+1}^{\infty} \left[(k-\alpha p)(p-\beta p+\beta k)\right] k^{n+m-1} a_k b_k \quad \leq \quad p^{n+m+1}(1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. Assume that $f \in AS_g^*(m,n,p,\alpha,\beta)$.Then, in view of (1.3) to (1.5), we have

$$\frac{1}{p}\operatorname{Re}\left[\frac{(1-\beta) D^{n+m}(f*g)(z) + \frac{\beta}{p} D^{n+m+1}(f*g)(z)}{(1-\beta) D^{n+m-1}(f*g)(z) + \frac{\beta}{p} D^{n+m}(f*g)(z)}\right]$$

$$= \frac{1}{p}\operatorname{Re}\left[\frac{(1-\beta) p^{n+m} z^{p} - \sum_{k=2p+1}^{\infty}(1-\beta) k^{n+m} a_k b_k z^{k} + \frac{\beta}{p}\left(p^{n+m+1} z^{p} - \sum_{k=2p+1}^{\infty} k^{n+m+1} a_k b_k z^{k}\right)}{(1-\beta) p^{n+m-1} z^{p} - \sum_{k=2p+1}^{\infty}(1-\beta) k^{n+m-1} a_k b_k z^{k} + \frac{\beta}{p}\left(p^{n+m} z^{p} - \sum_{k=2p+1}^{\infty} k^{n+m} a_k b_k z^{k}\right)}\right]$$

$$= \frac{1}{p}\,\mathrm{Re}\left[\frac{p^{n+m}z^{p}-\sum_{k=2p+1}^{\infty}\left(1-\beta+\frac{\beta}{p}k\right)k^{n+m}a_{k}b_{k}z^{k}}{(1-\beta)\,p^{n+m-1}z^{p}-\sum_{k=2p+1}^{\infty}\left(1-\beta+\frac{\beta}{p}k\right)k^{n+m-1}a_{k}b_{k}z^{k}}\right]>\alpha$$

If we choose z to be real and let $r\to 1^{-}$, the last inequality leads us to desired to assertion (2.1) of Theorem 2.1 .

Conversely, assume that (2.1) holds for $f(z)\in A_{p}$, let us define the function F(z) by

$$F(z)=\frac{1}{p}\frac{(1-\beta)\,D^{n+m}\,(f*g)\,(z)+\frac{\beta}{p}D^{n+m+1}\,(f*g)\,(z)}{(1-\beta)\,D^{n+m-1}\,(f*g)\,(z)+\frac{\beta}{p}D^{n+m}\,(f*g)\,(z)}-\alpha$$

it suffices to show thats

$$\left|\frac{F(z)-1}{F(z)+1}\right|<1 \qquad (z\in U)\,.$$

We note that

$$\left|\frac{F(z)-1}{F(z)+1}\right|$$

$$=\left|\frac{(1-\beta)\,D^{n+m}\,(f*g)\,(z)+\frac{\beta}{p}D^{n+m+1}\,(f*g)\,(z)-p\,(\alpha+1)\left[(1-\beta)\,D^{n+m-1}\,(f*g)\,(z)+\frac{\beta}{p}D^{n+m}\,(f*g)\,(z)\right]}{(1-\beta)\,D^{n+m}\,(f*g)\,(z)+\frac{\beta}{p}D^{n+m+1}\,(f*g)\,(z)-p\,(\alpha-1)\left[(1-\beta)\,D^{n+m-1}\,(f*g)\,(z)+\frac{\beta}{p}D^{n+m}\,(f*g)\,(z)\right]}\right|$$

$$=\left|\frac{-\alpha p^{n+m}-\sum_{k=2p+1}^{\infty}(k-\alpha p-p)\left(1-\beta+\frac{\beta}{p}k\right)k^{n+m-1}a_{k}b_{k}z^{k-p}}{(2-\alpha)p^{n+m}-\sum_{k=2p+1}^{\infty}(k-\alpha p-p)\left(1-\beta+\frac{\beta}{p}k\right)k^{n+m-1}a_{k}b_{k}z^{k-p}}\right|$$

The last expression is bounded above by 1, if

$$\alpha p^{n+m+1}+\sum_{k=2p+1}^{\infty}\left[(k-\alpha p-p)\,(p-\beta p+\beta k)\right]k^{n+m-1}a_{k}b_{k}$$

$$\leq\ (2-\alpha)\,p^{n+m+1}-\sum_{k=2p+1}^{\infty}\left[(k-\alpha p-p)\,(p-\beta p+\beta k)\right]k^{n+m-1}a_{k}b_{k}$$

which is equivalent to our condition (2.1). This completes the proof of our theorem.

The result is sharp for the functions
$f\,(z)=z^{p}-\frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]k^{n+m-1}b_{k}}z^{k}.$

Corollary 2.2. *Let $f(z)$ given by (1.3). If $f\in AS_{g}^{*}(m,n,p,\alpha,\beta)$, then*

$$a_{k}\leq\frac{p^{n+m+1}\,(1-\alpha)}{\left[(k-\alpha p)\,(p-\beta p+\beta k)\right]k^{n+m-1}b_{k}}$$

with equality only for functions of the form

$$f_{k}\,(z)=z^{p}-\frac{p^{n+m+1}\,(1-\alpha)}{\left[(k-\alpha p)\,(p-\beta p+\beta k)\right]k^{n+m-1}b_{k}}z^{k}.$$

Proof. If $f\in AS_{g}^{*}(m,n,p,\alpha,\beta)$, then by making use of (2.1), we obtain

$$\left[(k-\alpha p)\,(p-\beta p+\beta k)\right]k^{n+m-1}a_{k}b_{k}\leq\sum_{k=2p+1}^{\infty}\left[(k-\alpha p)\,(p-\beta p+\beta k)\right]k^{n+m-1}a_{k}b_{k}$$

$$\leq\ p^{n+m+1}\,(1-\alpha)$$

or

$$a_k \le \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k}$$

clearly for

$$f_k(z) = z^p - \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k} z^k,$$

we have

$$a_k \le \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k}.$$

Theorem 2.3. *Let* $0 \le \alpha_1 \le \alpha_2 < 1, m, n, p \in N$ *and* $0 \le \beta \le 1$ *then*

$$AS_g^*(m,n,p,\alpha_1,\beta) \supseteq AS_g^*(m,n,p,\alpha_2,\beta).$$

Proof. Let the function $f(z)$ defined by (1.3) in the class $AS_g^*(m,n,p,\alpha_2,\beta)$. Then by the Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty}[(k-\alpha_2 p)(p-\beta p+\beta k)]\,k^{n+m-1}a_k b_k \quad \le \quad p^{n+m+1}(1-\alpha_2)$$

consequently

$$\sum_{k=2p+1}^{\infty}[(k-\alpha_1 p)(p-\beta p+\beta k)]\,k^{n+m-1}a_k b_k \le \sum_{k=2p+1}^{\infty}[(k-\alpha_2 p)(p-\beta p+\beta k)]\,k^{n+m-1}a_k b_k$$

$$\le \quad p^{n+m+1}(1-\alpha_2).$$

This completes the proof of the Theorem 2.3 with the aid of the Theorem 2.1.

Theorem 2.4. *Let* $0 \le \alpha < 1, m, n, p \in N$ *and* $0 \le \beta \le 1$ *then*

$$AS_g^*(m,n+1,p,\alpha,\beta) \subseteq AS_g^*(m,n,p,\alpha,\beta).$$

Proof. Let the function $f(z)$ defined by (1.3) in the class $AS_g^*(m,n+1,p,\alpha,\beta)$. Then by the Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty}[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m}a_k b_k \quad \le \quad p^{n+m+2}(1-\alpha)$$

consequently

$$\sum_{k=2p+1}^{\infty}[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}a_k b_k \le \sum_{k=2p+1}^{\infty}[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m}a_k b_k$$

$$\le \quad p^{n+m+2}(1-\alpha).$$

This completes the proof of the Theorem 2.4 with the aid of the Theorem 2.1.

3 Disortion Inequalities

In this section, we shall prove distortion theorems for the functions belonging to the class $AS_g^*(m,n,p,\alpha,\beta)$.

Theorem 3.1. *Let the functions* $f(z)$ *of the form (1.3) be in the class* $AS_g^*(m,n,p,\alpha,\beta)$. *Then for* $|z| = r < 1$, *we have*

$$|f(z)| \ge r^p - \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)]\,(2p+1)^{n+m-1}\,b_{2p+1}} r^{2p+1} \tag{3.1}$$

and

$$|f(z)| \le r^p + \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-1} b_{2p+1}} r^{2p+1}. \tag{3.2}$$

The inequalities (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-1} b_{2p+1}} z^{2p+1}.$$

Proof. Since $f(z) \in AS_g^*(m,n,p,\alpha,\beta)$,we apply Theorem 2.1, we obtain

$$(2p+1-\alpha p)(p+\beta p+\beta)(2p+1)^{n+m-1} b_{2p+1} \sum_{k=2p+1}^{\infty} a_k$$

$$\le \sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)] k^{n+m-1} a_k b_k \quad \le \quad p^{n+m+1}(1-\alpha).$$

Thus, we obtain

$$\sum_{k=2p+1}^{\infty} a_k \le \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-1} b_{2p+1}}. \tag{3.3}$$

From (1.3) and (3.3), we have

$$|f(z)| \le |z|^p + |z|^{2p+1} \sum_{k=2p+1}^{\infty} a_k \le r^p + \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-1} b_{2p+1}} r^{2p+1}$$

$$|f(z)| \ge |z|^p - |z|^{2p+1} \sum_{k=2p+1}^{\infty} a_k \ge r^p - \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-1} b_{2p+1}} r^{2p+1}.$$

This completes the proof of Theorem 3.1.

Theorem 3.2. *Let the functions $f(z)$ of the form (1.3) be in the class $AS_g^*(m,n,p,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$\left|f'(z)\right| \ge pr^{p-1} - \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-2} b_{2p+1}} r^{2p} \tag{3.4}$$

and

$$\left|f'(z)\right| \le pr^p + \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-2} b_{2p+1}} r^{2p}. \tag{3.5}$$

The inequalities (3.4) and (3.5) are attained for the function $f(z)$ given by

$$f(z) = z^{p-1} - \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-2} b_{2p+1}} z^{2p}.$$

Proof. From Theorem 2.1 and (3.3), we have

$$\sum_{k=2p+1}^{\infty} k a_k \le \frac{p^{n+m+1}(1-\alpha)}{[(2p+1-\alpha p)(p+\beta p+\beta)](2p+1)^{n+m-2} b_{2p+1}}.$$

and the remaining part of the proof is similar to the proof of the Theorem 3.1 .

4 Extreme Points

Theorem 4.1. *Let $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} b_k} z^k$$

$$(b_k \geq 0, 0 \leq \beta \leq 1, 0 \leq \alpha < 1, m, n, p \in N).$$

Then $f(z) \in AS_g^*(m,n,p,\alpha,\beta)$ if and only if it can be expressed in the following form

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_p \geq 0, \lambda_k \geq 0$ and $\lambda_p + \sum_{k=2p+1}^{\infty} \lambda_k = 1.$

Proof. Suppose that

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z) = z^p - \sum_{k=2p+1}^{\infty} \lambda_k \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} b_k} z^k.$$

Then from Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty} [(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} \lambda_k \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} b_k} z^k$$

$$= \sum_{k=2p+1}^{\infty} \lambda_k p^{n+m+1}(1-\alpha) \leq p^{n+m+1}(1-\alpha)(1-\lambda_p) \leq p^{n+m+1}(1-\alpha).$$

Thus , in view of Theorem 2.1, we find that $f(z) \in AS_g^*(m,n,p,\alpha,\beta)$.

Conversely, suppose that $f(z) \in AS_g^*(m,n,p,\alpha,\beta)$. Then, since

$$a_k \leq \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} b_k} \qquad (p \in N),$$

we may set

$$\lambda_k = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} b_k}{p^{n+m+1}(1-\alpha)} a_k \qquad (p \in N)$$

and

$$\lambda_p = 1 - \sum_{k=2p+1}^{\infty} \lambda_k.$$

Thus, clearly, we have

$$f(z) = \lambda_p z^p + \sum_{k=2p+1}^{\infty} \lambda_k f_k(z).$$

This completes the proof of theorem.

Corollary 4.2. *The extreme points of the class $AS_g^*(m,n,p,\alpha,\beta)$ are given by*

$$f_p(z) = z^p$$

and

$$f_k(z) = z^p - \frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\, k^{n+m-1} b_k} z^k, \qquad (k \geq 2p+1, p \in N). \tag{4.1}$$

Theorem 4.3. *The class $AS_g^*(m,n,p,\alpha,\beta)$ is a convex set .*

Proof. Suppose that each of the functions $f_i(z), \quad (i = 1, 2)$ given by

$$f_i(z) = z^p - \sum_{k=2p+1}^{\infty} a_{k,i} z^k, \qquad (a_{k,i} \geq 0)$$

is in the class $AS_g^*(m, n, p, \alpha, \beta)$. It is sufficient to show that the function $g(z)$ defined by

$$g(z) = \eta f_1(z) + (1 - \eta) f_2(z), \qquad (0 \leq \eta < 1)$$

is also in the class $AS_g^*(m, n, p, \alpha, \beta)$. Since

$$g(z) = \eta \left(z^p - \sum_{k=2p+1}^{\infty} a_{k,1} z^k \right) + (1 - \eta) \left(z^p - \sum_{k=2p+1}^{\infty} a_{k,2} z^k \right)$$

$$= z^p - \sum_{k=2p+1}^{\infty} \left[\eta a_{k,1} + (1 - \eta) a_{k,2} \right] z^k$$

with the aid of Theorem 2.1, we have

$$\sum_{k=2p+1}^{\infty} \left[(k - \alpha p) (p - \beta p + \beta k) \right] k^{n+m-1} \left[\eta a_{k,1} + (1 - \eta) a_{k,2} \right] b_k$$

$$= \eta \sum_{k=2p+1}^{\infty} \left[(k - \alpha p) (p - \beta p + \beta k) \right] k^{n+m-1} a_{k,1} b_k + (1 - \eta) \sum_{k=2p+1}^{\infty} \left[(k - \alpha p) (p - \beta p + \beta k) \right] k^{n+m-1} a_{k,2} b_k$$

$$\leq \eta p^{n+m+1} (1 - \alpha) + (1 - \eta) p^{n+m+1} (1 - \alpha) \;=\; p^{n+m+1} (1 - \alpha).$$

5 Integral Means Inequalities

In 1925, Littelewood prove the following subordination lemma.

Lemma 5.1. *(Littlewood [3]) If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^{\mu} \, d\theta \;\leq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta.$$

We will make use of Lemma 5.1 to prove the following theorem.

Theorem 5.2. *Let $f(z) \in AS_g^*(m, n, p, \alpha, \beta)$ and $f_k(z)$ is defined by (4.1). If there exist an analytic function $w(z)$ given by*

$$[w(z)]^{k-p} = \frac{\left[(k - \alpha p) (p - \beta p + \beta k) \right] k^{n+m-1} b_k}{p^{n+m+1} (1 - \alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p},$$

then for $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\mu} d\theta \;\leq \int_0^{2\pi} \left| g(re^{i\theta}) \right|^{\mu} d\theta. \qquad (\mu > 0).$$

Proof. We must show that

$$\int_0^{2\pi} \left| 1 - \sum_{k=2p+1}^{\infty} a_k z^{k-p} \right|^{\mu} d\theta \;\leq \int_0^{2\pi} \left| 1 - \frac{p^{n+m+1} (1 - \alpha)}{\left[(k - \alpha p) (p - \beta p + \beta k) \right] k^{n+m-1} b_k} z^{k-p} \right|^{\mu} d\theta.$$

By applying Littlewood's subordination lemma, it would suffice to show that

$$1-\sum_{k=2p+1}^{\infty} a_k z^{k-p} \quad \prec \quad 1-\frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k} z^{k-p}.$$

By setting

$$1-\sum_{k=2p+1}^{\infty} a_k z^{k-p} = 1-\frac{p^{n+m+1}(1-\alpha)}{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k}\,[w(z)]^{k-p},$$

we find that

$$[w(z)]^{k-p} = \frac{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k}{p^{n+m+1}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p}$$

which readily yields $w(0)=0$.

Furthermore, using(2.1) we obtain

$$|w(z)|^{k-p} = \left| \frac{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k}{p^{n+m+1}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k z^{k-p} \right|$$

$$\leq \frac{[(k-\alpha p)(p-\beta p+\beta k)]\,k^{n+m-1}b_k}{p^{n+m+1}(1-\alpha)} \sum_{k=2p+1}^{\infty} a_k |z|^{k-p}$$

$$\leq |z|^{k-p} < 1.$$

This completes the proof of the theorem.

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