Palestine Journal of Mathematics

Vol. 8(1)(2019), 71–85

Approximation by Double Cesàro Submethods of Double Fourier Series for Lipschitz Fuctions

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Communicated by Jawad Abuhlail

MSC 2010 Classifications: Primary 40C05, 42A25; Secondary 42A10, 44A9.

Keywords and phrases: Double Sequence, Fourier Series, Submethod, Lipschitz class, Dirichlet kernel.

Abstract In this paper we investigate the rate of uniform approximation by double Cesàro submethod of the rectangular partial sums of the double Fourier series of a function $f(x,y)$ belonging to the class $Lip\alpha$, $0<\alpha\leq 1$, on the two dimensional torus $-\pi<x,y\leq\pi$. We extend two theorems on four dimensional Cesàro submethods of Moricz [6] et al. (1984), where they have extended two theorems of Goel [2] and Holland [4] using d-dimensional cases, where d is an integer greater than 2, are straightforward.

1 Introduction

Some equivalance results for Cesàro submethods have been studied by Goffman and Petersen [3], Armitage and Maddox [1] and Osikiewicz[7]. Ünver, in [8], continued to work on four dimensional Cesàro submethods and to give some inclusion results. Moricz [6] have extend some results of Goel [2] and Holland [4] to general matrices. The aim of our paper generalize the two theorems of Moricz[6] under weaker assumptions and give sharper estimates.

Let $\{p_{jk}\ :\ j,k=0,1,\ldots\}$ be a double sequence of nonnegative numbers $p_{00}>0$. Its partial sum is defined as

$$P_{mn}=\sum_{j=0}^{m}\sum_{k=0}^{n}p_{jk}\qquad (m,n=0,1,\ldots).$$

$\{s_{jk}\ :\ j,k=0,1,\ldots\}$ be a double sequence of complex numbers, the Nörlund means N_{mn} are defined by

$$N_{mn}=\frac{1}{P_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n}p_{m-j,n-k}s_{jk}.$$

Let $f(x,y)$ be a complex valued function of period 2π with respect to each of the variable and integrable defined on the two dimensional real torus $Q:=\{(x,y)\in\mathbb{R}\ :\ -\pi<x\leq\pi,\ \ -\pi<y\leq\pi\}$. i.e., $f\in L^1(Q)$. We consider the double Fourier series of f defined by

$$f(x,y)\approx\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}c_{jk}e^{i(jx+ky)}\tag{1.1}$$

where

$$c_{jk}=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)e^{-i(ju+kv)}dudv\qquad (j,k=\ldots,-1,0,1,\ldots).$$

We write the double sequence of (symmetric) rectangular partial sums for the series (1.1) as follows:

$$s_{mn}(x,y)=\sum_{j=-m}^{m}\sum_{k=-n}^{n}c_{jk}e^{i(jx+ky)}\qquad (m,n=0,1,\ldots).$$

We say that the function f belongs to Lipschitz class of order α for some $\alpha>0$, if

$$\begin{aligned} w(\delta;f) &= \sup_{(x,y)\in Q}\ \sup_{\{u^2+v^2\}^{1/2}\leq\delta}|f(x+u,y+v)-f(x,y)| \\ &\leq C\delta^{\alpha}\qquad (\delta>0)\end{aligned}\tag{1.2}$$

where the constant C does not depend on δ. The quantity $w(\delta; f)$ is called the (total) modulus of continuity of the function f.

Clearly, if $f \in Lip\alpha$ for some $\alpha > 0$, then f is necessarily continuous everywhere. Only the case $0 < \alpha \leq 1$ is interesting. If $\alpha > 1$, then $\partial f/\partial x$ and $\partial f/\partial y$ exist and are zero everywhere, so f must be a constant.

Condition (1.2) can be rewritten as

$$|f(x+u, y+v) - f(x,y)| \leq C\{u^2 + v^2\}^{\alpha/2}$$

for every real x, y, u and v; or equivalently,

$$|f(x+u, y+v) - f(x,y)| \leq C(|u|^\alpha + |v|^\alpha).$$

From these inequalities obviously yields

$$|\phi(u,v)| \leq C(|u|^\alpha + |v|^\alpha). \tag{1.3}$$

1.1 Estimation of The Kernel

We will use some well-known estimates. $D_j(u)$ is the Dirichlet kernels in terms of u;

$$D_j(u) = \frac{1}{2} + \sum_{\sigma=1}^{j} cos\sigma u = \frac{sin(j+\frac{1}{2})u}{2sin\frac{1}{2}u} \qquad (j = 0, 1, \ldots).$$

For $j = 0, 1, \ldots$

$$|D_j(u)| < j + 1 \qquad \text{for every } u. \tag{1.4}$$

For $a, b = 0, 1, \ldots;\ a \leq b$,

$$\sum_{j=a}^{b} sin\left(j + \frac{1}{2}\right) u = \frac{cosau - cos(b+1)u}{2sin\frac{1}{2}u},$$

whence, on account of the inequality

$$\frac{sinu}{u} \geq \frac{2}{\pi} \qquad 0 < u \leq \frac{\pi}{2},$$

we obtain

$$\left| \sum_{j=a}^{b} sin\left(j + \frac{1}{2}\right) u \right| \leq \frac{\pi}{u} \qquad \text{for} \quad 0 < u \leq \pi. \tag{1.5}$$

Similarly,

$$\begin{aligned} \left| \sum_{j=a}^{b} D_j(u) \right| &= \frac{cosau - cos(b+1)u}{(2sin\frac{1}{2}u)^2} \\ &\leq \frac{\pi^2}{2u^2} \qquad \text{for} \quad 0 < u \leq \pi. \end{aligned} \tag{1.6}$$

We recall that the double index sequence $\beta = \beta(m,n)$ is defined as $\beta(m,n) = (\lambda(m), \mu(n))$ where $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequence of positive integers. Let $x = (x_{jk})$ be a double sequence. Then, the Cesàro submethod $C_\beta := (C_\beta, 1, 1)$ is defined to be

$$(C_\beta x)_{mn} = \frac{1}{\lambda(m)\mu(n)} \sum_{(j,k)=(1,1)}^{(\lambda(m),\mu(n))} x_{jk}.$$

Since $\{(C_\beta x)_{mn}\}$ is a subsequence of $\{(C_1 x)_{mn}\}$, the method C_β is RH-regular for any β [8].

2 Main results

Motivating by Moricz [6] we make the following definition:

Definition 2.1. Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers and $\{p_{jk} : \ j,k = 0,1,\ldots\}$ be a double sequence of nonnegative numbers $p_{00} > 0$. Set

$$P_{\lambda(m)\mu(n)} := \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} p_{jk} \qquad (m,n = 0,1,\ldots).$$

$\{s_{jk} : \ j,k = 0,1,\ldots\}$ be any double sequence of complex numbers, the N_λ^μ-method is defined as

$$N_\lambda^\mu := \frac{1}{P_{\lambda(m)\mu(n)}} \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} p_{\lambda(m)-j,\mu(n)-k} s_{jk} \qquad (n,m = 0,1,\ldots).$$

It is clear that N_λ^μ method is regular.

The N_λ^μ method for the sequence $\{s_{mn}(x,y)\}$ are defined as

$$N_\lambda^\mu(x,y) = \frac{1}{P_{\lambda(m)\mu(n)}} \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} p_{\lambda(m)-j,\mu(n)-k} s_{jk}(x,y) \qquad (m,n = 0,1,\ldots).$$

The following representation is important,

$$N_\lambda^\mu(x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) K_\lambda^\mu(u,v) du dv \qquad (m,n = 0,1,\ldots), \tag{2.1}$$

where N_λ^μ- kernel $K_\lambda^\mu(u,v)$ is defined by

$$K_\lambda^\mu(u,v) = \frac{1}{P_{\lambda(m)\mu(n)}} \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} p_{\lambda(m)-j,\mu(n)-k} D_j(u) D_k(v) \qquad (m,n = 0,1,\ldots).$$

From (2.1), we have

$$N_\lambda^\mu(x,y) - f(x,y) = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \phi(u,v) K_\lambda^\mu(u,v) du dv$$

where

$$\begin{aligned} \phi(u,v) = \quad & \frac{1}{4} \{ f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) \\ & + f(x-u, y-v) - 4f(x,y) \} . \end{aligned}$$

We will use the notations

$$\Delta_{10} p_{jk} = p_{jk} - p_{j+1,k},$$
$$\Delta_{01} p_{jk} = p_{jk} - p_{j,k+1},$$

and

$$\Delta_{11} p_{jk} = p_{jk} - p_{j+1,k} - p_{j,k+1} + p_{j+1,k+1} \quad (j,k = 0,1,\ldots).$$

The double sequence $\{p_{jk}\}$ is nondecreasing if $\Delta_{10} p_{jk} \leq 0$, and is nonincreasing if $\Delta_{01} p_{jk} \geq 0$ and $\Delta_{10} p_{jk} \geq 0$ for every $j,k = 0,1,\ldots$. We also set,

$$h_{\lambda(m)\mu(n)} = \frac{1}{P_{\lambda(m)\mu(n)}} \sum_{k=0}^{\mu(n)} p_{\lambda(m),k}$$

$$t_{\lambda(m)\mu(n)} = \frac{1}{P_{\lambda(m)\mu(n)}} \sum_{j=0}^{\lambda(m)} p_{j,\mu(n)} \qquad (m,n = 0,1,\ldots)$$

Let us consider the case where $\{p_{jk}\}$ is nondecreasing. Then

$$\begin{aligned}(\lambda(m)+1)h_{\lambda(m)\mu(n)} &= \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} p_{\lambda(m),k} \\ &\geq \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} p_{jk} = 1 \qquad (2.2)\end{aligned}$$

and similarly,

$$(\mu(n)+1)t_{\lambda(m)\mu(n)} \geq 1. \qquad (2.3)$$

We have also

$$P_{\lambda(m)\mu(n)} \leq (\lambda(m)+1)(\mu(n)+1)p_{\lambda(m),\mu(n)} \qquad (m,n=0,1,\ldots).$$

In the sequel, we need the opposite inequality:

$$\frac{(\lambda(m)+1)(\mu(n)+1)p_{\lambda(m),\mu(n)}}{P_{\lambda(m)\mu(n)}} = O(1). \qquad (2.4)$$

The opposite inequality is provided for the following sequence defined below,

$$p_{jk} = (j+1)^{\gamma}(k+1)^{\rho} \qquad (\gamma,\rho \geq 1).$$

Condition (2.4) implies that

$$\begin{aligned}(\lambda(m)+1)h_{\lambda(m)\mu(n)} &= \frac{(\lambda(m)+1)}{P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)} p_{\lambda(m),k} \qquad (2.5)\\ &\leq \frac{(\lambda(m)+1)}{P_{\lambda(m)\mu(n)}}(\mu(n)+1)p_{\lambda(m),\mu(n)} = O(1) \qquad (2.6)\end{aligned}$$

and

$$(\mu(n)+1)t_{\lambda(m)\mu(n)} = O(1). \qquad (2.7)$$

In particular, the conditions of regularity are satisfied:

$$\lim_{n,m\to\infty} h_{\lambda(m)\mu(n)} = \lim_{n,m\to\infty} t_{\lambda(m)\mu(n)} = 0.$$

Thus, we may assume that

$$h_{\lambda(m)\mu(n)} < \pi \quad \text{and} \quad t_{\lambda(m)\mu(n)} < \pi \quad (m,n=0,1,\ldots).$$

The following result is an analogue of a result of Moore [[5], page 39] which almost follows the same lines, so the details are omitted.

Theorem 2.2. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers. If $\{p_{jk} \geq 0 :\ j,k = 0,1,\ldots,\ p_{00} > 0\}$, then the necessary and sufficient conditions for the regularity of the N_{λ}^{μ}-method of summability are*

$$\lim_{m,n\to\infty} \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)} p_{\lambda(m)-j,k} = 0 \qquad (j=0,1,\ldots;\lambda(m)\geq j)$$

and

$$\lim_{m,n\to\infty} \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)} p_{j,\mu(n)-k} = 0 \qquad (k=0,1,\ldots;\mu(n)\geq k).$$

Lemma 2.3. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers and $\{p_{jk} > 0 \ : \ j,k = 0,1,\ldots\}$ be a nondecreasing double sequence such that $\Delta_{11}p_{jk}$ is of fixed sign. Then*

$$|K_\lambda^\mu(u,v)| \le (\lambda(m)+1)(\mu(n)+1)\ , \quad \forall u,v \tag{2.8}$$

$$\le \frac{\pi^2}{2}\frac{1}{P_{\lambda(m)\mu(n)}u^2}\sum_{k=0}^{\mu(n)}(k+1)p_{\lambda(m),\mu(n)-k}\ , \quad \forall v \ \textit{and} \ 0<u\le\pi \tag{2.9}$$

$$\le \frac{\pi^2}{2}\frac{1}{P_{\lambda(m)\mu(n)}v^2}\sum_{j=0}^{\lambda(m)}(j+1)p_{\lambda(m)-j,\mu(n)}\ , \quad \forall u \ \textit{and} \ 0<v\le\pi \tag{2.10}$$

$$\le \frac{3\pi^4}{4u^2v^2}\frac{p_{\lambda(m)\mu(n)}}{P_{\lambda(m)\mu(n)}}\ , \quad 0<u,v\le\pi. \tag{2.11}$$

Proof. From (1.4),

$$\begin{aligned}
|K_\lambda^\mu(u,v)| &\le \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}p_{\lambda(m)-j,\mu(n)-k}|D_j(u)||D_k(v)| \\
&\le \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}(j+1)(k+1)p_{\lambda(m)-j,\mu(n)-k} \\
&\le \frac{(\lambda(m)+1)(\mu(n)+1)}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}p_{jk} \\
&= (\lambda(m)+1)(\mu(n)+1)
\end{aligned}$$

which is (2.8).

Again from (1.4), we have

$$\begin{aligned}
\left|P_{\lambda(m)\mu(n)}K_\lambda^\mu(u,v)\right| &\le \sum_{k=0}^{\mu(n)}\left|\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}D_j(u)\right||D_k(v)| \\
&\le \sum_{k=0}^{\mu(n)}(k+1)\left|\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}D_j(u)\right|. \qquad (2.12)
\end{aligned}$$

By Abel's transformation, we can rewrite the inner sum as follows, for each k

$$\begin{aligned}
\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}D_j(u) = \quad & p_{0,\mu(n)-k}\sum_{l=0}^{\lambda(m)}D_l(u) \\
& -\sum_{j=1}^{\lambda(m)}\Delta_{10}p_{\lambda(m)-j,\mu(n)-k}\sum_{l=0}^{j-1}D_l(u).
\end{aligned}$$

Because we regard $\{p_{jk}\}$ as nondecreasing in j and (1.6), we get

$$\begin{aligned}
\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}D_j(u) &\le \frac{\pi^2}{2u^2}\left(-\sum_{j=1}^{\lambda(m)}\Delta_{10}p_{\lambda(m)-j,\mu(n)-k}+p_{0,\mu(n)-k}\right) \\
&= \frac{\pi^2}{2u^2}p_{\lambda(m),\mu(n)-k}. \qquad (2.13)
\end{aligned}$$

(2.12) and (2.13) yields (2.9). (2.10) can be shown in similar way.

We perform a double Abel's transformation to prove (2.11):

$$\begin{aligned} P_{\lambda(m)\mu(n)} K_{\lambda}^{\mu}(u,v) &= \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \Delta_{11} p_{\lambda(m)-j,\mu(n)-k} \sum_{a=0}^{j-1} D_a(u) \sum_{b=0}^{k-1} D_b(v) \\ &- \sum_{j=0}^{\lambda(m)} \Delta_{10} p_{\lambda(m)-j,0} \sum_{a=0}^{j-1} D_a(u) \sum_{b=0}^{\mu(n)} D_b(v) \\ &- \sum_{k=0}^{\mu(n)} \Delta_{01} p_{0,\mu(n)-k} \sum_{a=0}^{\lambda(m)} D_a(u) \sum_{b=0}^{k-1} D_b(v) \\ &+ p_{00} \sum_{a=0}^{\lambda(m)} D_a(u) \sum_{b=0}^{\mu(n)} D_b(v) \end{aligned}$$

whence, from (1.6)

$$\begin{aligned} P_{\lambda(m)\mu(n)} \left| K_{\lambda}^{\mu}(u,v) \right| &\leq \frac{\pi^4}{4u^2v^2} \left(\sum_{j=1}^{\lambda(m)} \sum_{k=1}^{\mu(n)} \left| \Delta_{11} p_{\lambda(m)-j,\mu(n)-k} \right| \right. \\ &\left. + \sum_{j=0}^{\lambda(m)} \Delta_{10} p_{\lambda(m)-j,0} + \sum_{k=0}^{\mu(n)} \Delta_{01} p_{0,\mu(n)-k} + p_{00} \right). \end{aligned} \tag{2.14}$$

Because of $\Delta_{11} p_{jk}$ is fixed, we get

$$\begin{aligned} \sum_{j=1}^{\lambda(m)} \sum_{k=1}^{\mu(n)} \left| \Delta_{11} p_{\lambda(m)-j,\mu(n)-k} \right| &= \left| \sum_{j=1}^{\lambda(m)} \sum_{k=1}^{\mu(n)} \Delta_{11} p_{\lambda(m)-j,\mu(n)-k} \right| \\ &= \left| p_{\lambda(m)\mu(n)} - p_{\lambda(m)0} - p_{0\mu(n)} + p_{00} \right|. \end{aligned}$$

Again (2.14), if $\Delta_{11} p_{jk} \geq 0$,

$$\begin{aligned} P_{\lambda(m)\mu(n)} \left| K_{\lambda}^{\mu}(u,v) \right| &\leq \frac{\pi^4}{4u^2v^2} \left[\left(p_{\lambda(m)\mu(n)} - p_{\lambda(m)0} - p_{0\mu(n)} + p_{00} \right) \right. \\ &\left. + \left(p_{\lambda(m)0} - p_{00} \right) + \left(p_{0\mu(n) - p_{00}} \right) + p_{00} \right] \\ &= \frac{\pi^4 p_{\lambda(m)\mu(n)}}{4u^2v^2} \end{aligned}$$

and if $\Delta_{11} p_{jk} \leq 0$,

$$\begin{aligned} P_{\lambda(m)\mu(n)} \left| K_{\lambda}^{\mu}(u,v) \right| &\leq \frac{\pi^4}{4u^2v^2} \left[\left(-p_{\lambda(m)\mu(n)} + p_{\lambda(m)0} + p_{0\mu(n)} - p_{00} \right) \right. \\ &\left. + \left(p_{\lambda(m)0} - p_{00} \right) + \left(p_{0\mu(n) - p_{00}} \right) + p_{00} \right] \\ &= \frac{\pi^4}{4u^2v^2} \left(-p_{\lambda(m)\mu(n)} + 2p_{\lambda(m)0} + 2p_{0\mu(n)} - 2p_{00} \right) \\ &\leq \frac{\pi^4 p_{\lambda(m)\mu(n)}}{4u^2v^2} \end{aligned}$$

□

Lemma 2.4. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers and $\{p_{jk} > 0 \;:\; j,k = 0,1,\ldots\}$ be a nonincreasing double sequence such that*

$\Delta_{11}p_{jk}$ *is of fixed sign and let* $\sigma = [1/u]$, $\tau = [1/v]$ *where* $[.]$ *means the integral part. Then*

$$|K^{\mu}_{\lambda}(u,v)| \leq (\lambda(m)+1)(\mu(n)+1)\ , \qquad \forall u, v \tag{2.15}$$

$$\leq \frac{\pi(\pi+1)}{2P_{\lambda(m)\mu(n)}u}\sum_{k=0}^{\mu(n)}(k+1)\sum_{j=0}^{\sigma}p_{j,\mu(n)-k}\ , \qquad \forall v\ \ and\ \ 0<u\leq\pi \tag{2.16}$$

$$\leq \frac{\pi(\pi+1)}{2P_{\lambda(m)\mu(n)}u}\sum_{j=0}^{\lambda(m)}(j+1)\sum_{k=0}^{\tau}p_{\lambda(m)-j,k}\ , \qquad \forall u\ \ and\ \ 0<v\leq\pi \tag{2.17}$$

$$\leq \frac{\pi^2(1+2\pi+3\pi^2)}{4}\frac{P_{\sigma\tau}}{P_{\lambda(m)\mu(n)}uv}\ , \qquad 0<u,v\leq\pi. \tag{2.18}$$

Proof.

$$\begin{aligned} P_{\lambda(m)\mu(n)}\,|K^{\mu}_{\lambda}(u,v)| &\leq \sum_{k=0}^{\mu(n)}(k+1)\left|\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}D_j(u)\right| \\ &\leq \frac{\pi}{2u}\sum_{k=0}^{\mu(n)}(k+1)\left|\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\right| \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\left|\sum_{j=0}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\right| \\ \leq\ & \sum_{j=0}^{\sigma}p_{j,\mu(n)-k}+\left|\sum_{j=\sigma+1}^{\lambda(m)}p_{\lambda(m)-j,\mu(n)-k}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\right| \end{aligned} \tag{2.20}$$

Using Abel's transformation, we have

$$\begin{aligned} &\sum_{j=\sigma+1}^{\lambda(m)}p_{j,\mu(n)-k}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u \\ =\ &\sum_{j=\sigma+1}^{\lambda(m)-1}\Delta_{10}p_{j,\mu(n)-k}\sum_{l=\sigma+1}^{j}\sin\left(\lambda(m)-l+\frac{1}{2}\right)u \\ &+p_{\lambda(m),\mu(n)-k}\sum_{l=\sigma+1}^{\lambda(m)}\sin\left(\lambda(m)-l+\frac{1}{2}\right)u. \end{aligned} \tag{2.21}$$

From (1.5), p_{jk} is nonincreasing in j and $\frac{1}{u}<\sigma+1$, we have

$$\begin{aligned} \left|\sum_{j=\sigma+1}^{\lambda(m)}p_{j,\mu(n)-k}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\right| &\leq \frac{\pi}{u}p_{\sigma+1,\mu(n)-k} \\ &\leq \pi(\sigma+1)p_{\sigma+1,\mu(n)-k} \\ &\leq \pi\sum_{j=0}^{\sigma}p_{j,\mu(n)-k} \end{aligned} \tag{2.22}$$

Combining (2.14), (2.20) and (2.22) yield (2.16).

(2.17) can be proved similarly. Now let's prove (2.18) begining the following inequalitiy,

$$
\begin{aligned}
P_{\lambda(m)\mu(n)}\left|K_{\lambda}^{\mu}(u,v)\right| &= \left|\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} p_{jk} D_{\lambda(m)-j}(u) D_{\mu(n)-k}(v)\right| \\
&\leq \frac{\pi^2}{4uv}\left|\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} p_{jk} \sin\left(\lambda(m)-j+\frac{1}{2}\right)u \sin\left(\mu(n)-k+\frac{1}{2}\right)v\right|. \qquad (2.23)
\end{aligned}
$$

We divide the double sum into four parts:

$$
\begin{aligned}
&\left|\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} p_{jk} \sin\left(\lambda(m)-j+\frac{1}{2}\right)u \sin\left(\mu(n)-k+\frac{1}{2}\right)v\right| \\
\leq\ & \sum_{j=0}^{\sigma}\sum_{k=0}^{\tau} p_{jk} + \sum_{k=0}^{\tau}\left|\sum_{j=\sigma+1}^{\lambda(m)} p_{jk} \sin\left(\lambda(m)-j+\frac{1}{2}\right)u\right| \\
&+\sum_{j=0}^{\sigma}\left|\sum_{k=\tau+1}^{\mu(n)} p_{jk} \sin\left(\mu(n)-k+\frac{1}{2}\right)v\right| \\
&+\left|\sum_{j=\sigma+1}^{\lambda(m)}\sum_{k=\tau+1}^{\mu(n)} p_{jk} \sin\left(\lambda(m)-j+\frac{1}{2}\right)u \sin\left(\mu(n)-k+\frac{1}{2}\right)v\right| \\
=\ & P_{\sigma\tau} + A_1 + A_2 + A_3 \qquad (2.24)
\end{aligned}
$$

say.

For A_1 we can perform an Abel's transformation similar to (2.21) and conclude that

$$
\begin{aligned}
&\sum_{j=\sigma+1}^{\lambda(m)} p_{jk} \sin\left(\lambda(m)-j+\frac{1}{2}\right)u \\
\leq\ & \sum_{j=\sigma+1}^{\lambda(m)-1} \Delta_{10} p_{jk}\left|\sum_{l=\sigma+1}^{j} \sin\left(\lambda(m)-l+\frac{1}{2}\right)u\right| \\
& p_{\lambda(m),k}\left|\sum_{l=\sigma+1}^{\lambda(m)} \sin\left(\lambda(m)-l+\frac{1}{2}\right)u\right| \\
\leq\ & \frac{\pi}{u} p_{\sigma+1,k} \leq \pi(\sigma+1) p_{\sigma+1,k} \leq \pi\sum_{j=0}^{\sigma} p_{jk},
\end{aligned}
$$

which results in

$$
A_1 \leq \pi P_{\sigma\tau}\ . \qquad (2.25)
$$

Similarly,

$$
A_2 \leq \pi P_{\sigma\tau}\ . \qquad (2.26)
$$

For A_3, we perform a double Abel's transformation:

$$\begin{aligned}
&\sum_{j=\sigma+1}^{\lambda(m)}\sum_{k=\tau+1}^{\mu(n)} p_{jk}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\sin\left(\mu(n)-k+\frac{1}{2}\right)v\\
=&\sum_{j=\sigma+1}^{\lambda(m)-1}\sum_{k=\tau+1}^{\mu(n)-1}\Delta_{11}p_{jk}\sum_{a=\sigma+1}^{j}\sin\left(\lambda(m)-a+\frac{1}{2}\right)u\sum_{b=\tau+1}^{k}\sin\left(\mu(n)-b+\frac{1}{2}\right)v\\
&+\sum_{j=\sigma+1}^{\lambda(m)-1}\Delta_{10}p_{j\mu(n)}\sum_{a=\sigma+1}^{j}\sin\left(\lambda(m)-a+\frac{1}{2}\right)u\sum_{b=\tau+1}^{\mu(n)}\sin\left(\mu(n)-b+\frac{1}{2}\right)v\\
&+\sum_{k=\tau+1}^{\mu(n)-1}\Delta_{01}p_{\lambda(m)k}\sum_{a=\sigma+1}^{\lambda(m)}\sin\left(\lambda(m)-a+\frac{1}{2}\right)u\sum_{b=\tau+1}^{k}\sin\left(\mu(n)-b+\frac{1}{2}\right)v\\
&+p_{\lambda(m)\mu(n)}\sum_{a=\sigma+1}^{\lambda(m)}\sin\left(\lambda(m)-a+\frac{1}{2}\right)u\sum_{b=\tau+1}^{\mu(n)}\sin\left(\mu(n)-b+\frac{1}{2}\right)v
\end{aligned}$$

whence, by (1.5),

$$\begin{aligned}
&\left|\sum_{j=\sigma+1}^{\lambda(m)}\sum_{k=\tau+1}^{\mu(n)} p_{jk}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\sin\left(\mu(n)-k+\frac{1}{2}\right)v\right|\\
\leq&\ \frac{\pi^2}{uv}\left\{\left|\sum_{j=\sigma+1}^{\lambda(m)}\sum_{k=\tau+1}^{\mu(n)}\Delta_{11}p_{jk}\right|+\sum_{j=\sigma+1}^{\lambda(m)-1}\Delta_{10}p_{j\mu(n)}+\sum_{k=\tau+1}^{\mu(n)-1}\Delta_{01}p_{\lambda(m)k}+p_{\lambda(m)\mu(n)}\right\}\\
=&\ \frac{\pi^2}{uv}p_{\sigma+1,\tau+1}\qquad(\Delta_{11}p_{jk}\geq 0)\\
=&\ \frac{\pi^2}{uv}\left(-2p_{\lambda(m)\mu(n)}+2p_{\sigma+1,\mu(n)}+2p_{\lambda(m),\tau+1}-p_{\sigma+1,\tau+1}\right)\\
\leq&\ \frac{3\pi^2}{uv}p_{\sigma+1,\tau+1}\qquad(\Delta_{11}p_{jk}\leq 0).
\end{aligned}$$

Hence, the following inequality is hold:

$$\begin{aligned}
A_3 &\leq \frac{3\pi^2}{uv}p_{\sigma+1,\tau+1}\leq 3\pi^2(\sigma+1)(\tau+1)p_{\sigma+1,\tau+1}\\
&\leq 3\pi^2\sum_{j=0}^{\sigma}\sum_{k=0}^{\tau}p_{jk}=3\pi^2P_{\sigma\tau}. \qquad (2.27)
\end{aligned}$$

Putting (2.24)-(2.27) together yields

$$\begin{aligned}
&\left|\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} p_{jk}\sin\left(\lambda(m)-j+\frac{1}{2}\right)u\sin\left(\mu(n)-k+\frac{1}{2}\right)v\right|\\
\leq&\ (1+2\pi+3\pi^2)P_{\sigma\tau}.
\end{aligned}$$

Thus (2.23) implies (2.18). □

Theorem 2.5. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers and $\{p_{jk}>0\ :\ j,k=0,1,\ldots\}$ be a nondecreasing double sequence such that $\Delta_{11}p_{jk}$ is of fixed sign and condition (2.4) is satisfied. If $f\in Lip\alpha$ for some $0<\alpha<1$, then*

$$
\begin{aligned}
& sup_{(x,y)\in Q}\left|N_\lambda^\mu(x,y)-f(x,y)\right| \\
= & \begin{cases} O((h_{\lambda(m)\mu(n)})^\alpha+(t_{\lambda(m)\mu(n)})^\alpha) & , \quad 0<\alpha<1 \\ O(h_{\lambda(m)\mu(n)}\ \ln\frac{\pi}{h_{\lambda(m)\mu(n)}}+t_{\lambda(m)\mu(n)}\ \ln\frac{\pi}{t_{\lambda(m)\mu(n)}}) & , \quad \alpha=1\ . \end{cases}
\end{aligned} \tag{2.28}
$$

Proof.

$$
\begin{aligned}
& \frac{\pi^2}{4}\left|N_\lambda^\mu(x,y)-f(x,y)\right| \\
\leq & \left\{\int_0^{h_{\lambda(m)\mu(n)}}\int_0^{t_{\lambda(m)\mu(n)}}+\int_{h_{\lambda(m)\mu(n)}}^{\pi}\int_0^{t_{\lambda(m)\mu(n)}}+\int_0^{h_{\lambda(m)\mu(n)}}\int_{t_{\lambda(m)\mu(n)}}^{\pi}\right. \\
& \left.+\int_{h_{\lambda(m)\mu(n)}}^{\pi}\int_{t_{\lambda(m)\mu(n)}}^{\pi}\right\}|\phi(u,v)|\left|K_\lambda^\mu(u,v)\right|dudv \\
= & I_1+I_2+I_3+I_4
\end{aligned} \tag{2.29}
$$

say.

Each time $\phi(u,v)$ is estimated by (1.3) and the appropriate estimate of Lemma 2.3 is substituted for the kernel $K_\lambda^\mu(u,v)$. By (2.8), for $\alpha>0$

$$
\begin{aligned}
I_1 & \leq (\lambda(m)+1)(\mu(n)+1)\int_0^{h_{\lambda(m)\mu(n)}}\int_0^{t_{\lambda(m)\mu(n)}}(u^\alpha+v^\alpha)dudv \\
& = \frac{1}{\alpha+1}(\lambda(m)+1)(\mu(n)+1)h_{\lambda(m)\mu(n)}t_{\lambda(m)\mu(n)}\left(h^\alpha_{\lambda(m)\mu(n)}+t^\alpha_{\lambda(m)\mu(n)}\right).
\end{aligned}
$$

By (2.5) and (2.7),

$$
I_1=O\left(h^\alpha_{\lambda(m)\mu(n)}+t^\alpha_{\lambda(m)\mu(n)}\right). \tag{2.30}
$$

By (2.9),

$$
I_2\leq\frac{\pi^2}{2}\frac{1}{P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)}(k+1)p_{\lambda(m),\mu(n)-k}\int_{h_{\lambda(m)\mu(n)}}^{\pi}\int_0^{t_{\lambda(m)\mu(n)}}\frac{u^\alpha+v^\alpha}{u^2}dvdu
$$

whence for $0<\alpha<1$,

$$
I_2\leq\frac{\pi^2}{2}\frac{t_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)}P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)}(k+1)p_{\lambda(m),\mu(n)-k}\left(\frac{h^\alpha_{\lambda(m)\mu(n)}}{1-\alpha}+\frac{t^\alpha_{\lambda(m)\mu(n)}}{1+\alpha}\right),
$$

while for $\alpha=1$,

$$
I_2\leq\frac{\pi^2}{2}\frac{t_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)}P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)}(k+1)p_{\lambda(m),\mu(n)-k}\left(h_{\lambda(m)\mu(n)}\ln\frac{\pi}{h_{\lambda(m)\mu(n)}}+\frac{1}{2}t_{\lambda(m)\mu(n)}\right).
$$

Using (2.7)

$$
\begin{aligned}
\frac{t_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)}P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)}(k+1)p_{\lambda(m),\mu(n)-k} & \leq \frac{t_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)}P_{\lambda(m)\mu(n)}}(\mu(n)+1)\sum_{k=0}^{\mu(n)}p_{\lambda(m),k} \\
& = (\mu(n)+1)t_{\lambda(m)\mu(n)}=O(1).
\end{aligned}
$$

So,

$$
I_2=\begin{cases} O\left(\left(h_{\lambda(m)\mu(n)}\right)^\alpha+\left(t_{\lambda(m)\mu(n)}\right)^\alpha\right) & , \quad 0<\alpha<1 \\ O\left(h_{\lambda(m)\mu(n)}\ \ln\frac{\pi}{h_{\lambda(m)\mu(n)}}+t_{\lambda(m)\mu(n)}\right) & , \quad \alpha=1\ . \end{cases} \tag{2.31}
$$

Similarly, using (2.10),

$$I_3 = \begin{cases} O\left(\left(h_{\lambda(m)\mu(n)}\right)^{\alpha} + \left(t_{\lambda(m)\mu(n)}\right)^{\alpha}\right) & , \quad 0 < \alpha < 1 \\ O\left(h_{\lambda(m)\mu(n)} + t_{\lambda(m)\mu(n)} \ln \frac{\pi}{t_{\lambda(m)\mu(n)}}\right) & , \quad \alpha = 1\,. \end{cases} \tag{2.32}$$

By (2.11),

$$I_2 \leq \frac{3\pi^4}{4} \frac{p_{\lambda(m)\mu(n)}}{P_{\lambda(m)\mu(n)}} \int_{h_{\lambda(m)\mu(n)}}^{\pi} \int_{t_{\lambda(m)\mu(n)}}^{\pi} \frac{u^{\alpha} + v^{\alpha}}{u^2 v^2} du dv$$

whence for $0 < \alpha < 1$,

$$\frac{3\pi^4}{4(1-\alpha)} \frac{p_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)} t_{\lambda(m)\mu(n)} P_{\lambda(m)\mu(n)}} \left(\left(h_{\lambda(m)\mu(n)}\right)^{\alpha} + \left(t_{\lambda(m)\mu(n)}\right)^{\alpha}\right)$$

while for $\alpha = 1$,

$$\frac{3\pi^4}{4} \frac{p_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)} t_{\lambda(m)\mu(n)} P_{\lambda(m)\mu(n)}} \left(h_{\lambda(m)\mu(n)} \ln \frac{\pi}{h_{\lambda(m)\mu(n)}} + t_{\lambda(m)\mu(n)} \ln \frac{\pi}{t_{\lambda(m)\mu(n)}}\right).$$

By (2.2), (2.3) and (2.4),

$$\frac{p_{\lambda(m)\mu(n)}}{h_{\lambda(m)\mu(n)} t_{\lambda(m)\mu(n)} P_{\lambda(m)\mu(n)}} = \frac{(\lambda(m)+1)(\mu(n)+1) p_{\lambda(m)\mu(n)}}{(\lambda(m)+1) h_{\lambda(m)\mu(n)} (\mu(n)+1) t_{\lambda(m)\mu(n)} P_{\lambda(m)\mu(n)}} = O(1)$$

Consequently,

$$I_4 = \begin{cases} O\left(\left(h_{\lambda(m)\mu(n)}\right)^{\alpha} + \left(t_{\lambda(m)\mu(n)}\right)^{\alpha}\right) & , \quad 0 < \alpha < 1 \\ O\left(h_{\lambda(m)\mu(n)} \ln \frac{\pi}{h_{\lambda(m)\mu(n)}} + t_{\lambda(m)\mu(n)} \ln \frac{\pi}{t_{\lambda(m)\mu(n)}}\right) & , \quad \alpha = 1\,. \end{cases} \tag{2.33}$$

Collecting (2.29)-(2.33) together yields (2.28).

□

Theorem 2.6. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers and $\{p_{jk} > 0 \ : \ j,k = 0,1,\ldots\}$ be a nonincreasing double sequence such that $\Delta_{11} p_{jk}$ is of fixed sign. If $f \in Lip\alpha$ for some $0 < \alpha \leq 1$, then*

$$\sup_{(x,y)\in Q} \left|N_{\lambda}^{\mu}(x,y) - f(x,y)\right| \tag{2.34}$$

$$= O\left\{\frac{1}{P_{\lambda(m)\mu(n)}} \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \left(\frac{P_{jk}}{(\lambda(m)+1)^{\alpha+1}(\mu(n)+1)} + \frac{P_{jk}}{(\lambda(m)+1)(\mu(n)+1)^{\alpha+1}}\right)\right\}.$$

In the special case where

$$\lim_{m,n\to\infty} p_{mn} > 0, \tag{2.35}$$

we have

$$\frac{1}{(\lambda(m)+1) h_{\lambda(m)\mu(n)}} \leq \frac{p_{00}}{p_{mm}} = O(1) \quad \text{and} \quad \frac{1}{(\mu(n)+1) t_{\lambda(m)\mu(n)}} = O(1)$$

and the right-hand side of (2.34) reduces to that of (2.28).

Proof. We use (2.29) with $h_{\lambda(m)\mu(n)}$ and $t_{\lambda(m)\mu(n)}$ replaced by $\pi/(\lambda(m)+1)$ and $\pi/(\mu(n)+1)$, respectively. For brevity, we denote by Ψ_{mn} the quantity in braces on the right-hand side of (2.34).

From (2.15), for $\alpha > 0$

$$\begin{aligned} I_1 &\leq (\lambda(m)+1)(\mu(n)+1)\int_0^{\pi/(\lambda(m)+1)}\int_0^{\pi/(\mu(n)+1)}(u^\alpha+v^\alpha)dudv \\ &= \frac{\pi^{\alpha+2}}{\alpha+1}\left(\frac{1}{(\lambda(m)+1)^\alpha}+\frac{1}{(\mu(n)+1)^\alpha}\right). \end{aligned} \tag{2.36}$$

Since p_{jk} is nonincreasing, we clearly have

$$P_{jk} \geqq (j+1)(k+1)p_{jk} \qquad (j,k=0,1,\ldots).$$

Therefore,

$$\begin{aligned} \frac{1}{(\lambda(m)+1)^\alpha} &= \frac{1}{(\lambda(m)+1)^\alpha}\frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}p_{jk} \\ &\leq \frac{1}{(\lambda(m)+1)^\alpha}\frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}\frac{P_{jk}}{(j+1)(k+1)} \\ &\leq \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}\frac{P_{jk}}{(j+1)^{\alpha+1}(k+1)} \end{aligned}$$

and similarly,

$$\frac{1}{(\mu(n)+1)^\alpha} \leq \frac{1}{P_{\lambda(m)\mu(n)}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)}\frac{P_{jk}}{(j+1)(k+1)^{\alpha+1}}.$$

Combining (2.36) with the last two inequalities results in

$$I_1 = O(\Psi_{mn}). \tag{2.37}$$

By (2.16),

$$\begin{aligned} I_2 &\leq \frac{\pi(\pi+1)}{2P_{\lambda(m\mu(n))}}\sum_{k=0}^{\mu(n)}\int_{\frac{\pi}{\lambda(m)+1}}^{\pi}\int_0^{\frac{\pi}{\mu(n)+1}}\frac{u^\alpha+v^\alpha}{u}\sum_{j=0}^{\sigma}p_{j,\mu(n)-k}dvdu \\ &= \frac{\pi(\pi+1)}{2P_{\lambda(m\mu(n))}}\sum_{k=0}^{\mu(n)}(k+1)\left\{\frac{\pi}{\mu(n)+1}\int_{\frac{\pi}{\lambda(m)+1}}^{\pi}u^{\alpha-1}\sum_{j=0}^{\sigma}p_{j,\mu(n)-k}du\right. \\ &\left.+\frac{\pi^{\alpha+1}}{(\alpha+1)(\mu(n)+1)^{\alpha+1}}\int_{\frac{\pi}{\lambda(m)+1}}^{\pi}\frac{1}{u}\sum_{j=0}^{\sigma}p_{j,\mu(n)-k}du\right\} \end{aligned} \tag{2.38}$$

In each integration replace u by $\frac{1}{y}$ (remembering that $\sigma = [1/u]$) to get

$$\begin{aligned} I_2 &= \frac{O(1)}{P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)}(k+1)\left\{\frac{1}{\mu(n)+1}\int_{\frac{1}{\pi}}^{\frac{\lambda(m)+1}{\pi}}\frac{1}{y^{\alpha+1}}\sum_{j=1}^{[u]}p_{j,\mu(n)-k}dy\right. \\ &\left.+\frac{1}{(\mu(n)+1)^{\alpha+1}}\int_{\frac{1}{\pi}}^{\frac{\lambda(m)+1}{\pi}}\frac{1}{y}\sum_{j=1}^{[u]}p_{j,\mu(n)-k}dy\right\} \end{aligned}$$

Then making a simple approximation to the integrals involved yields

$$\begin{aligned} I_2 &= \frac{O(1)}{P_{\lambda(m)\mu(n)}}\sum_{k=0}^{\mu(n)}(k+1)\left\{\frac{1}{\mu(n)+1}\sum_{l=0}^{\lambda(m)}\frac{1}{(l+1)^{\alpha+1}}\sum_{j=0}^{l}p_{j,\mu(n)-k}\right. \\ &\left.+\frac{1}{(\mu(n)+1)^{\alpha+1}}\sum_{l=0}^{\lambda(m)}\frac{1}{l+1}\sum_{j=0}^{l}p_{j,\mu(n)-k}\right\}. \end{aligned} \tag{2.39}$$

The first sum on the right is equal to

$$\begin{aligned} A &= \frac{1}{(\mu(n)+1)P_{\lambda(m)\mu(n)}} \sum_{k=0}^{\mu(n)} (k+1) \sum_{l=0}^{\lambda(m)} \frac{1}{(l+1)^{\alpha+1}} \sum_{j=0}^{l} p_{j,\mu(n)-k} \\ &= \frac{1}{(\mu(n)+1)P_{\lambda(m)\mu(n)}} \sum_{l=0}^{\lambda(m)} \frac{1}{(l+1)^{\alpha+1}} \sum_{j=0}^{l} \sum_{k=0}^{\mu(n)} (k+1) p_{j,\mu(n)-k}. \end{aligned}$$

Using the identity

$$\sum_{k=0}^{\mu(n)} (k+1) p_{j,\mu(n)-k} = \sum_{k=0}^{\mu(n)} \sum_{r=0}^{\mu(n)-k} p_{jr},$$

we can write

$$\begin{aligned} A &= \frac{1}{(\mu(n)+1)P_{\lambda(m)\mu(n)}} \sum_{l=0}^{\lambda(m)} \frac{1}{(l+1)^{\alpha+1}} \sum_{k=0}^{\mu(n)} \sum_{j=0}^{l} \sum_{r=0}^{\mu(n)-k} p_{jr} \\ &= \frac{1}{(\mu(n)+1)P_{\lambda(m)\mu(n)}} \sum_{l=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \frac{P_{l,\mu(n)-k}}{(l+1)^{\alpha+1}} \\ &= \frac{1}{(\mu(n)+1)P_{\lambda(m)\mu(n)}} \sum_{l=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \frac{P_{lk}}{(l+1)^{\alpha+1}} \\ &\leq \frac{1}{P_{\lambda(m)\mu(n)}} \sum_{l=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \frac{P_{lk}}{(l+1)^{\alpha+1}(k+1)}. \end{aligned} \tag{2.40}$$

The second sum in the right hand side of (2.39) can be dominated in a similar manner:

$$\begin{aligned} &\frac{1}{(\mu(n)+1)^{\alpha+1}P_{\lambda(m)\mu(n)}} \sum_{k=0}^{\mu(n)} (k+1) \sum_{l=0}^{\lambda(m)} \frac{1}{l+1} \sum_{j=0}^{l} p_{j,\mu(n)-k} \\ \leq\; &\frac{1}{P_{\lambda(m)\mu(n)}} \sum_{l=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \frac{P_{lk}}{(\mu(n)+1)^{\alpha+1}(l+1)}. \end{aligned} \tag{2.41}$$

From (2.39)-(2.41) it follows that

$$I_2 = O(\Psi_{mn}). \tag{2.42}$$

In an analogous way, by (2.17),

$$I_3 = O(\Psi_{mn}). \tag{2.43}$$

Using that (2.18),

$$I_4 = \frac{O(1)}{P_{\lambda(m)\mu(n)}} \int_{\frac{\pi}{\lambda(m)+1}}^{\pi} \int_{\frac{\pi}{\mu(n)+1}}^{\pi} \frac{u^\alpha + v^\alpha}{uv} P_{\sigma\tau} du dv.$$

We replace u by $1/y$, v by $1/w$, keeping in mind that $\sigma = [1/u]$ and $\tau = [1/v]$. As a result we obtain

$$I_4 = \frac{O(1)}{P_{\lambda(m)\mu(n)}} \int_{\frac{1}{\pi}}^{\frac{\lambda(m)+1}{\pi}} \int_{\frac{1}{\pi}}^{\frac{\mu(n)+1}{\pi}} \left(\frac{1}{y^{\alpha+1}w} + \frac{1}{yw^{\alpha+1}} \right) P_{[y][w]} dy dw.$$

$$I_4 = \frac{O(1)}{P_{\lambda(m)\mu(n)}} \sum_{j=0}^{\lambda(m)} \sum_{k=0}^{\mu(n)} \left(\frac{1}{(j+1)^{\alpha+1}(k+1)} + \frac{1}{(j+1)(k+1)^{\alpha+1}} \right) P_{jk} = O(\Psi_{mn}) \tag{2.44}$$

Combining (2.29), (2.37), (2.42)-(2.44) results in (2.34). □

Corollary 2.7. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers and $\{p_{jk} > 0 \ : \ j,k = 0,1,\ldots\}$ be a nonincreasing double sequence such that $\Delta_{11}p_{jk}$ is of fixed sign and condition (2.35) satisfied. If $f \in Lip\alpha$, $0 < \alpha \leq 1$, then statement (2.28) holds.*

The (C,β,γ)-summability, $\beta,\gamma > -1$, is a particular case of the Nörlund summability, where $\{p_{jk}\}$ is given by

$$p_{jk} = A_j^{\beta-1}A_k^{\gamma-1} \qquad (j,k = 0,1,2,\ldots) \tag{2.45}$$

where

$$A_l^{\alpha} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+l)}{l!}$$

for $l = 1,2,\ldots$ and $A_0^{\alpha} = 1$. Then, as is known,

$$P_{mn} = A_m^{\beta}A_n^{\gamma} \qquad (m,n = 0,1,2,\ldots).$$

Clearly, if we take $\{p_{jk}\}$ as (2.45), then we have

$$P_{\lambda(m)\mu(n)} = \sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} A_j^{\beta-1}A_k^{\gamma-1} = A_{\lambda(m)}^{\beta}A_{\mu(n)}^{\gamma}.$$

Furthermore, there exist two positive constants C_1 and C_2 such that

$$C_1 \leq \frac{A_l^{\alpha}}{(l+1)^{\alpha}} \leq C_2 \qquad (l = 0,1,\ldots;\alpha > -1)$$

(see [9]).

The approximation rate for (C,β,γ) summability immediately follows from Theorem 2.28 (for $\beta,\gamma \geq 1$) and Theorem 2.34 (for $\alpha \leq \beta,\gamma \leq 1$).

Corollary 2.8. *Let the index sequences $\lambda(m)$ and $\mu(n)$ are strictly increasing single sequences of positive integers. If $f \in Lip\alpha$, $0 < \alpha \leq 1$, and $\beta,\gamma \geq \alpha$, then*

$$\sup_{(x,y)\in Q}\left|\frac{1}{A_{\lambda(m)}^{\beta}A_{\mu(n)}^{\gamma}}\sum_{j=0}^{\lambda(m)}\sum_{k=0}^{\mu(n)} A_{\lambda(m)-1}^{\beta-1}A_{\mu(n)-1}^{\gamma-1}s_{jk}(x,y) - f(x,y)\right|$$
$$= \begin{cases} O\left(\frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{1}{(\mu(n)+1)^{\alpha}}\right) & , \ \textit{if } \beta > \alpha \textit{ and } \gamma > \alpha \\ O\left(\frac{ln(\lambda(m)+2)}{(\lambda(m)+1)^{\alpha}} + \frac{1}{(\mu(n)+1)^{\alpha}}\right) & , \ \textit{if } \beta = \alpha \textit{ and } \gamma > \alpha \\ O\left(\frac{ln(\lambda(m)+2)}{(\lambda(m)+1)^{\alpha}} + \frac{ln(\mu(n)+2)}{(\mu(n)+1)^{\alpha}}\right) & , \ \textit{if } \beta = \alpha = \gamma. \end{cases}$$

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Received: June 10, 2017.
Accepted: December 12, 2017