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Noetherian modules with prime nilradical

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Abstract This paper is devoted to studying Φ-Noetherian modules as a new class of Noetherian modules. A module M is Φ-Noetherian if $Nil(M)$ is divided prime and each submodule that properly contains $Nil(M)$ is finitely generated. If M is a Φ-Noetherian module and $X_1, ..., X_n$ are indeterminates, then a submodule N of $M[X_1, ..., X_n]$ which contains a nonnil element of M is finitely generated.

1 Introduction

We assume throughout this paper all rings are commmutative with $1 \neq 0$ and all modules are unitary. Let R be a ring with identity and $Nil(R)$ be the set of nilpotent elements of R. Recall from [20] and [10], that a prime ideal P of R is called a divided prime ideal if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. Badawi in [9], [11], [10], [14], [15] and [16] investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring with $1 \neq 0$ and $Nil(R)$ is a divided prime ideal of $R\}$. Anderson and Badawi in [6] and [7] generalized the concept of Prüfer, Dedekind, Krull and Bezout domain to context of rings that are in the class $\mathcal{H}$. Also, Lucas and Badawi in [12] generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. Let R be a ring, $Z(R)$ the set of zero divisors of R and $S = R \setminus Z(R)$. Then $T(R) := S^{-1}R$ denoted the total quotient ring of R. We start by recalling some background material. A nonzero divisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. An ideal I of a ring R is said to be a nonnil ideal if $I \nsubseteq Nil(R)$. If I is a nonnil ideal of $R \in \mathcal{H}$, then $Nil(R) \subset I$. In particular, it holds if I is a regular ideal of a ring $R \in \mathcal{H}$. Recall from [6] that for a ring $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ given by $\phi(a/b) = a/b$, for $a \in R$ and $b \in R \setminus Z(R)$, is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$ and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$.

For a nonzero ideal I of R let $I^{-1} = \{x \in T(R) : xI \subseteq R\}$. It is obvious that $II^{-1} \subseteq R$. An ideal I of R is called invertible, if $II^{-1} = R$. An integral domain R is called a Dedekind domain if every nonzero ideal of R is invertible. Recall from [22] that a ring R is called a Dedekind ring if every regular ideal of R is invertible. An integral domain R is called almost Dedekind if for each nonzero prime ideal P of R, R_P is a Dedekind domain. We generaliz the concept of almost Dedekind domains to the context of commutative rings with zero divisors. A ring R is an almost Dedekind if for each regular prime ideal P of R, R_P is a Dedeking ring. Let $R \in \mathcal{H}$. Then a nonnil ideal I of R is called ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. Recall from [7] that R is called ϕ-Dedekind ring if every nonnil ideal of R is ϕ-invertible.

Let R be a ring and M be an R-module. Then M is a multiplication R-module if every submodule N of M has the form IM for some ideal I of R. If M be a multiplication R-module and N a submodule of M, then $N = IM$ for some ideal I of R. Hence $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. Therefore $N = (N :_R M)M$ [17]. Let M be a multiplication R-module, $N = IM$ and $L = JM$ be submodules of M for ideals I and J of R. Then, the product of N and L is denoted by $N.L$ or NL and is defined by IJM [5]. An R-module M is called a cancellation module if $IM = JM$ for two ideals I and J of R implies $I = J$ [1]. By [25, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It follows that if M is a finitely generated faithful multiplication R-module, then $(IN :_R M) = I(N :_R M)$ for all ideals I of R and all submodules N of M. If R is an integral domain and M a faithful multiplication R-module, then M is a finitely generated R-module [18].

Let M be an R-module and set

$$T = \{t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0\} = (R \setminus Z(M)) \cap (R \setminus Z(R)).$$

Then T is a multiplicatively closed subset of R with $T \subseteq S$, and if M is torsion-free then $T = S$. In particular, $T = S$ if M is a faithful multiplication R-module [18, Lemma 4.1]. Let N be a nonzero submodule of M. Then we write $N^{-1} = (M :_{R_T} N) = \{x \in R_T : xN \subseteq M\}$ and $N_\nu = (N^{-1})^{-1}$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that N is invertible in M if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in M. An R-module M is called a Dedekind module if every nonzero submodule of M is invertible, [24]. If N is an invertible submodule of a faithful multiplication module M over an integral domain R, then $(N :_R M)$ is invertible [3]. Let R be a ring and M an R-module. Then M is said to be an almost Dedekind module if for each prime ideal P of R, M_P is an R_P-module. Clearly Dedekind modules are almost Dedekind, [4].
Let M be an R-module. An element $r \in R$ is said to be zero divisor on M if $rm = 0$ for some $0 \neq m \in M$. The set of zero divisors of M is denoted by $Z_R(M)$ (briefly, $Z(M)$). It is easy to see that $Z(M)$ is not necessarily an ideal of R, but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. A submodule N of M is called a nilpotent submodule if $[N :_R M]^n N = 0$ for some positive integer n. An element $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M [2]. We let $Nil(M)$ to denote the set of all nilpotent elements of M; then $Nil(M)$ is a submodule of M provided that M is a faithful module, and if in addition M is multiplication, then $Nil(M) = Nil(R)M = \bigcap P$, where the intersection runs over all prime submodules of M, [2, Theorem 6]. If M contains no nonzero nilpotent elements, then M is called a reduced R-module. A submodule N of M is said to be a nonnil submodule if $N \nsubseteq Nil(M)$. Recall that a submodule N of M is prime if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$. If N is a prime submodule of M, then $p := [N :_R M]$ is a prime ideal of R. In this case we say that N is a p-prime submodule of M. Let N be a submodule of multiplication R-module M, then N is a prime submodule of M if and only if $[N :_R M]$ is a prime ideal of R if and only if $N = pM$ for some prime ideal p of R with $[0 :_R M] \subseteq p$, [18, Corollary 2.11]. Recall from [4] that a prime submodule P of M is called a divided prime submodule if $P \subset Rm$ for every $m \in M \setminus P$; thus a divided prime submodule is comparable to every submodule of M.
Now assume that $T^{-1}(M) = \mathfrak{T}(M)$. Set

$$\mathbb{H} = \{M \mid M \text{ is an } R-\text{module and } Nil(M) \text{ is a divided prime submodule of } M\}$$

and

$$\mathbb{H}_0 = \{M \in \mathbb{H} \mid Nil(M) = Z(M)M\}.$$

For an R-module $M \in \mathbb{H}$, $Nil(M)$ is a prime submodule of M. So $P := [Nil(M) :_R M]$ is a prime ideal of R. If M is an R-module and $Nil(M)$ is a proper submodule of M, then $[Nil(M) :_R M] \subseteq Z(R)$. Consequently, $R \setminus Z(R) \subseteq R \setminus [Nil(M) :_R M]$. In particular, $T \subseteq R \setminus [Nil(M) :_R M]$ [26]. Recall from [26] that we can define a mapping $\Phi : \mathfrak{T}(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ which is clearly an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M in to M_P given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule N of M is said to be Φ-invertible if $\Phi(N)$ is an invertible submodule of $\Phi(M)$ [28]. An R-module M is called a Φ-Dedekind module if every nonnil submodule of M is Φ-invertible [28]. Ahmad in [26], introduced a new class of modules which is closely related to the class of Noetherian modules. A module M is called a Φ-Noetherian if every nonnil submodule of M is finitely generated. In this paper we find some properties of this class of modules.

2 Some properties of Φ-Noetherian modules

Theorem 2.1. *[26, Theorem 11] Let R be a ring and M be a finitely generated faithful multiplication R-module with $M \in \mathbb{H}_0$. The following are equivalent:*
(1) *M is a Φ-Noetherian R-module;*
(2) *$\frac{M}{Nil(M)}$ is a Noetherian R-module;*

(3) $\frac{\Phi(M)}{Nil(\Phi(M))}$ *is a Noetherian R-module;*
(4) $\Phi(M)$ *is a Φ-Noetherian R-module;*
(5) *each nonnil prime submodule of M is finitely generated.*

Proposition 2.2. *Let R be a ring and not an integral domain and let $M \in \mathbb{H}$ be a finitely genetated faithful multiplication R-module. Then $Nil(M)$ is finitely generated if and only if M is an Artinian module with only maximal submodule $Nil(M)$. Inparticular, If M is a Noetherian module, then M is an Artinian module with only maximal submodule $Nil(M) \neq (0)$.*

Proof. By [25, Proposition 13], [26, Proposition 1] and [13, Proposition 2.3], we have $Nil(R) = (Nil(M) :_R M)$ is finitely generated if and only if R is a local Artinian ring with maximal ideal $Nil(R)$. Hence $Nil(M) = Nil(R)M$ is finitely generated if and only if M is an Artinian module with only maximal submodule $Nil(M)$, because M is faithful multiplication. For next statement, Since M is Noetherian faithful multiplication, R is Noetherian. Thus, by [13, Proposition 2.3], R is a local ring with maximal ideal $Nil(R) \neq (0)$. Therefore, M is an Artinian module with only maximal submodule $Nil(M) \neq (0)$. □

Proposition 2.3. *Let R be a ring and $M \in \mathbb{H}_0$ be a Φ-Noetherian faithful multiplication R-module and let N be a proper submodule of M. If $N \subset Nil(M)$, then $\frac{M}{N}$ is a Φ-Noetherian module. If $N \nsubseteq Nil(M)$, then $Nil(M) \subset N$ and $\frac{M}{N}$ is a Noetherian module. Moreover, if $Nil(M) \subset N$, then M/N is both Noetherian module and Φ-Noetherian module if and only if N is either a prime submodule or a primary submodule whose radical is a maximal submodule.*

Proof. If $N \subset Nil(M)$, then $Nil(\frac{M}{N}) = \frac{Nil(M)}{N}$ is a divided prime submodule of $\frac{M}{N}$. Hence, $\frac{M}{N} \in \mathbb{H}$. Since $\frac{\frac{M}{N}}{Nil(\frac{M}{N})}$ is module-isomorphic to $\frac{M}{Nil(M)}$ and $\frac{M}{Nil(M)}$ is Noetherian module by Theorem 2.1, we conclude that $\frac{M}{N}$ is a Φ-Noetherian module.
Now, suppose that $N \nsubseteq Nil(M)$. Since $Nil(M)$ is a divided prime submodule of M, $Nil(M) \subset N$. Let Q be a prime submodule of $\frac{M}{N}$. Then $Q = \frac{P}{N}$ for some nonnil prime submodule P of M such that $N \subseteq P$. Since P is finitely generated, Q is finitely generated. Therefore $\frac{M}{N}$ is Noetherian module.
The third statement follows from Proposition 2.2. □

Corollary 2.4. *Let R be a ring and $M \in \mathbb{H}_0$ be a finitely generated faithful multiplication R-module. Then a homomorphic image of M is either a Φ-Noetherian module or a Noetherian module.*

Lemma 2.5. *Let R be an integral domain and M be a faithful multiplication R-module. Then R is an almost-Dedekind domain if and only if M is an almost-Dedekind module.*

Proof. Let R be an almost-Dedekind domain. Then R_P is a Dedekind domain for each nonzero prime ideal P of R. Hence, by [4], M_P is a Dedekind module. Therefore, M is an almost-Dedekind module. The converse is similar. □

It is clear that if M is a Φ-Dedekind module, then M is a Φ-Prüfer module.

Theorem 2.6. *Let R be a ring and $M \in \mathbb{H}_0$ be a finitely generated faithful multiplication R-module. If M is a Φ-Noetherian and Φ-Prüfer module, then M is a Φ-Dedekind module.*

Proof. Suppose that M is Φ-Noetherian and Φ-Prüfer module. Then, by [26, Theorem 7] and [28, Theorem 2.11] $\frac{M}{Nil(M)}$ is a Noetherian and Prüfer module. Thus $\frac{M}{Nil(M)}$ is a Dedekind module. Therefore, M is a Φ-Dedekind module. □

Proposition 2.7. *Let R be an integral domain and M be a faithful multiplication R-module. If M is an almost-Dedekind module but not Dedekind module, then M is locally Noetherian module but not Noetherian module.*

Proof. Suppose that M is an almost-Dedekind module but not Dedekind. Then, by Lemma 2.5 and [4], R is an almost-Dedekind domain but not Dedekind. Hence, by [13], R is a locally Noetherian ring but not Noetherian. Therefore, M is a locally Noetherian module but not Noetherian. □

Proposition 2.8. *Let R be an integral domain and M be a faithful multiplication R-module. If M is a locally Noetherian module and each nonzero element is contained in at most finitely many maximal submodules, then M is a Noetherian module.*

Proof. Suppose that M is a locally Noetherian module and each nonzero element is contained in at most finitely many maximal submodules. Then R is a locally Noetherian domain and each nonzero element is contained in at most finitely many maximal ideals. Hence, by [23, Exercisee ♯10, page 73], R is a Noetherian domain. Therefore, M is a Noetherian module, because M is faithful multiplication. □

Lemma 2.9. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following are hold:*
(1) *If $R \in \mathcal{H}$ is a ϕ-Noetherian ring, Then M is a Φ-Noetherian module.*
(2) *If $M \in \mathbb{H}$ is a Φ-Noetherian module, then R is a ϕ-Noetherian ring.*

Proof. Since $Nil(R) \subseteq Ann(\frac{M}{Nil(R)M}) = Ann(\frac{M}{Nil(M)})$, we have:
(1) Let $R \in \mathcal{H}$. Then, by [26, Proposition 3], $M \in \mathbb{H}$. If R is a ϕ-Noetherian ring, then [16, Theorem 2.2], $\frac{R}{Nil(R)}$ is a Noetherian domain. So, $\frac{M}{Nil(M)}$ is a Noetherian module. Therefore, by [26, Theorem 7], M is a Φ-Noetherian module.
(2) Let $M \in \mathbb{H}$. Then, by [26, Proposition 3], $R \in \mathcal{H}$. If M is a Φ-Noetherian module, then by [26, Theorem 7], $\frac{M}{Nil(M)}$ is a Noetherian module. So, $\frac{R}{Nil(R)}$ is a Noetherian domain. Therefore, by [16, Theorem 2.2], R is a ϕ-Noetherian ring. □

Proposition 2.10. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module. Let M_P be a Φ-Noetherian module for every maximal ideal P of R and each nonnil element of M lies in only a finite number of maximal submodules of M. Then M is a Φ-Noetherian module.*

Proof. Since $M \in \mathbb{H}$, by [26, Proposition 3], $R \in \mathcal{H}$. Suppose that for every maximal ideal P of R, M_P is a Φ-Noetherian module and each nonnil element of M lies in only a finite number of maximal submodules of M. Hence, by Lemma 2.9, R_P is a ϕ-Noetherian module for every maximal ideal P of R and each nonnil element of M lies in only a finite number of maximal ideals of M. Thus, by [13, Proposition 2.6], R is a ϕ-Noetherian ring. Therefore, by Lemma 2.9, M is a Φ-Noetherian module. □

Proposition 2.11. *Let R be a ring, $M \in \mathbb{H}_0$ be a Φ-Noetherian faithful multiplication R-module and P a prime submodule of M. If P is minimal over an submodule generated by n or fewer elements, then $ht(P) \leq n$. In particular, if P is a prime minimal submodule over a nonnil element of M, then $ht(P) = 1$.*

Proof. The module $\frac{M}{Nil(M)}$ is Noetherian module by Theorem 2.1. Assume P is minimal over the submodule $N = (a_1, ...a_n)$. If $N \subset Nil(M)$, there is nothing to prove since we would have $N = Nil(M)$, the prime of height 0. Thus we may assume N is not nilpotent. Since $Nil(M)$ is divided, $Nil(M) \subset N$. Thus $\frac{N}{Nil(M)}$ can be generated by n (or fewer) elements. Since M is Notherian, $ht(\frac{P}{Nil(M)}) \leq n$. Hence $ht(P) \leq n$. □

Proposition 2.12. *Let R be a ring and $M \in \mathbb{H}$ be a faithful multiplication R-module such that satisfy the ascending chain condition on radical submodules. If M has an infinite number of prime submodules of height 1, then their intersection is $Nil(M)$.*

Proof. Suppose that M satisfy the ascending chain condition on radical submodules and M has an infinite number of prime submodules of height 1. Then R satisfy the ascending chain condition on radical ideals and R has an infinite number of prime ideals of height 1. Hence, their intersection is $Nil(R)$, by [23, Theorem 145]. Therefore, the intersection of an infinite number of prime submodule of height 1 is $Nil(M)$. □

Proposition 2.13. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module and P a nonnil prime submodule of M with $ht(P) = n$. Then there exist nonnil elements $a_1, ..., a_n$ in M such that P is minimal over the submodule $(a_1, ..., a_n)$, and for any $1 \leq i \leq n$, every nonnil prime submodule of M minimal over $(a_1, ..., a_n)$ has height i.*

Proposition 2.14. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module and N a proper submodule of M generated by n elements. If P is a prime submodule of M conating N with $ht(\frac{P}{N}) = k$, then $ht(P) \leq n + k$.*

Proof. Suppose that M is a Φ-Noetherian module and N a proper submodule of M generated by n elements. Then, by Lemma 2.9, R is a ϕ-Noetherian ring and $(N :_R M)$ is a proper ideal of R generated by n elements. If P is a prime submodule of M conatining N with $ht(\frac{P}{N}) = k$, then $(P :_R M)$ is a prime ideal of R containing $(N :_R M)$ with $ht(\frac{(P:_R M)}{(N:_R M)}) = k$. Hence, by [13, Proposition 2.10], $ht(P :_R M) = n + k$. Therefore, $ht(P) = n + k$. □

Proposition 2.15. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module. Let P be a prime submodule of M with $ht(P) = n$ and Q be a prime submodule of $M[X]$ such that $P \neq Q$ and $PM[X] \subsetneq Q$. Then $ht(PM[X]) = n$ and $ht(Q) = n + 1$.*

Proof. Since $Nil(M)$ is the minimal prime of M, $Nil(M[X]) = Nil(R)M[X]$ is the minimal prime of $M[X]$. We assume that $K = \frac{M}{Nil(M)}$ and $K[X] = \frac{M[X]}{Nil(M[X])}$. Hence, by Theorem 2.1, K is a Noetherian module. Moreover, $\frac{P}{Nil(M)}$ is a prime submodule of M with $ht(\frac{P}{Nil(M)}) = n$ and $\frac{Q}{Nil(M[X])}$ is a prime submodule of $M[X]$ such that $\frac{P}{Nil(M)} \neq \frac{Q}{Nil(M[X])}$ and $(\frac{P}{Nil(M)})K[X] \subsetneq \frac{Q}{Nil(M[X])}$. Therefore, $ht((\frac{P}{Nil(M)})K[X]) = ht(PM[X]) = n$ and $ht(\frac{Q}{Nil(M[X])}) = ht(Q) = n + 1$. □

Proposition 2.16. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module. Let P be a prime submodule of M with $ht(P) = n$ and Q be a prime submodule of $M[X_1, ..., X_m]$ such that $P \neq Q$ and $PM[X_1, ..., X_m] \subsetneq Q$. Then $ht(PM[X_1, ..., X_m]) = n$ and $ht(Q) \leqslant n + m$. Moreover, $ht(PM[X_1, ..., X_m] + (X_1, ..., X_m)M[X_1, ..., X_m]) = n + m$.*

Corollary 2.17. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module and $dim(M) = n$. Then $dim(M[X_1, ..., X_m]) = n + m$ for each integer $m > 0$.*

Proposition 2.18. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module. If N is a submodule of $M[X_1, ..., X_n]$ for which $N \cap M$ is not contained in $Nil(M)$, then N is a finitely generated submodule of $M[X_1, ..., X_n]$.*

Proof. If $N \cap M$ is not contained in $Nil(M)$, then any single nonnil element in this intersection is enough to generate the nilradical of $M[X_1, ..., X_n]$. Since $\frac{M}{Nil(M)}$ is a Noetherian module, $(\frac{N}{Nil(M)})[X_1, ..., X_n]$ is finitely generated. Let $\{f_1, ..., f_m\} \subset N$ generate the image of N modulo $Nil(M)[X_1, ..., X_n]$. To get a finite set of generators for N, simply add any single nonnil element $k \in N \cap M$ to the set $\{f_1, ..., f_m\}$. Since $kNil(M) = Nil(M)$, the set $\{k, f_1, ..., f_m\}$ is a finite set of generators for N. □

Corollary 2.19. *Let R be a ring, $M \in \mathbb{H}$ be a Φ-Noetherian faithful multiplication R-module and let P be a prime submodule of $M[X_1, ..., X_n]$. If $ht(P) > n$, then P is finitely generated.*

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