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MONOTONE ITERATIVE METHOD FOR WEIGHTED FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACE

Mohammed Benyoub and Samir Benaissa

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Abstract This paper deals with some existence results for a class of fractional differential equations involving Riemann-Liouville fractional derivative, by using the lower and upper solution method and the measure of noncompactness in the weighted space of continuous functions, we prove the existence of maximal and minimal solutions. Finally an example is provided to illustrate our results.

1 Introduction

Differential equations of fractional order have been recently proved to be valuable tools in the modeling of many physical phenomena [20]. There has been a significant theoretical development in fractional differential equations in recent years, see the monographs of Abbas *et al.* [1], Kilbas *et al.* [15], Podlubny [26], Somko *et al.* [27]. The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool for proving the existence of solutions for nonlinear ordinary differential equations in abstract spaces [25]. Also many people paid attention to the existence result of solution of the initial value problem for fractional differential equations involving Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, see [5, 34].
In [31], the lower and upper solution method was used to study the IVP

$$^{L}D^{\alpha}_{0^+}x(t) = f(t, x(t)), \quad t \in (0, 1), (0 < \alpha < 1),$$

$$x(0) = 0,$$

where $^{L}D^{\alpha}$ is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $I^{1-\alpha}_{0^+}$ is Riemann-Liouville integral of order $1 - \alpha$ and $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and $f(t, \cdot)$ is nondecreasing for each $t \in [0, 1]$.
Very recently Zhang [32], discussed the existence and uniqueness of solution of the initial value problem

$$^{L}D^{\alpha}_{0^+}x(t) = f(t, x(t)), \quad (0 < \alpha < 1, t > 0),$$

$$I^{1-\alpha}_{0^+}x(t) \mid_{t=0} = x_0,$$

was obtained under the assumption that $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is Lipchitz continuous, by using the Banach contraction mapping principal. In [22] a new proof of the maximum principle was given by using the completely monotonicity of the Mittag-Leffler type function.
Let X be a general Banach space and let $0 < \alpha < 1$. The objective of the paper was discussed by using the method of lower and upper solutions and its associated monotone iterative method of fractional differential equations

$$^{L}D^{\alpha}_{0^+}x(t) = f(t, x(t)), \quad t \in J' := (0, b], \tag{1.1}$$

$$\lim_{t \to 0^+} t^{1-\alpha}x(t) = x_0, \tag{1.2}$$

where $^{L}D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$,
$f : [0, b] \times X \to X$ is a continuous function.
Our aim in this paper is obtain similar results in more general setting, namely when the function right-hand side has values on infinite dimensional Banach space.
This paper is organized as follows. In Section 2, we recall some notion of fractional calculus and theory of measure noncompactness. In, Section 3, we prove the main results. Finally an illustrative example is given in Section 4.

2 Preliminaries

In this section, we introduce the notations, definitions, and preliminary facts that will be used in remainder of this paper.
Let $J := [0, b], b > 0$ and $(X, \| \cdot \|)$ be a Banach space, $C(J, X)$ be the space of X-valued continuous functions on J endowed with the uniform norm topology

$$\|x\|_\infty = \sup\{\|x(t)\|, t \in J\}.$$

$L^1(J,X)$ the space of X-valued Bochner integrable functions on J with norm

$$\|f\|_{L^1} = \int_0^b \|f(t)\|dt.$$

We consider the Banach space of continuous functions

$$C_{1-\alpha}(J,X) = \{x \in C(J',X) : \lim_{t\to 0^+} t^{1-\alpha}x(t) \text{ exists } \}.$$

A norm in this space is given by

$$\|x\|_\alpha = \sup_{t\in J} t^{1-\alpha}\|x\|.$$

For Ω a subset of the space $C_{1-\alpha}(J,X)$, define Ω_α by

$$\Omega_\alpha = \{x_\alpha, x \in \Omega\},$$

where

$$x_\alpha(t) = \begin{cases} t^{1-\alpha}x(t), & \text{if } t \in (0,b]; \\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t), & \text{if } t = 0. \end{cases}$$

It is clear that $x_\alpha \in C(J,X)$.

Lemma 2.1. *[34, Lemma 1] A set $\Omega \subset C_{1-\alpha}(J,X)$ is relatively compact if and only if Ω_α is relatively compact in $C(J,X)$.*

Definition 2.2. Let $0 < \alpha < 1$. A function $h : J \to X$ has a fractional integral if the following integral

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} h(s)ds, \tag{2.1}$$

is defined for $t \geq 0$, where $\Gamma(\cdot)$ is the gamma function.
The Reimann-Liouville derivative of h of order α is defined as

$$^LD^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds = \frac{d}{dt} I^{1-\alpha} h(t), \tag{2.2}$$

provided it is well defined for $t \geq 0$. The previous integral is taken in Bochner sense. Let $\phi_\alpha(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_\alpha = \begin{cases} \dfrac{t^{1-\alpha}}{\Gamma(\alpha)}, & \textit{if } t > 0, \\ 0, & \textit{if } t \leq 0. \end{cases}$$

Then

$$I^\alpha x(t) = (\phi_\alpha * x)(t),$$

and

$$^LD^\alpha x(t) = \frac{d}{dt}(\phi_{1-\alpha} * x)(t).$$

Lemma 2.3. *[9] Let $\alpha, \beta \in \mathbb{R}_+$. Then*

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.3}$$

and hence

$$\int_0^x t^{\alpha-1}(x-t)^{\beta-1}dt = x^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2.4}$$

The integral in the first equation of Lemma is known as Beta function $\mathcal{B}(\alpha,\beta)$.
We recall Gronwall's Lemma for singular kernels whose proof can be found in [30].

Lemma 2.4. *Let $v : J \to [0,\infty)$ be a real function and $\omega(\cdot)$ is a nonnegative, locally integrable function on J and there are constants λ and $0 < \alpha < 1$ such that*

$$v(t) \leq \omega(t) + \lambda \int_0^t (t-s)^{-\alpha} v(s)ds.$$

Then there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq \omega(t) + K\lambda \int_0^t (t-s)^{-\alpha} \omega(s)ds,$$

for every $t \in J$.

Next, we recall some definitions and properties of measure of noncompactness.

Definition 2.5. [4] Let X be a Banach space, $\mathcal{P}(X)$ denote the collection of all nonempty subsets of X, and $(\mathcal{A}, \geq)$ a partially ordered set A map $\beta : \mathcal{P}(X) \longrightarrow \mathcal{A}$ is called a measure of noncompactness on X, MNC for short, if

$$\beta(\overline{\mathrm{co}}\Omega) = \beta(\Omega)$$

for every $\Omega \in \mathcal{P}(X)$, where $\overline{\mathrm{co}}\Omega$ is the closure of convex hull of Ω.

Definition 2.6. [13] A measure of noncompactness β is called

(1) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}(X)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$,

(2) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X$, $\Omega \in \mathcal{P}(X)$,

(3) invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subseteq X$ and $\Omega \in \mathcal{P}(X)$,

(4) regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω,

(5) algebraically semiadditive if $\beta(\gamma_1 + \gamma_2) \leq \beta(\gamma_1) + \beta(\gamma_2)$, where $\gamma_1 + \gamma_2 = \{x + y : x \in \gamma_1, y \in \gamma_2\}$,

(6) $\beta(\lambda\gamma) \leq |\lambda|\beta(\gamma)$ for any $\lambda \in \mathbb{R}$.

(7) If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subsets and $\lim_{n\to+\infty} \beta(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact.

One of the most important examples of the measure of noncompactness possessing all these properties is the Hausdoff measure of noncopactness defined by:

$$\chi(\Omega) = \inf\{\epsilon > 0 : \Omega \ \text{ has a finite } \epsilon\text{−net}\}.$$

For any $W \subset C(J, X)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t x(s)ds : x \in W, \ \text{ for } t \in J = [0, b] \right\},$$

where $W(s) = \{x(s) \in X : x \in w\}$.

Lemma 2.7. *[11] If $W \subset C(J, X)$ is bounded and equicontinuous then $\beta(W(t))$ is continuous on J and*

$$\beta\left(\int_0^t W(s)ds\right) \leq \int_0^t \beta(W(s))ds, \ \ \textit{for } t \in [0, b].$$

Lemma 2.8. *[15] The linear initial value problem*

$$^{L}D^{\alpha}_{0^+} x(t) + \lambda x(t) = p(t), \ \ t \in (0, b],$$

$$\lim_{t\to 0^+} t^{1-\alpha} x(t) = x_0,$$

where $\lambda \geq 0$ is a constant and $p \in L^1(J, X)$, has the following integral representation for a solution

$$x(t) = \Gamma(\alpha) x_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) p(s)ds.$$

Where $E_{\alpha,\alpha}(t)$ is a Mittag-Leffler function.

Lemma 2.9. *[28] For $0 < \alpha \leq 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\lambda t^{\alpha})$ satisfies*

$$0 \leq E_{\alpha,\alpha}(-\lambda t^{\alpha}) \leq \frac{1}{\Gamma(\alpha)}, \ \ t \in [0, \infty), \lambda \geq 0.$$

Lemma 2.10. *[12] Suppose that X is an ordered Banach space $u_0, y_0 \in X$, $u_0 \leq y_0$, $D = [u_0, y_0]$, $N : D \to X$ is an increasing completely continuous operator and*

$$u_0 \leq N u_0, \ \ y_0 \geq N y_0.$$

Then the operator N has a minimal fixed u^* and a maximal fixed y^*. If we let

$$u_n = N u_{n-1} \ \ , \ \ y_n = N y_{n-1}, \ \ n = 1, 2 \cdots,$$

then

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq y_n \leq \cdots \leq y_2 \leq y_1 \leq y_0,$$

$$u_n \to u^*, y_n \to y^*.$$

Definition 2.11. A function $v(\cdot) \in C_{1-\alpha}(J, X)$ is called as a lower solution of (1.1)-(1.2) if it satisfies

$$ {}^{L}D_{0^+}^{\alpha} v(t) \leq f(t, v(t)), \quad t \in (0, b], \tag{2.5}$$

$$\lim_{t \to 0^+} t^{1-\alpha} v(t) \leq x_0. \tag{2.6}$$

Definition 2.12. A function $w(\cdot) \in C_{1-\alpha}(J, X)$ is called as an upper solution of (1.1)-(1.2) if it satisfies

$$ {}^{L}D_{0^+}^{\alpha} w(t) \geq f(t, w(t)), \quad t \in (0, b], \tag{2.7}$$

$$\lim_{t \to 0^+} t^{1-\alpha} w(t) \geq x_0. \tag{2.8}$$

3 Main Results

Before stating and proving the main results, we introduce following assumptions

(H_1) The map $f : [0, b] \times X \to X$ is continuous.

(H_2) There exists a constant $c > 0$ such that

$$\|f(t, x)\| \leq c(1 + t^{1-\alpha}\|x\|) \ \text{ for all } \ t \in [0, b] \ \text{ and } \ x \in X.$$

(H_3) there exists a constant $c_1 > 0$, and let $F(t, x) = f(t, x) + \lambda x(t)$ such that for each nonempty, bounded set $\Omega \subset C_{1-\alpha}(J, X)$

$$\beta(F(t, \Omega(t)) \leq c_1 \beta(\Omega(t)), \quad \text{for all} \ \ t \in [0, b],$$

where β is measure of noncompactness in X.

(H_4) Assume that $f : [0, b] \times X \to X$ satisfies

$$f(t, x) - f(t, v) + \lambda(x - v) \geq 0 \ \text{ for } \ \widehat{x} \leq v \leq x \leq \tilde{x},$$

where $\lambda \geq 0$ is a constant and $\widehat{x}$, $\tilde{x}$ are lower and upper solutions of problem (1.1)-(1.2) respectively.

Theorem 3.1. *Suppose (H_1)-(H_3) holds. The function $x(\cdot) \in C_{1-\alpha}(J, X)$ solves problem (1.1)-(1.2) if and only if it a fixed point of the operator N defined by*

$$\begin{aligned} N(x)(t) &= \Gamma(\alpha) x_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})[f(s, x(s)) + \lambda x(s)]ds. \end{aligned}$$

Proof. It's clear that the operator N is well defined, i.e., for every $x \in C_{1-\alpha}(J, X)$ and $t > 0$, the integral

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha})[f(s, x(s)) + \lambda x(s)]ds,$$

belongs to $C_{1-\alpha}(J, X)$.

Step 1. N is continuous.
Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C_{1-\alpha}(J, X)$. Then

$$\begin{aligned} & t^{1-\alpha}\|N(x_n)(t) - N(x)(t)\| \\ & \leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|f(s, x_n(s)) - f(s, x(s))\|ds \\ & \quad + \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|x_n(s) - x(s)\|ds \\ & \leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha}\|f(s, x_n(s)) - f(s, x(s))\|ds \\ & \quad + \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha}\|x_n(s) - x(s)\|ds \\ & \leq \frac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds\|f(s, x_n(s)) - f(s, x(s))\|_{\alpha} \\ & \quad + \frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds\|x_n(s) - x(s)\|_{\alpha} \\ & \leq \frac{b^{\alpha}}{\Gamma(\alpha)} B(\alpha, \alpha)\|f(\cdot, x_n(\cdot)) - f(\cdot, x(\cdot))\|_{\alpha} + \frac{\lambda b^{\alpha}}{\Gamma(\alpha)} B(\alpha, \alpha)\|x_n(\cdot) - x(\cdot)\|_{\alpha} \end{aligned}$$

Using the hypothesis (H_2) we have

$$\|N(x_n)(t) - N(x)(t)\|_\alpha \longrightarrow 0 \;\text{ as }\; n \to +\infty.$$

Step 2. N maps bounded sets into bounded sets in $C_{1-\alpha}(J, X)$.
Indeed, it enough to show that there exists a positive constant l such that for each $x \in B_r = \{x \in C_{1-\alpha}(J, X) : \|x\|_\alpha \le r\}$ one has $\|N(x)\|_\alpha \le l$.
Let $x \in B_r$. Then for each $t \in (0, b]$, by (H_2) we have

$$\begin{aligned}
t^{1-\alpha}\|Nx(t)\| &\le \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, x(s))\|ds \\
&+\frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)\|ds \\
&\le \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}c(1 + s^{1-\alpha}\|x(s)\|)ds \\
&+\frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}s^{1-\alpha}\|x(s)\|ds \\
&\le \|x_0\| + \frac{ct^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+r)ds \\
&+\frac{\lambda t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}r ds \\
&\le \|x_0\| + \frac{cb^{1-\alpha}(1+r)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&+\frac{\lambda b^{1-\alpha}r}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds,
\end{aligned}$$

$$\|N(x)\|_\alpha \le \|x_0\| + \frac{cb(1+r)}{\Gamma(\alpha+1)} + \frac{\lambda b^\alpha r\Gamma(\alpha)}{\Gamma(2\alpha)} := l.$$

Step 3. N maps bounded sets into equicontinuous sets.
Let $t_1, t_2 \in (0, b]$, $t_1 \le t_2$, let B_r be a bounded set in $C_{1-\alpha}(J, X)$ as in step 2, and let $x \in B_r$, we have

$$\begin{aligned}
&\|t_2^{1-\alpha}N(x)(t_2) - t_1^{1-\alpha}N(x)(t_1)\| \\
&\le \Gamma(\alpha)\|x_0\|\left[E_{\alpha,\alpha}(-\lambda t_2^\alpha) - E_{\alpha,\alpha}(-\lambda t_1^\alpha)\right] \\
&+\left(t_2^{1-\alpha} - t_1^{1-\alpha}\right)\left\|\int_0^{t_1}[(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1}\right. \\
&E_{\alpha,\alpha}(-\lambda(t_1-s)^\alpha)]f(s, x(s))ds\| \\
&+t_2^{1-\alpha}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha)f(s, x(s))ds\right\| \\
&+\left(t_2^{1-\alpha} - t_1^{1-\alpha}\right)\left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha) - (t_1-s)^{\alpha-1}\right.\right. \\
&E_{\alpha,\alpha}(-\lambda(t_1-s)^\alpha)]\,(\lambda x(s))ds\| \\
&+t_2^{1-\alpha}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_2-s)^\alpha)(\lambda x(s))ds\right\| \\
&\le I_1 + \frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}](c(1 + s^{1-\alpha}\|x(s)\|))ds \\
&+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(c(1 + s^{1-\alpha}\|x(s)\|))ds \\
&+\frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}](\lambda s^{\alpha-1}s^{1-\alpha}\|x(s)\|)ds \\
&+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(\lambda s^{\alpha-1}s^{1-\alpha}\|x(s)\|)ds \\
&\le I_1 + \frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha+1)}(c(1+r))\left[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha\right] + \frac{t_2^{1-\alpha}}{\Gamma(\alpha+1)}(c(1+r))\left[((t_2-t_1)^\alpha\right] \\
&+\frac{\Gamma(\alpha)(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(2\alpha)}(\lambda r))\left[(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha\right] + \frac{\Gamma(\alpha)t_2^{1-\alpha}}{\Gamma(2\alpha)}(\lambda r))\left[((t_2-t_1)^\alpha\right]
\end{aligned}$$

where

$$I_1 = \Gamma(\alpha)\|x_0\|[E_{\alpha,\alpha}(-\lambda t_2^\alpha) - E_{\alpha,\alpha}(-\lambda t_1^\alpha)].$$

Appling by the function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ is uniformly continuous on $[0,b]$, we have I_1 tend to zero independently of $x \in B_r$ as $t_2 \to t_1$.
Thus $\|t_2^{1-\alpha}N(x)(t_2) - t_1^{1-\alpha}N(x)(t_1)\|$ tend to zero independently of $x \in B_r$ as $t_2 \to t_1$, which means that the set NB_r is equicontinuous.
Define $B_{r_0} = \{x \in C_{1-\alpha} : \|x\|_\alpha \le r_0\}$, where $r_0 > 0$ is taken so that

$$r_0 \ge \left(\|x_0\| + \frac{cb}{\Gamma(\alpha+1)}\right)(1-L)^{-1},$$

such that

$$\frac{cb}{\Gamma(\alpha+1)} + \frac{\lambda b^\alpha \Gamma(\alpha)}{\Gamma(2\alpha)} \le L < 1.$$

Then B_{r_0} is closed convex bounded and hence $NB_{r_0} \subset B_{r_0}$.
Now we prove that there exists a compact subset $M \subset B_{r_0} \subset$ such that $NM \subset M$. We first costruct a series of sets $\{M_n\} \subset B_{r_0}$ by

$$M_0 = B_{r_0},\ M_1 = \overline{\text{conv}}NM_0,\ M_{n+1} = \overline{\text{conv}}NM_n,\ n = 1, 2\cdots.$$

From the above proof it is easy to see $M_{n+1} \subset M_n$ for $n = 1, 2\cdots$ and each $\widehat{M}_n$ is equicontinuous. Further from Definition 2.6 and Lemma 2.7 we can derive that

$$\begin{aligned}
\beta\left(\widehat{M}_{n+1}(t)\right) &= \beta(t^{1-\alpha}M_{n+1}(t)) = \beta(t^{1-\alpha}NM_n(t))\\
&\le \beta\left[\Gamma(\alpha)x_0E_{\alpha,\alpha}(-\lambda t^\alpha) + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}(E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)F(s, M_n(s))ds\right]\\
&\le c_1\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\beta(M_n(s))ds.
\end{aligned}$$

Define the function $F_n(t) = \beta(M_n(t))$ for $n = 1, 2\cdots$ we get

$$F_{n+1}(t) \le c_1\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F_n(s)ds, \tag{3.1}$$

for $n = 1, 2, \cdots$ the fact $M_{n+1} \subset M_n$.
Taking limit as $n \to \infty$ in (3.1) we get

$$F(t) \le c_1\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)ds,$$

for all $t \in J$. An application of Lemma 2.4 yields $F(t) = 0$ for all $t \in J$.
Therefore, $\cap_{n=1}^\infty M_n = M$ is nonempty and compact in $C_{1-\alpha}(J,X)$ due to Definition 2.6, and $NM \subset M$ by definition of M_n.
Up to now we have verified that there exists a nonempty bounded convex and compact subset M such that $NM \subset M$. An employment of Schauder's fixed point theorem shows that there exists at least a fixed point x of N in M. Combining with the fact that $\lim\limits_{t\to0^+} E_{\alpha,\alpha}(-\lambda t^\alpha) = E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$ yields that $\lim\limits_{t\to0^+} t^{1-\alpha}(Nx)(t) = x_0$. The proof is complete. □

Theorem 3.2. *Assume (H_1)-(H_4), hold, and $v, w \in C_{1-\alpha}(J,X)$ are lower and upper solutions of (1.1)-(1.2) respectively such that*

$$v(t) \le w(t),\quad 0 \le t \le b.$$

Then, the fractional IVP (1.1)-(1.2) has a minimal solution u^* and a maximal solution y^* such that

$$u^* = \lim_n N^n v,\quad y^* = \lim_n N^n w.$$

Proof. Suppose that functions $v, w \in C_{1-\alpha}(J,X)$ are lower and upper solution of IVP (1.1)-(1.2). We consider in $C_{1-\alpha}(J,X)$ the order induced by the sector $D = [v,w]$ define $[v,w] = \{x \in C_{1-\alpha}(J,X) : v \le x \le w\}$, then there are $v \le Nv$, $w \ge Nw$. In fact, by the definition of the lower solution, there exist $p(t) \ge 0$ and $\epsilon \ge 0$, we have

$${}^L D_{0^+}^\alpha v(t) = f(t, v(t)) - p(t),\quad t \in (0, b],$$

$$\lim_{t\to0^+} t^{1-\alpha}v(t) = x_0 - \epsilon.$$

Using Theorem 3.1 and Lemma 2.9, one has

$$\begin{aligned}
v(t) &= \Gamma(\alpha)(x_0 - \epsilon)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\left[f(s, v(s)) + \lambda v(s) - p(s)\right]ds\\
&\le (Nv)(t).
\end{aligned}$$

Similarly, there is $w \geq Nw$.
The operator $N : D \to C_{1-\alpha}(J, X)$ is increasing and completely continuous by the use of Lemma 2.10 the existence of u^*, y^* is obtained. The proof is complete. □

4 An example

As an application of our results we consider the following fractional equation

$$ {}^{L}D^{\alpha}_{0^+}x(t) = \frac{1}{e^{t^2}+1}\left\{\ln(|x_k|+1) + \frac{1}{1+k}\right\}_{k\in\mathbb{N}}, \quad t \in J = [0,1], \tag{4.1} $$

$$ \lim_{t\to 0^+} t^{1-\alpha}x(t) = x_0, \tag{4.2} $$

c_0 represents the space of all sequences converging to zero, which is a Banach space with respect to the norm

$$ \|x\| = \sup_k |x_k|. $$

Let $t \in J$ and $x = \{x_k\}_k \in c_0$, we have

$$ \begin{aligned} \|f(t,x)\|_\infty &= \frac{1}{e^{t^2}+1}\| \ln(|x_k|+1) + \tfrac{1}{k+1}\|_\infty \\ &\leq \frac{1}{e^{t^2}+1}\left(\sup_k |x_k| + 1\right) \\ &\leq \frac{1}{e^{t^2}+1}(1+\|x\|_\infty). \end{aligned} $$

Hence condition $(H_1)-(H_2)$ are satisfied with $c = \dfrac{1}{e+1}$, for all $t \in [0,1]$.
So, that function F by defined

$$ F(t,x(t)) = \frac{1}{e^{t^2}+1}\left\{\ln(|x_k|+1) + \frac{1}{k+1}\right\}_{k\in\mathbb{N}} + \lambda x(t), \quad \text{for all} \quad t \in [0,1]. $$

We recall that the measure of noncompactness β in space c_0 can be computed by means of the formula

$$ \beta(\Omega) = \lim_{n\to+\infty} \sup_{x\in\Omega} \|(I-P_n)x\|_\infty. $$

Where Ω is a bounded subset in c_0 and P_n is the projection onto the linear span of n vectors, we get

$$ \beta(F(t,\Omega)) \leq c_1\beta(\Omega(t)) \quad \text{for all} \quad t \in [0,1], $$

with $c = (e+1)^{-1}$. Therefore $\beta(F(t,\Omega(t)) \leq c_1\beta(\Omega(t))$, with $c_1 = \max(c,\lambda)$ due to (H_3) and definition . Then by Theorem 3.2 the problem (4.1)-(4.2) has a lower and upper solutions.

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Author information

Mohammed Benyoub, Department of mathematics, École normale supérieure Laghouat, B.P.4033, 03000, Laghouat, Algeria,.
E-mail: mohamedbenyoub64@yahoo.com

Samir Benaissa, Laboratory of Statistics and Stochastic Processes, Department of Probability and Statistics, Djillali Liabes University, Sidi Bel Abbes 22000, Algeria,.
E-mail: benaissamir@yahoo.fr

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