Palestine Journal of Mathematics

Vol. 9(1)(2020) , 12–14

ON ANTI-DIAGONALS RATIO INVARIANCE WITH EXPONENTIATION OF A 2×2 MATRIX: TWO NEW PROOFS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11C20.

Keywords and phrases: Anti-diagonals ratio invariance.

Abstract We state and prove (first by construction, and then by induction) the invariance, with respect to matrix power, of the anti-diagonals ratio of a general 2×2 matrix, adding these proofs to others found previously.

1 Introduction

Let

$$\mathbf{M} = \mathbf{M}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

be a general 2×2 matrix (assuming $A, B, C, D \neq 0$), with $\text{Tr}\{\mathbf{M}\} = A + D$ and $|\mathbf{M}| = AD - BC$. A somewhat unusual result holds which remains little known in the literature.

Theorem 1.1. *Unless otherwise indeterminate, the ratio of the two anti-diagonal terms in $\mathbf{M}^n$ is the quantity B/C, being invariant with respect to integer power $n \geq 1$.*

In a previous publication [1] the observation was formulated in four different ways (as two direct proofs, and two inductive ones), while in [2] it was extended to establish invariance of all anti-diagonal ratios within an arbitrary dimension tri-diagonal matrix. The article [3] delivers yet another (direct) proof for the case when $D = A$ and the matrix $\mathbf{M}$ becomes slightly specialised. This paper offers two further proofs of Theorem 1.1 accordingly.

2 The Proofs

2.1 Proof I

Proof. Consider, writing the 2-square identity matrix as $\mathbf{I}_2$,

$$\sum_{i \geq 0} (\mathbf{M}z)^i = [\mathbf{I}_2 - \mathbf{M}z]^{-1} = \frac{1}{1 - f(z; A, B, C, D)} \mathbf{E}(z; A, B, C, D) \tag{I.1}$$

after a little algebra, where $f(z; A, B, C, D) = z(\text{Tr}\{\mathbf{M}\} - |\mathbf{M}|z)$ and

$$\mathbf{E}(z; A, B, C, D) = \begin{pmatrix} 1 - Dz & Bz \\ Cz & 1 - Az \end{pmatrix}. \tag{I.2}$$

In the l.h.s. sum of (I.1), the terms within $\mathbf{M}^n$ each have an accompanying multiplier z^n. To capture the corresponding z^n terms in the r.h.s. version of the matrix $\mathbf{M}^n$ would require some effort (by writing $[1 - f(z; A, B, C, D)]^{-1}$ as $\sum_{j \geq 0} f^j(z; A, B, C, D) = \sum_{j \geq 0} z^j (\text{Tr}\{\mathbf{M}\} - |\mathbf{M}|z)^j$[1]), but their precise identification is unnecessary for it is enough to see that the anti-diagonal terms of $\mathbf{E}$ alone are Bz and Cz—each possessing the *same* power of z—whence the

[1]And, further, expressing the summand thereof as a binomially expanded sum $\sum_{p=0}^{j} \binom{j}{p} (\text{Tr}\{\mathbf{M}\})^{j-p} (-|\mathbf{M}|)^p z^{j+p}$.

anti-diagonals ratio of the r.h.s. form of $\mathbf{M}^n$ will (through cancellation) simplify to just B/C; since the power n is arbitrary, the result is established. □

Remark 2.1. It is clear that if $D = A$ then the diagonals ratio of $\mathbf{M}^n(A, B, C, A)$ is unity (since the diagonal elements of $\mathbf{E}$ are then each $1 - Az$), a fact mentioned (in passing) in [1, p. 362] and established formally in [3].

Line of Argument: Examples. We offer some detail, by way of examples, to the logical argument outlined in the proof for the reader's benefit. To assist in this, we write the function $f(z; A, B, C, D)$ as $f(z; A, B, C, D) = z(\alpha + \beta z)$ (where $\alpha = \alpha(A, D) = \mathrm{Tr}\{\mathbf{M}\} = A + D$ and $\beta = \beta(A, B, C, D) = -|\mathbf{M}| = -(AD - BC)$), and split the matrix $\mathbf{E}$ as

$$\mathbf{E}(z; A, B, C, D) = \mathbf{I}_2 + \mathbf{E}_s z, \tag{2.1}$$

where $\mathbf{E}_s(A, B, C, D) = \left(\begin{smallmatrix} -D & B \\ C & -A \end{smallmatrix}\right)$. Then (I.1) can be written as

$$\begin{aligned}\sum_{i\geq 0}(\mathbf{M}z)^i &= [1 + z(\alpha + \beta z) + z^2(\alpha + \beta z)^2 + z^3(\alpha + \beta z)^3 + \cdots](\mathbf{I}_2 + \mathbf{E}_s z) \\ &= [1 + \alpha z + (\alpha^2 + \beta)z^2 + \alpha(\alpha^2 + 2\beta)z^3 + \cdots](\mathbf{I}_2 + \mathbf{E}_s z). \end{aligned}\tag{2.2}$$

Thus, gathering up the r.h.s. terms in z^3, for instance, (2.2) delivers

$$\begin{aligned}(\mathbf{M}z)^3 &= \alpha(\alpha^2 + 2\beta)z^3 \cdot \mathbf{I}_2 + (\alpha^2 + \beta)z^2 \cdot \mathbf{E}_s z \\ &= \begin{pmatrix} \alpha(\alpha^2 + 2\beta) - D(\alpha^2 + \beta) & B(\alpha^2 + \beta) \\ C(\alpha^2 + \beta) & \alpha(\alpha^2 + 2\beta) - A(\alpha^2 + \beta) \end{pmatrix} z^3, \end{aligned}\tag{2.3}$$

so that $\mathbf{M}^3$ has anti-diagonals ratio B/C. Note that (from the above example) in terms of A, B, C, D,

$$\mathbf{M}^3 = \begin{pmatrix} A^3 + 2ABC + BCD & B(A^2 + AD + BC + D^2) \\ C(A^2 + AD + BC + D^2) & D^3 + ABC + 2BCD \end{pmatrix} \tag{2.4}$$

which, as an aside, illustrates Remark 2.1 (because interchanging A and D in one diagonal term generates the other; the variables $\alpha(A, D)$ and $\beta(A, B, C, D)$ are both symmetric in A, D).

Similarly, collecting up the r.h.s. terms in z^2 from (2.2) we find

$$\begin{aligned}(\mathbf{M}z)^2 &= (\alpha^2 + \beta)z^2 \cdot \mathbf{I}_2 + \alpha z \cdot \mathbf{E}_s z \\ &= \begin{pmatrix} \alpha^2 + \beta - D\alpha & B\alpha \\ C\alpha & \alpha^2 + \beta - A\alpha \end{pmatrix} z^2 \\ &= \begin{pmatrix} A^2 + BC & B(A + D) \\ C(A + D) & BC + D^2 \end{pmatrix} z^2, \end{aligned}\tag{2.5}$$

with $\mathbf{M}^2$ having anti-diagonals ratio B/C and Remark 2.1 confirmed again.

It is seen that (2.2) offers trivially the self-consistent equation

$$\mathbf{M}z = \alpha z \cdot \mathbf{I}_2 + 1 \cdot \mathbf{E}_s z = \begin{pmatrix} \alpha - D & B \\ C & \alpha - A \end{pmatrix} z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} z. \tag{2.6}$$

Remark 2.2. It is of interest that the actual entries of the exponentiated matrix $\mathbf{M}$ can be deduced (at low powers n, at least) from visual inspection of a (two vertex) graph, where A represents a single path (self-loop) from vertex 1 to itself, B a path from vertex 1 to vertex 2, C a path from vertex 2 to vertex 1, and D a loop from vertex 2 to itself. With $n = 2$, a 2-path route from vertex 1 to itself is achieved by the ordered moves AA or BC whose sum $A^2 + BC$, when treated algebraically, matches the $(1, 1)$ entry in $\mathbf{M}^2$ (2.5). The $(2, 1)$ entry in $\mathbf{M}^3$ (2.4) is the result of algebraicising as a sum the 3-path options CAA, DCA, CBC or DDC from vertex 2 to vertex 1, giving the expression $C(A^2 + AD + BC + D^2)$.

2.2 Proof II

Proof. Let $\mathcal{M}_2[\mathbb{F}]$ be the set of 2×2 matrices with entries from a field $\mathbb{F}$ (the set forming a vector space over $\mathbb{F}$ of dimension $2^2 = 4$), and define $G : \mathcal{M}_2[\mathbb{F}] \to \mathbb{F}$ to be the linear map

$$G(\mathbf{S}) = \mathbf{S}_{1,2} - t\mathbf{S}_{2,1} \tag{II.1}$$

acting on any matrix $\mathbf{S} \in \mathcal{M}_2[\mathbb{F}]$, where $\mathbf{S}_{i,j}$ is the row i, column j, element of $\mathbf{S}$, and $t = B/C \in \mathbb{F}$. We seek to show that $G(\mathbf{M}^n) = 0$ for $n \geq 1$. Noting that

$$\begin{aligned} G(\mathbf{M}^1) &= G(\mathbf{M}) = \mathbf{M}_{1,2} - t\mathbf{M}_{2,1} = B - t \cdot C = 0, \\ G(\mathbf{M}^2) &= (\mathbf{M}^2)_{1,2} - t(\mathbf{M}^2)_{2,1} = B(A + D) - t \cdot C(A + D) = 0, \end{aligned} \tag{II.2}$$

we assume the result holds for some arbitrary values of $n = k, k - 1$ $(k \geq 2)$—in other words, $0 = G(\mathbf{M}^k) = G(\mathbf{M}^{k-1})$. Our inductive step proceeds as follows, based on the familiar result (Cayley-Hamilton) $\mathbf{M}^2 = \text{Tr}\{\mathbf{M}\}\mathbf{M} - |\mathbf{M}|\mathbf{I}_2$ which reads

$$\mathbf{M}^{k+1} = \text{Tr}\{\mathbf{M}\}\mathbf{M}^k - |\mathbf{M}|\mathbf{M}^{k-1} \tag{II.3}$$

on multiplying throughout by $\mathbf{M}^{k-1}$. Thus,

$$\begin{aligned} G(\mathbf{M}^{k+1}) &= G(\text{Tr}\{\mathbf{M}\}\mathbf{M}^k - |\mathbf{M}|\mathbf{M}^{k-1}) \\ &= \text{Tr}\{\mathbf{M}\}G(\mathbf{M}^k) - |\mathbf{M}|G(\mathbf{M}^{k-1}) \end{aligned} \tag{II.4}$$

(by the linearity of G) $= \text{Tr}\{\mathbf{M}\} \cdot 0 - |\mathbf{M}| \cdot 0$ (by assumption) $= 0$, as required. □

Remark 2.3. It would seem appropriate to mention that the inductive approach deployed in Proof II—which connects with Proof IV of [1] in appealing to (II.3)—lends itself to a proof strategy for the corresponding version of Theorem 1.1 in the tri-diagonal matrix case [2] alluded to (where such a matrix, of dimension n, has $n-1$ linear maps to consider, each associated with a different anti-diagonals ratio that remains invariant w.r.t. matrix power); we leave the interested reader to think about this as an exercise.

Theorem 1.1 continues to be a surprising result within linear algebra, and work continues on new formulations. The authors are most grateful to Dr. Richard Pinch for suggesting, through private communication, the two proofs presented here.

References

[1] P. J. Larcombe, A note on the invariance of the general 2×2 matrix anti-diagonals ratio with increasing matrix power: four proofs, *Fib. Quart.* **53**, 360–364 (2015).

[2] P. J. Larcombe and E. J. Fennessey, A new tri-diagonal matrix invariance property, *Palest. J. Math.* **7**, 9–13 (2018).

[3] P. J. Larcombe and E. J. Fennessey, A note on two rational invariants for a particular 2×2 matrix, *Palest. J. Math.* **7**, 410–413 (2018).

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Received: October 2, 2018.
Accepted: December 19, 2019.