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CERTAIN INTEGRAL REPRESENTATIONS OF SOME QUADRUPLE HYPERGEOMETRIC SERIES

Maged G. Bin-Saad and Jihad A. Younis

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Abstract A new class of quadruple hypergeometric series is presented. We also give integral representations of Euler- type and Laplace-type for the new class of series.

1 Introduction

In [3], Exton introduced twenty one complete quadruple hypergeometric functions, which he denoted by symbols $K_1, K_2, ..., K_{21}$. In [7], eighty three complete quadruple hypergeometric functions given by $F_1^{(4)}, F_2^{(4)}, ..., F_{83}^{(4)}$ were defined by Sharma and Parihar. Very recently, Bin-Saad et al. [1] introduced five new quadruple hypergeometric functions whose names are $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$ to investigate their five Laplace integral representations which include the confluent hypergeometric functions ${}_0F_1, {}_1F_1$, the Humbert functions Φ_2, Φ_3 and Ψ_2 in their kernels, we recall these quadruple hypergeometric functions are defined by

$$X_6^{(4)}\left(a_1, a_1, a_2, a_1, a_1, a_2, a_2, a_2; c_1, c_1, c_2, c_2; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+2p}}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.1}$$

$$X_7^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_1; c_1, c_2, c_3, c_4; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.2}$$

$$X_8^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_1; c_1, c_1, c_2, c_3; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.3}$$

$$X_9^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_1; c_2, c_1, c_1, c_3; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.4}$$

$$X_{10}^{(4)}\left(a_1, a_1, a_1, a_2, a_1, a_2, a_3, a_3; c_1, c_2, c_3, c_4; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \tag{1.5}$$

Motivated by the works [1], [3] and [7], we consider five new hypergeometric series of four variables as below:

$$X_{16}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_3; x, y, z, u\right)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.6}$$

$$X_{17}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.7}$$

$$X_{18}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.8}$$

$$X_{19}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.9}$$

$$X_{20}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \tag{1.10}$$

This paper is devoted to obtain several integral representations for the new quadruple series defined above. In section 2, we present five integral representations of Euler-type for each series $X_i^{(4)}(\ i = 16, 17, 18, 19, 20)$ in terms of the Gaussian hypergeometric function ${}_2F_1$ [10], the Appell's functions of two variables F_1, F_2 and F_4 (see, for details [10] and [11]), the Horn's functions H_3 and H_4 of two variables (see[10]), the Exton's triple series X_3, X_4, X_5, X_6, X_7 and X_{17} (see [4]), the Lauricella's triple series F_E and F_F (cf. [5, 6], the Lauricella's quadruple series $F_C^{(4)}$ (for more details see [10]) and the quadruple series $X_{16}^{(4)}$, and in the next section, Laplace -type integrals are obtained for each series $X_i^{(4)}(\ i = 16, 17, 18, 19, 20)$.

2 Integral representations of Euler-Type

Now, by means of the Gaussian hypergeometric function ${}_2F_1$, Appell hypergeometric functions F_1, F_2 and F_4, Horn's functions H_3 and H_4 of two variables, the Exton's triple functions X_3, X_4, X_5, X_6, X_7 and X_{17}, the Lauricella's triple functions F_E and F_F and the quadruple series $F_C^{(4)}$ and $X_{16}^{(4)}$, we investigate some further integral representations of Euler-type for $X_i^{(4)}(\ i = 16, 17, 18, 19, 20)$ as follows:

$$X_{16}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \frac{\Gamma(c_2)}{\Gamma(a_3)\Gamma(c_2-a_3)}$$
$$\times \int_0^1 \alpha^{a_3-1}(1-\alpha)^{c_2-a_3-1}(1-\alpha z)^{-a_1} X_3\left(a_1,a_2;c_1,c_3;\frac{x}{(1-\alpha z)^2},\frac{y}{(1-\alpha z)},\frac{u}{(1-\alpha z)}\right) d\alpha$$
$$(Re(a_3) > 0, Re(c_2-a_3) > 0), \tag{2.1}$$

$$X_{16}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \frac{2\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times \int_0^{\frac{\pi}{2}} \left(sin^2\alpha\right)^{a_1-\frac{1}{2}} \left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}} \left(1+xsin^2\alpha\, tan^2\alpha\right)^{c_1-a_1-1}$$

$$\times\left(1-ysin^2\alpha\right)^{-a_2}F_2\left(1+a_1-c_1,a_3,a_2;c_2,c_3;\lambda_1 z,\lambda_2 u\right)d\alpha$$
$$\left(\lambda_1=-\frac{tan^2\alpha}{\left(1+xsin^2\alpha\,tan^2\alpha\right)},\lambda_2=-\frac{tan^2\alpha}{\left(1+xsin^2\alpha\,tan^2\alpha\right)\left(1-ysin^2\alpha\right)}\right),$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0\right), \tag{2.2}$$

$$X_{16}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(c_3-a_2)}$$
$$\times\int_0^\infty\left(e^{-\alpha}\right)^{a_2}\left(1-e^{-\alpha}\right)^{c_3-a_2-1}\left(1-ue^{-\alpha}\right)^{-a_1}X_6\left(a_1,1+a_2-c_3,a_3;c_1,c_2;\right.$$
$$\left.\frac{x}{\left(1-ue^{-\alpha}\right)^2},\frac{-ye^{-\alpha}}{\left(1-e^{-\alpha}\right)\left(1-ue^{-\alpha}\right)},\frac{z}{\left(1-ue^{-\alpha}\right)}\right)d\alpha$$
$$\left(Re(a_2)>0,Re(c_3-a_2)>0\right), \tag{2.3}$$

$$X_{16}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right)=\frac{2\Gamma(c_1)M^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{1+a_1+a_2-2c_1}$$
$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]^{c_1-a_1-1}$$
$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]^{-a_2}F_2\left(1+a_1-c_1,a_3,a_2;\right.$$
$$\left.c_2,c_3;\lambda_1 z,\lambda_2 u\right)d\alpha$$
$$\left(\lambda_1=\frac{-M\left(cos^2\alpha+Msin^2\alpha\right)tan^2\alpha}{\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]},\right.$$
$$\left.\lambda_2=\frac{-M\left(cos^2\alpha+Msin^2\alpha\right)^2tan^2\alpha}{\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]}\right),$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0,M>0\right), \tag{2.4}$$

$$X_{16}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right)=\frac{2\Gamma(c_1)\left(1+M\right)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}\left(1+Msin^2\alpha\right)^{1+a_1+a_2-2c_1}$$
$$\times\left[\left(1+Msin^2\alpha\right)+\left(1+M\right)^2xsin^2\alpha\,tan^2\alpha\right]^{c_1-a_1-1}$$
$$\times\left[\left(1+Msin^2\alpha\right)-\left(1+M\right)ysin^2\alpha\right]^{-a_2}F_2\left(1+a_1-c_1,a_3,a_2;\right.$$
$$\left.c_2,c_3;\lambda_1 z,\lambda_2 u\right)d\alpha$$
$$\left(\lambda_1=\frac{-\left(1+M\right)\left(1+Msin^2\alpha\right)tan^2\alpha}{\left[\left(1+Msin^2\alpha\right)+\left(1+M\right)^2xsin^2\alpha\,tan^2\alpha\right]},\right.$$
$$\left.\lambda_2=\frac{-\left(1+M\right)\left(1+Msin^2\alpha\right)^2tan^2\alpha}{\left[\left(1+Msin^2\alpha\right)+\left(1+M\right)^2xsin^2\alpha\,tan^2\alpha\right]\left[\left(1+Msin^2\alpha\right)-\left(1+M\right)ysin^2\alpha\right]}\right),$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0,M>-1\right), \tag{2.5}$$

$$X_{17}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)$$
$$=\frac{8\Gamma(a_1+a_2+a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(c_1-a)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}$$
$$\times\left(cos^2\alpha\right)^{a_2-\frac{1}{2}}\left(sin^2\beta\right)^{a_1+a_2-\frac{1}{2}}\left(cos^2\beta\right)^{a_3-\frac{1}{2}}\left(sin^2\gamma\right)^{a-\frac{1}{2}}\left(cos^2\gamma\right)^{c_1-a-\frac{1}{2}}$$
$$\times F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_2,a,c_1-a,c_3;\lambda_1x,\lambda_2y,\lambda_3z,\lambda_4u\right)$$
$$\times d\alpha d\beta d\gamma$$
$$\left(\lambda_1=4sin^4\alpha\ sin^4\beta,\ \lambda_2=sin^22\alpha\ sin^4\beta\ sin^2\gamma,\right.$$
$$\left.\lambda_3=sin^2\alpha\ sin^22\beta\ cos^2\gamma,\ \lambda_4=sin^22\alpha\ sin^4\beta\right),$$
$$\left(Re(a_i)>0,(i=1,2,3),Re(a)>0,Re(c_1-a)>0\right), \tag{2.6}$$

$$X_{17}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)$$
$$=\frac{\Gamma(c_3)\left(S-T\right)^{a_2}\left(R-T\right)^{c_3-a_2}}{\Gamma(a_2)\Gamma(c_3-a_2)\left(S-R\right)^{c_3-a_1-1}}\int_R^S\left(\alpha-R\right)^{a_2-1}\left(S-\alpha\right)^{c_3-a_2-1}$$
$$\times\left(\alpha-T\right)^{a_1-c_3}\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)u\right]^{-a_1}$$
$$\times X_7\left(a_1,1+a_2-c_3,a_3;c_2,c_1;\lambda_1x,\lambda_2y,\lambda_3z\right)d\alpha$$
$$\left(\lambda_1=\frac{\left(S-R\right)^2\left(\alpha-T\right)^2}{\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)u\right]^2}\right.,$$
$$\lambda_2=-\frac{\left(S-R\right)\left(S-T\right)\left(\alpha-R\right)\left(\alpha-T\right)}{\left(R-T\right)\left(S-\alpha\right)\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)u\right]},$$
$$\left.\lambda_3=\frac{\left(S-R\right)\left(\alpha-T\right)}{\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)u\right]}\right),$$
$$\left(Re(a_2)>0,Re(c_3-a_2)>0,T<R<S\right), \tag{2.7}$$

$$X_{17}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_3)\Gamma(c_1-a_3)}$$
$$\times\int_0^1\alpha^{a_3-1}\left(1-\alpha\right)^{c_1-a_3-1}\left(1-\alpha z\right)^{-a_1}X_4\left(a_1,a_2;c_2,c_1-a_3,c_3;\frac{x}{\left(1-\alpha z\right)^2},\right.$$
$$\left.\frac{\left(1-\alpha\right)y}{\left(1-\alpha z\right)},\frac{u}{\left(1-\alpha z\right)}\right)d\alpha$$
$$\left(Re(a_3)>0,Re(c_1-a_3)>0\right), \tag{2.8}$$

$$X_{17}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}$$
$$\times\int_0^1\alpha^{a_1-1}\left[\left(1-\alpha\right)+\alpha^2x\right]^{c_2-a_1-1}F_F\left(1+a_1-c_2,1+a_1-c_2,1+a_1-c_2,\right.$$
$$\left.a_2,a_3,a_2;c_3,c_1,c_1;\lambda u,\lambda z,\lambda y\right)d\alpha$$
$$\left(\lambda=-\frac{\alpha}{\left[\left(1-\alpha\right)+\alpha^2x\right]}\right),$$
$$\left(Re(a_1)>0,Re(c_2-a_1)>0\right), \tag{2.9}$$

$$X_{17}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_1)\left(1+M\right)^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^1\alpha^{a_1-1}\left(1-\alpha\right)^{c_1-a_1-1}\left(1+M\alpha\right)^{a_2+a_3-c_1}\left[\left(1+M\alpha\right)-\left(1+M\right)\alpha y\right]^{-a_2}$$
$$\times\left[\left(1+M\alpha\right)-\left(1+M\right)\alpha z\right]^{-a_3}H_4\left(1+a_1-c_1,a_2;c_2,c_3;\lambda_1 x,\lambda_2 u\right)d\alpha$$
$$\left(\lambda_1=\frac{\left(1+M\right)^2\alpha^2}{\left(1-\alpha\right)^2},\lambda_2=-\frac{\left(1+M\right)\alpha\left(1+M\alpha\right)}{\left(1-\alpha\right)\left[\left(1+M\alpha\right)-\left(1+M\right)\alpha y\right]}\right),$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0,M>-1\right),\tag{2.10}$$

$$X_{18}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^\infty\left(e^{-\alpha}\right)^{a_1}\left[\left(1-e^{-\alpha}\right)+xe^{-2\alpha}\right]^{c_1-a_1-1}\left(1-ye^{-\alpha}\right)^{-a_2}F_1\left(1+a_1-c_1,\right.$$
$$\left.a_3,a_2;c_2;-\frac{ze^{-\alpha}}{\left[\left(1-e^{-\alpha}\right)+xe^{-2\alpha}\right]},-\frac{ue^{-\alpha}}{\left[\left(1-e^{-\alpha}\right)+xe^{-2\alpha}\right]\left(1-ye^{-\alpha}\right)}\right)d\alpha$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0\right),\tag{2.11}$$

$$X_{18}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right)$$
$$=\frac{\Gamma(c_2)\left(S-T\right)^{a_3}\left(R-T\right)^{c_2-a_3}}{\Gamma(a_3)\Gamma(c_2-a_3)\left(S-R\right)^{c_2-a_1-1}}\int_R^S\left(\alpha-R\right)^{a_3-1}\left(S-\alpha\right)^{c_2-a_3-1}$$
$$\times\left(\alpha-T\right)^{a_1-c_2}\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)z\right]^{-a_1}$$
$$\times X_3\left(a_1,a_2;c_1,c_2-a_3;\lambda_1 x,\lambda_2 y,\lambda_3 u\right)d\alpha$$
$$\left(\lambda_1=\frac{\left(S-R\right)^2\left(\alpha-T\right)^2}{\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)z\right]^2},\right.$$
$$\lambda_2=\frac{\left(S-R\right)\left(\alpha-T\right)}{\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)z\right]},$$
$$\left.\lambda_3=\frac{\left(R-T\right)\left(S-\alpha\right)}{\left[\left(S-R\right)\left(\alpha-T\right)-\left(S-T\right)\left(\alpha-R\right)z\right]}\right),$$
$$\left(Re(a_3)>0,Re(c_2-a_3)>0,T<R<S\right),\tag{2.12}$$

$$X_{18}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{2\Gamma(c_1)M^{a_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{1+a_1+a_2-2c_1}$$
$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]^{c_1-a_1-1}$$
$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]^{-a_2}F_1\left(1+a_1-c_1,a_3,a_2;\right.$$
$$\left.c_2;\lambda_1 z,\lambda_2 u\right)d\alpha$$
$$\left(\lambda_1=\frac{-M\left(cos^2\alpha+Msin^2\alpha\right)tan^2\alpha}{\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]},\right.$$
$$\left.\lambda_2=\frac{-M\left(cos^2\alpha+Msin^2\alpha\right)^2tan^2\alpha}{\left[\left(cos^2\alpha+Msin^2\alpha\right)+M^2xsin^2\alpha\,tan^2\alpha\right]\left[\left(cos^2\alpha+Msin^2\alpha\right)-Mysin^2\alpha\right]}\right),$$

$$\left(Re(a_1) > 0, Re(c_1 - a_1) > 0, M > 0\right), \tag{2.13}$$

$$X_{18}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_2; x, y, z, u\right) = \frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2 - a_1)}$$
$$\times \int_0^1 \alpha^{a_1-1}(1-\alpha)^{c_2-a_1-1}(1-\alpha z)^{-a_3}(1-\alpha u)^{-a_2} H_3\left(1 + a_1 - c_2, a_2; c_1;\right.$$
$$\left.\frac{\alpha^2 x}{(1-\alpha)^2}, -\frac{\alpha y}{(1-\alpha)(1-\alpha u)}\right) d\alpha$$
$$\left(Re(a_1) > 0, Re(c_2 - a_1) > 0\right), \tag{2.14}$$

$$X_{18}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_2, c_2; x, y, z, u\right) = \frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2 - a_1)}$$
$$\times \int_0^\infty \alpha^{a_1-1}(1+\alpha)^{a_2+a_3-c_2}\left[(1+\alpha) - \alpha z\right]^{-a_3}\left[(1+\alpha) - \alpha u\right]^{-a_2}$$
$$\times H_3\left(1 + a_1 - c_2, a_2; c_1; \alpha^2 x, -\frac{\alpha(1+\alpha) y}{\left[(1+\alpha) - \alpha u\right]}\right) d\alpha$$
$$\left(Re(a_1) > 0, Re(c_2 - a_1) > 0\right), \tag{2.15}$$

$$X_{19}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u\right) = \frac{\Gamma(c_1)}{\Gamma(a)\Gamma(c_1 - a)}$$
$$\times \int_0^\infty \left(e^{-\alpha}\right)^a\left(1 - e^{-\alpha}\right)^{c_1-a-1} X_{16}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; a, a, c_1 - a, c_2;\right.$$
$$\left.xe^{-\alpha}, ye^{-\alpha}, z\left(1 - e^{-\alpha}\right), u\right) d\alpha$$
$$\left(Re(a) > 0, Re(c_1 - a) > 0\right), \tag{2.16}$$

$$X_{19}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u\right) = \frac{\Gamma(c_1)}{\Gamma(a_3)\Gamma(c_1 - a_3)}$$
$$\times \int_0^1 \alpha^{a_3-1}(1-\alpha)^{c_1-a_3-1}(1-\alpha z)^{-a_1} X_3\left(a_1, a_2; c_1 - a_3, c_2; \frac{(1-\alpha)x}{(1-\alpha z)^2},\right.$$
$$\left.\frac{(1-\alpha)y}{(1-\alpha z)}, \frac{u}{(1-\alpha z)}\right) d\alpha$$
$$\left(Re(a_3) > 0, Re(c_1 - a_3) > 0\right), \tag{2.17}$$

$$X_{19}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u\right) = \frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2 - a_1)}$$
$$\times \int_0^\infty \left(e^{-\alpha}\right)^{c_2-a_2-1}\left(e^\alpha - 1\right)^{c_2-a_1-1}\left(e^\alpha - u\right)^{-a_2} X_5\left(1 + a_1 - c_2, a_2, a_3; c_1;\right.$$
$$\left.\frac{x}{(e^\alpha - 1)^2}, -\frac{ye^\alpha}{(e^\alpha - 1)(e^\alpha - u)}, -\frac{z}{(e^\alpha - u)}\right) d\alpha$$
$$\left(Re(a_1) > 0, Re(c_2 - a_1) > 0\right), \tag{2.18}$$

$$X_{19}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u\right) = \frac{2\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1 - a_1)}$$
$$\times \int_0^{\frac{\pi}{2}} \left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a_1-\frac{1}{2}}\left(1 + xsin^2\alpha\, tan^2\alpha\right)^{c_1-a_1-1}$$

$$\times \left(1 - ysin^2\alpha\right)^{-a_2} \left(1 - zsin^2\alpha\right)^{-a_3} {}_2F_1 \left(1 + a_1 - c_1, a_2; c_2; \right.$$

$$\left. -\frac{u\,tan^2\alpha}{(1 + xsin^2\alpha\,tan^2\alpha)\,(1 - ysin^2\alpha)}\right) d\alpha$$

$$\left(Re(a_1) > 0, Re(c_1 - a_1) > 0\right), \tag{2.19}$$

$$X_{19}^{(4)}\left(a_1, a_1, a_1, a_1, a_1, a_2, a_3, a_2; c_1, c_1, c_1, c_2; x, y, z, u\right)$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1 - a_1)\Gamma(c_2 - a_2)} \int_0^1 \int_0^1 \alpha^{a_1 - 1}\beta^{a_2 - 1}$$

$$\times \left[(1 - \alpha) + \alpha^2 x + \alpha\beta u\right]^{c_1 - a_1 - 1} \left[(1 - \beta) + \alpha\beta y\right]^{c_2 - a_2 - 1} (1 - \alpha z)^{-a_3}\, d\alpha d\beta$$

$$\left(Re(a_1) > 0, Re(a_2) > 0, Re(c_1 - a_1) > 0, Re(c_2 - a_2) > 0\right), \tag{2.20}$$

$$X_{20}^{(4)}\left(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_2, c_3, c_4; x, y, z, u\right)$$

$$= \frac{4\Gamma(a_1 + a_2 + a_3)\,(1 + M_1)^{a_1}\,(1 + M_2)^{a_1 + a_2}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$

$$\times \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\left(sin^2\alpha\right)^{a_1 - \frac{1}{2}} \left(cos^2\alpha\right)^{a_2 - \frac{1}{2}}}{(1 + M_1 sin^2\alpha)^{a_1 + a_2}} \frac{\left(sin^2\beta\right)^{a_1 + a_2 - \frac{1}{2}} \left(cos^2\beta\right)^{a_3 - \frac{1}{2}}}{(1 + M_2 sin^2\alpha)^{a_1 + a_2 + a_3}}$$

$$\times F_C^{(4)}\left(\frac{a_1 + a_2 + a_3}{2}, \frac{a_1 + a_2 + a_3}{2} + \frac{1}{2}; c_1, c_2, c_3, c_4; \lambda_1 x, \lambda_2 y, \lambda_3 z, \lambda_4 u\right) d\alpha d\beta$$

$$\left(\lambda_1 = \frac{4\,(1 + M_1)^2\,(1 + M_2)^2\, sin^4\alpha\, sin^4\beta}{(1 + M_1 sin^2\alpha)^2\,(1 + M_2 sin^2\alpha)^2}, \lambda_2 = \frac{(1 + M_1)\,(1 + M_2)^2\, sin^2 2\alpha\, sin^4\beta}{(1 + M_1 sin^2\alpha)^2\,(1 + M_2 sin^2\alpha)^2}, \right.$$

$$\left. \lambda_3 = \frac{(1 + M_2)\, cos^2\alpha\, sin^2 2\beta}{(1 + M_1 sin^2\alpha)\,(1 + M_2 sin^2\alpha)^2}, \lambda_4 = \frac{(1 + M_1)\,(1 + M_2)^2\, sin^2 2\alpha\, sin^4\beta}{(1 + M_1 sin^2\alpha)^2\,(1 + M_2 sin^2\alpha)^2}\right),$$

$$\left(Re(a_i) > 0, (i = 1, 2, 3) > 0, M_1 > -1, M_2 > -1\right), \tag{2.21}$$

$$X_{20}^{(4)}\left(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_2, c_3, c_4; x, y, z, u\right) = \frac{\Gamma(c_3)\,(1 + M)^{a_3}}{\Gamma(a_3)\Gamma(c_3 - a_3)}$$

$$\times \int_0^1 \alpha^{a_3 - 1}\,(1 - \alpha)^{c_3 - a_3 - 1}\,(1 + M\alpha)^{a_3 - c_3}\left[(1 + M\alpha) - (1 + M)\,\alpha z\right]^{-a_2}$$

$$\times X_4\left(a_1, a_2; c_1, c_2, c_4; x, \lambda y, \lambda u\right) d\alpha$$

$$\left(\lambda = \frac{(1 + M\alpha)}{\left[(1 + M\alpha) - (1 + M)\,\alpha z\right]}\right),$$

$$\left(Re(a_3) > 0, Re(c_3 - a_3) > 0, M > -1\right), \tag{2.22}$$

$$X_{20}^{(4)}\left(a_1, a_1, a_2, a_1, a_1, a_2, a_3, a_2; c_1, c_2, c_3, c_4; x, y, z, u\right) = \frac{2\Gamma(c_4)}{\Gamma(a_1)\Gamma(c_4 - a_1)}$$

$$\times \int_0^{\frac{\pi}{2}} \left(sin^2\alpha\right)^{a_1 - \frac{1}{2}} \left(cos^2\alpha\right)^{c_4 - a_1 - \frac{1}{2}} \left(1 - usin^2\alpha\right)^{-a_2} X_{17}\left(1 + a_1 - c_4, a_2, a_3; \right.$$

$$\left. c_1, c_2, c_3; x\, tan^4\alpha, -\frac{y\, tan^2\alpha}{(1 - usin^2\alpha)}, \frac{z}{(1 - usin^2\alpha)}\right) d\alpha$$

$$\left(Re(a_1) > 0, Re(c_4 - a_1) > 0\right), \tag{2.23}$$

$$X_{20}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_0^\infty\left(e^{-\alpha}\right)^{a_1}\left[\left(1-e^{-\alpha}\right)+xe^{-2\alpha}\right]^{c_1-a_1-1}F_E\left(a_2,a_2,a_2,a_3,1+a_1-c_1,\right.$$
$$\left.1+a_1-c_1;c_3,c_2,c_4;z,\lambda y,\lambda u\right)d\alpha$$
$$\left(\lambda=-\frac{e^{-\alpha}}{\left[\left(1-e^{-\alpha}\right)+xe^{-2\alpha}\right]}\right),$$
$$\left(Re(a_1)>0,Re(c_1-a_1)>0\right),\tag{2.24}$$

$$X_{20}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)$$
$$=\frac{\Gamma(c_1)\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1-a_1)\Gamma(c_3-a_3)}\int_0^1\int_0^1\alpha^{a_1-1}\beta^{a_3-1}\left(1-\beta\right)^{c_3-a_3-1}$$
$$\times\left[(1-\alpha)+\alpha^2x\right]^{c_1-a_1-1}\left(1-\beta z\right)^{-a_2}F_4\left(1+a_1-c_1,a_2;c_2,c_4;\lambda y,\lambda u\right)d\alpha d\beta$$
$$\left(\lambda=-\frac{\alpha}{\left[(1-\alpha)+\alpha^2x\right]\left(1-\beta z\right)}\right),$$
$$\left(Re(a_1)>0,Re(a_3)>0,Re(c_1-a_1)>0,Re(c_3-a_3)>0\right).\tag{2.25}$$

Proof. Once substituting the series definition of the special function in each integrand and then, changing the order of the integral and the summation, and finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals (see, for example, [2, p. 9-11], [8, 9, Section 1.1] and [11, p. 26 and p. 86, Problem 1]), we derive each of the integral representations from (2.1) to (2.25).

$$B(a,b)=\begin{cases}\displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}\,dt & \left(Re(a)>0,Re(b)>0\right),\\ \displaystyle\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases}\tag{2.26}$$

$$B(a,b)=\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}\,d\alpha=\int_0^\infty(e^{-\alpha})^a(1-e^{-\alpha})^{b-1}\,d\alpha$$
$$\left(Re(a)>0,Re(b)>0\right),\tag{2.27}$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(sin\alpha)^{2a-1}(cos\alpha)^{2b-1}\,d\alpha=\int_0^\infty\frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\,d\alpha$$
$$\left(Re(a)>0,Re(b)>0\right),\tag{2.28}$$

$$B(a,b)=\frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S\frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha\;(T<R<S)$$
$$=(1+M)^a\int_0^1\frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)^{a+b}}\,d\alpha\;(M>-1)$$
$$\left(Re(a)>0,Re(b)>0\right).\tag{2.29}$$

□

3 Integrals of Laplace-Type

We represent the quadruple series $X_{16}^{(4)}, X_{17}^{(4)}, X_{18}^{(4)}, X_{19}^{(4)}, X_{20}^{(4)}$ in terms of integrals by means of Laplace transform. The Laplace integral representations of these quadruple series are given as follows:

$$X_{16}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right){}_1F_1\left(a_3;c_2;sz\right){}_0F_1\left(-;c_3;stu\right)dsdt,$$
$$\left(Re(a_1)>0,Re(a_2)>0\right), \tag{3.1}$$

$$X_{17}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}\Phi_3\left(a_3;c_1;sz,sty\right){}_0F_1\left(-;c_2;s^2x\right){}_0F_1\left(-;c_3;stu\right)dsdt,$$
$$\left(Re(a_1)>0,Re(a_2)>0\right), \tag{3.2}$$

$$X_{18}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}{}_0F_1\left(-;c_1;s^2x+sty\right)\Phi_3\left(a_3;c_2;sz,stu\right)dsdt,$$
$$\left(Re(a_1)>0,Re(a_2)>0\right), \tag{3.3}$$

$$X_{19}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}s^{a_1-1}t^{a_2-1}v^{a_3-1}{}_0F_1\left(-;c_1;s^2x+sty+svz\right)$$
$$\times{}_0F_1\left(-;c_2;stu\right)dsdtdv,$$
$$\left(Re(a_1)>0,Re(a_2)>0,Re(a_3)>0\right), \tag{3.4}$$

$$X_{20}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^\infty\int_0^\infty e^{-(s+t)}s^{a_1-1}t^{a_2-1}{}_0F_1\left(-;c_1;s^2x\right)\Psi_2^{(3)}\left(a_2;c_2,c_3,c_4;sy,tz,su\right)dsdt,$$
$$\left(Re(a_1)>0,Re(a_2)>0\right), \tag{3.5}$$

where ${}_0F_1, {}_1F_1, \Phi_3$ and $\Psi_2^{(3)}$ denote the confluent hypergeometric functions defined, respectively, by

$${}_0F_1\left(-;c;x\right)=\sum_{m=0}^\infty\frac{1}{(c)_m}\frac{x^m}{m!},$$

$${}_1F_1\left(a;c;x\right)=\sum_{m=0}^\infty\frac{(a)_m}{(c)_m}\frac{x^m}{m!},$$

$$\Phi_3\left(a;c;x,y\right)=\sum_{m,n=0}^\infty\frac{(a)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}$$

and

$$\Psi_2^{(3)}\left(a;b,c,d;x,y,z\right)=\sum_{m,n,p=0}^\infty\frac{(a)_{m+n+p}}{(b)_m(c)_n(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

Proof. It is noted that each of the integral representations (3.1) to (3.5) can be proved mainly by expressing the series definition of the involved special functions in each integrand and changing the order of the integral sign and the summation, and finally using the following well-known integral formula [2]:

$$\Gamma(a)=\int_0^\infty e^{-t}\,t^{a-1}\,dt. \quad (\Re(a)>0)$$

□

4 Concluding Remarks

Integral representations for most of the special functions of mathematical physics and applied mathematics have been investigated in the existing literature. Here we have presented some integral representations for five new quadruple hypergeometric series.

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Author information

Maged G. Bin-Saad, Department of Mathematics, Aden University,Aden
Kohrmakssar P.O.Box 6014, Yemen.
E-mail: mgbinsaad@yahoo.com

Jihad A. Younis, Department of Mathematics, Aden University,Aden
Kohrmakssar P.O.Box 6014, Yemen.
E-mail: jihadalsaqqaf@gmail.com

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