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ON THE GENUS OF A LATTICE OVER AN ORDER OF A DEDEKIND DOMAIN

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Abstract The property of mutual embeddings of index not divisible by any prime in a given finite set of primes has been used successfully in the case of finitely generated groups with finite commutator subgroup to define a group structure on the non-cancellation set of such groups. If R is a Dedekind domain and $\mathcal{O}$ is an Order over R, it has been proved that lattices over $\mathcal{O}$ belonging to the same genus have mutual embeddings. This result is formulated in this article in terms of module index and thus, allows us to define an abelian monoid structure on the genus set of such modules. We construct also some homomorphisms between genera class groups.

1 Introduction

Given a localization in some category $\mathcal{C}$, one can introduce the concept of the genus $\mathcal{G}(X)$ of an object X of $\mathcal{C}$. We say that two objects X, Y in $\mathcal{C}$ belong to the same genus, or that $Y \in \mathcal{G}(X)$, if X_p is isomorphic to Y_p for each prime p, where X_p is the localization of X at a prime p.
In the category of groups, the theory of P-localization of groups, where P is a family of primes, appears to have been first discussed in [11, 12] by Kurosh and Lazard . In their work emphasis was placed on the explicit construction of the localization and properties of the localization G_P of the nilpotent group G were deduced from the construction, utilizing nilpotent group theory. Baumslag in [1] has given a comprehensive treatment of the main properties of nilpotent groups as they relate to the problem of localization. He has explicitly shown in [2] how to construct G_P in the case of an arbitrary nilpotent group G and an arbitrary family of primes P. Thus extending the generality of Malcev's original construction. Bousfield and Kan [3] exploit this general Mal'cev construction in their study of completion and localization.
In the 1970s, Hilton and Mislin became interested, through their work on the localization of nilpotent spaces, in the localization of nilpotent groups. Mislin [15] define the genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N to be the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. This version of genus became known as the *Mislin genus*, and other very useful variations of this concept came into being.
One of the important question here has been the description of the genus in terms of its algebraic structure: Is it a group? Is it finite? Is it abelian? Is it computable? In many interesting cases, the genus is finite. Below are some known results from the literature: In [7], Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup.
For groups in the class N_1, these are nilpotent groups N, given in terms of the associated short exact sequence (1.1) $TN \to N \to FN$, where TN is the torsion subgroup and FN the torsion-free quotient, by the conditions:

(a) TN and FN are commutative
(b) Relation (1.1) splits for the action $\omega : FN \to \text{Aut}TN$
(c) $\omega(FN)$ lies in the center of AutTN.
(It was observed in [6] that, in the presence of (a), (c) is equivalent to (c'))
(c') For all $\xi \in FN$, there exists an integer u such that $\cdot a = \omega(\xi)(a) = ua$ for all $a \in TN$. (Here, TN is written additively.)
It is shown in [8] that the genus $\mathcal{G}(N)$ of a group N in N_1 is trivial unless FN is cyclic. In

the case FN is cyclic generated by ξ, and d is the multiplicative order of u (see (c')) modulo m, where m is the exponent of TN, then the calculation of the genus yields (1.2) $\mathcal{G}(N) \cong (\mathbb{Z}_d)^*/\pm 1$ where $(\mathbb{Z}_d)^*$ is the multiplicative group of units in the ring $\mathbb{Z}_d$.
Also in [14], a group structure is defined on the restricted genus of a particular class of groups.
The localization of a ring results to local ring. A *local ring* is a ring with just one maximal ideal. Ever since Krull's paper (1938), local rings have occupied a central position in commutative algebra. The technique of localization reduces many problems in commutative algebra to problems about local rings. This often turns out to be extremely useful. Most of the problems with which commutative algebra has been successful are those that can be reduced to the local case. Despite this, localization as a general procedure was defined rather late. In the case of integral domains, it was described by Grell, a student of Noether in 1927. It was not defined for arbitrary commutative rings until the work of Chevalley (1944) and Uzkov (1948). The process of localization does not lose much information about the ring. For example, if R is an integral domain, the fields of fractions of R and R_p (the localization of R at a prime ideal p) are the same. This process permits operations which, from a geometrical point of view, provide information about the neighborhood of p in Spec(R) (the set of all primes ideals of R). The localization of an R-module M at a prime ideal p is the R_p-module M_p obtained by tensoring M with R_p. Modules having isomorphic localization are said to be in the same genus.
The concepts of genus and localization are closely related to that of cancellation of groups and modules. For example, cancellation holds for the class of finitely generated R-modules where R is a local noetherian commutative ring. Cancellation holds also in every genus of R-modules if R is commutative and has no nilpotent elements, see [5]. We recall that the cancellation question investigates the following: *If P and Q are finitely generated R-modules such that $P \oplus R \cong Q \oplus R$, are P and Q necessarily isomorphic?* In [13], a group structure is induced on the non-cancellation set of the localization of G at the set of primes $\{3, 7\}$.

Given R a Dedekind domain, K the quotient field of R and A a semisimple separable K-algebra; An R-*Order* (or Order over R) is a ring with $K\mathcal{O} = A$ and $1 \in \mathcal{O}$, such that $\mathcal{O}$ is finitely generated as an R-module. By $\mathcal{O}$-lattice, we mean a finitely generated (unital) $\mathcal{O}$-module, which is torsion-free as an R-module. The category of $\mathcal{O}$-lattices will be denoted by $\mathcal{L}_{\mathcal{O}}$.
It so happens that there are mutual embeddings for lattices belonging to the same genus as presented by Jacobinski in [9]. We formulate this result in terms of module index as presented in Theorem 2.4 of this study, which allows us to define an algebraic structure on the genus of such modules.
The property of mutual embeddings was successfully used by Witbooi [18] in the case of finitely generated group with finite commutator subgroup G to define a group structure on the non-cancellation set $\chi(G)$ (this is the set of all isomorphism classes of groups H such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$) which coincide with the Mislin genus when G is nilpotent.
In this paper, We generalize this result to $\mathcal{O}$-lattices. This enable us to define an abelian monoid structure on the genus set of such modules. We construct also some homomorphisms between genera class groups.

The rest of the paper is organized as follows:
The second section presents the embeddings property.
In the third section, we define a monoid structure on the genus of a $\mathcal{O}$-lattice.
The last section is on homomorphism of genera where we present some homomorphisms between genus class groups of lattices and some typical examples.

2 On the mutual embeddings with index module relatively prime to an ideal

Let $\mathcal{O}$ be any R-Order in A and let $\Gamma \supset \mathcal{O}$ be a maximal Order. Then there exists an $r \neq 0$ in R with $r\Gamma \subset \mathcal{O}$. If p is a prime ideal of R with $r \notin p$, then $r\Gamma_p \subset \mathcal{O}_p \subset \Gamma_p$ and r is a unit at p. Hence $\mathcal{O}_p \supset \Gamma_p$. There are only a finite number of primes of R containing r. Thus, except for those primes, $\mathcal{O}_p$ is a maximal Order. Two $\mathcal{O}$-lattices M and N are said to be in the same genus if and only if $M_p \cong N_p$ for every prime p of R. We denote by gen(M) the genus set of an $\mathcal{O}$-lattice M.

One of the most basic concepts in group theory is that of *order* of an element. For modules over a commutative ring, the analogous concept is the annihilator of an element.
We start with the definition of the *order ideal*. Let R be a Dedekind domain and $\mathcal{O}$ an R-order. For a finitely generated torsion $\mathcal{O}$-module T, we define an *order ideal* denoted by $\mathrm{Ord}_{\mathcal{O}}T$, to be the product of the annihilators of the composition factors of a composition series of T.

Let M be an $\mathcal{O}$-lattice and N a submodule of M such that M/N is a torsion module.

Definition 2.1. The *index module* of M and N, denoted by $[M : N]$, is the order ideal of the torsion module M/N, i.e. $[M : N] = \mathrm{ord}_{\mathcal{O}}(M/N)$.

In checking whether M and N have the same genus, we can restrict our attention to a finite set of primes of R. Let Γ be a maximal order containing $\mathcal{O}$. Then there is an $r \neq 0$ in R with $r\Gamma \subset \mathcal{O}$ (see [17, P. 104]). Let P be a finite non empty set of primes of R including all the primes containing r.

Lemma 2.2. *[17, Lemma 6.5]. Two $\mathcal{O}$-lattices M and N will have the same genus if and only if N_p is isomorphic to M_p for all $p \in P$.*

Proof. If K is the quotient field of R, then $K \otimes_R M$ is isomorphic to $K \otimes_R N$ since P is non empty. Now M_p and N_p $p \notin P$ are finitely generated torsion-free modules over a maximal R_p-order $\mathcal{O}_p = \Gamma_p$ which become isomorphic over K. Therefore, by [17, Theorem 5.27, P.101], M_p is isomorphic to N_p. Hence M and N are isomorphic at every prime of R if they are so at all $p \in P$. □

Recall now the following definition.

If R is any commutative domain, then the *Jordan-Zassenhaus Theorem holds for R* if for each order $\mathcal{O}$ in any semisimple separable K-algebra (K is the quotient field of R) and for each positive integer n, there are only finitely many isomorphism classes of $\mathcal{O}$-modules which are finitely generated and torsion-free of rank $\leq n$.

Lemma 2.3. *(Roiter Theorem) Let R be a Dedekind domain such that the Jordan-Zassenhaus Theorem holds for R. Let $\mathcal{O}$ be an order in a semisimple separable K-algebra A. Let $0 \neq I$ be an ideal of R. Let M and N be two $\mathcal{O}$-modules of the same genus. Then there exists an exact sequence $0 \to M \to N \to U \to 0$ where U is torsion with annihilator prime to I and with $U = \oplus U_i$ where the U_i are simple $\mathcal{O}$-modules and the annihilators of the U_i are pairwise relatively prime.*

The following theorem describe the mutual embeddings of torsion free finitely generated modules.

Theorem 2.4. *Let A be a separable semisimple K-algebra where K is the quotient field of a Dedekind domain R and $\mathcal{O}$ any R-order in A. Let $r \neq 0$ be an element of R such that $r\Gamma \subset \mathcal{O}$ where $\Gamma \supset \mathcal{O}$ is a maximal order. Let P be a finite non empty set of all primes p containing r. Let M and N be two torsion-free finitely generated $\mathcal{O}$-modules. If M_p is isomorphic to N_p for all primes p of R. Then N is isomorphic to a submodule L of M such that the index $[M : L]$ is relatively prime to any prime $p \in P$.*

Proof. Suppose that $M_p \cong N_p$ for all primes p of R. Applying Roiter's Theorem to the ideal (r), we obtain an exact sequence $0 \to N \xrightarrow{i} M \xrightarrow{j} X \to 0$, where X has annihilator prime to (r). Let $X = X_0 \supset \cdots \supset X_m = 0$ be a composition series for X and $M_i = j^{-1}(X_i)$. Therefore $M = M_0 \supset M_1 \supset \cdots \supset M_{m-1} \supset M_m = \mathrm{Ker} j \cong N$ has all quotients M_i/M_{i+1} simple. Take $L = \mathrm{Ker} j$. Moreover by Lemma 2.3, this embedding can be such that M/N is a direct sum of simple modules whose annihilators are relatively prime and prime to (r). Thus the module index $[M : N]$ is relatively prime to any $p \in P$. □

3 On the structure of the genus of a lattice over an order

Let R be an integral domain with fraction field K. A *fractional ideal* is a nonzero R-module I of K for which there exists a nonzero x in R such that $xI \subset R$. Clearly, a non-zero ideal of R may be viewed as a fractional ideal (take $x = 1$); for emphasis, such an ideal is called an *integral ideal*. One can multiply fractional ideals, and under multiplication, they form an abelian monoid with identity element R. In a Dedekind domain, every fractional ideal is invertible and the set of all fractional ideals $\text{Frac}(R)$ under multiplication then form a group.
Let J be the product of all prime ideals in the finite set P. That is, if $|P| = n$, let $J = p_1 p_2 \cdots p_n$. Let W be the set of all ideals of R relatively prime to J.
It is clear that W is closed under multiplication and has R as identity element. It is a submonoid of $\text{Frac}(R)$.
In order to explore an algebraic structure on $\text{gen}(M)$, we now construct the following map:

$$\varphi : W \to \text{gen}(M)$$

defined by $\varphi(I) = [N]$ for I in W and $N \in \text{gen}(M)$ such that $[M : N] = I$.

Lemma 3.1. *The map $\varphi : W \to \text{gen}(M)$ is a well-defined function and surjective.*

Proof. By Lemma 2.3 and Theorem 2.4, φ is well-defined and surjective. □

The surjectivity of φ, together with the monoid structure on W lead us to the right direction in finding an algebraic structure on $\text{gen}(M)$.
By construction, the kernel $Q = \ker \varphi$ coincide with $\varphi^{-1}(M)$ which is a submonoid of W. It follows that there is a bijection $\psi : W/Q \to \text{gen}(M)$ such that for the canonical epimorphism of semigroups $\eta : W \to W/Q$, we have: $\varphi = \psi \circ \eta$.
We use ψ to equip $\text{gen}(M)$ with a monoid structure.

Theorem 3.2. *Let R be a Dedekind domain and $\mathcal{O}$ an R-order in a separable semisimple K-algebra where K is the quotient field of R. Let $r \neq 0$ be an element of R such that $r\Gamma \subset \mathcal{O}$ where $\Gamma \supset \mathcal{O}$ is a maximal order. Let P be a finite non empty set of all primes p containing r. Let π be the product of all prime ideals in P. Then for any $\mathcal{O}$-lattice M, the function $Q \to \text{gen}(M)$ induces a monoid structure on $\text{gen}(M)$.*

Since $\text{gen}(M)$ does not admit a group structure, using a stable isomorphism relation, which is an equivalence relation, we can form a genus class group as presented in the next section. We will then construct some homomorphisms between genera.

4 Genus class group of an $\mathcal{O}$-lattice

Two left $\mathcal{O}$-lattices M, N are called stably isomorphic if

$$M \oplus \mathcal{O}^{(k)} \cong N \oplus \mathcal{O}^{(k)} \quad \text{for some } k$$

where $\mathcal{O}^{(k)}$ is the external direct sum of k copies of $\mathcal{O}$.
A left $\mathcal{O}$-lattice is said to be stably free if it is stably isomorphic to a finitely generated free $\mathcal{O}$-module. In other words, M is stably free if $M \oplus \mathcal{O}^{(m)} \cong \mathcal{O}^{(n)}$ for some positive integers m, n.
Let $[M]$ denote the stable isomorphism class of M.

Proposition 4.1. *[5]. The stable isomorphism classes of $\mathcal{O}$-lattices in $\text{gen}(M)$ form an abelian additive group $\mathcal{G}(M)$, called the genus class group of M. Addition is given by $[N] + [N'] = [L]$ whenever $N \oplus N' \cong M \oplus L$. The zero element is the class $[M]$.*

4.1 Homomorphisms of genera

It is always interesting to see how genera are related to each other by constructing some homomorphisms. In the case of groups, see for example [18].

Let M, N be two $\mathcal{O}$-lattices. Suppose that for each $M' \in \text{gen}(M)$ there exists $N' \in \text{gen}(N)$, unique up to isomorphism, such that

$$M' \oplus N \cong M \oplus N' \qquad (*)$$

. We can define a function

$$\zeta : \mathcal{G}(M) \to \mathcal{G}(N)$$

as follows $\zeta([M']) = [N'] \iff M' \oplus N \cong M \oplus N'$. If condition $(*)$ is fulfilled, we say that ζ is defined.

We now show that

Proposition 4.2. *If ζ is defined, then it is a group homomorphism.*

Proof. To see that ζ is a homomorphism, suppose that $\zeta[M'] = [N']$ and $\zeta[M''] = [N'']$. Let $[S] = [M'] + [M'']$ in $\mathcal{G}(M)$, and $[T] = [N'] + [N'']$ in $\mathcal{G}(N)$; we must show that $\zeta[S] = [T]$. By the definition of ζ, we have $M' \oplus N \cong M \oplus N'$ and $M'' \oplus N \cong M \oplus N''$, and by the definition of S and T, we have $M' \oplus M'' \cong M \oplus S$ and $N' \oplus N'' \cong N \oplus T$. Taking the direct sum of the first pair of isomorphisms and substituting the second pair of isomorphisms yields

$$M \oplus S \oplus N^{(2)} \cong M^{(2)} \oplus N \oplus T$$

which is equivalent to

$$(M \oplus N) \oplus (S \oplus N) \cong (M \oplus N) \oplus (M \oplus T)$$

By local cancellation, $S \oplus N$ and $M \oplus T$ are in the same genus $\mathcal{G}(M \oplus N)$ and since cancellation holds for modules in the same genus, we therefore have $S \oplus N \cong M \oplus T$. Thus $\zeta[S] = [T]$. □

In what follows we are going to give a specific case where ζ exists for $\mathcal{O}$-lattices M and N.

Definition Let M and N be $\mathcal{O}$-lattices. N is called a local direct factor of M, if for every p, N_p is isomorphic to a direct factor of M_p.

The next lemma gives us a decomposition of M into a direct sum with a direct summand locally isomorphic to N. Refer to [10] for the proof.

Lemma 4.3. *[10, Theorem 3.3]. Let M and N be $\mathcal{O}$-lattices and suppose that N is a local direct factor of M. Then there is a decomposition $M = N' \oplus L$, with N and N' in the same genus and L an $\mathcal{O}$-lattice.*

Therefore if N is a local direct factor of M, for any K in the genus of M, N_p is isomorphic to a direct factor Y^p of K_p and according to [10, Lemma 3.2], there is an $\mathcal{O}$-lattice Y such that $Y_p \cong Y^p$ for every p.

Thus we have the following proposition

Proposition 4.4. *Let M and N be $\mathcal{O}$-lattices. If N is a local direct factor of M, then ζ exists and it is well-defined.*

The next step is to define a crossing map $\mathcal{G}(M) \to \mathcal{G}(M \oplus N)$. Let us start with the following lemma.

Lemma 4.5. *[16, Proposition 1.3] Let L, M, N be left $\mathcal{O}$-lattices such that $L \in \text{gen}(M \oplus N)$. Then there exist $\mathcal{O}$-lattices X, Y such that $L \cong X \oplus Y$, $X \in \text{gen}M$, $Y \in \text{gen}N$.*

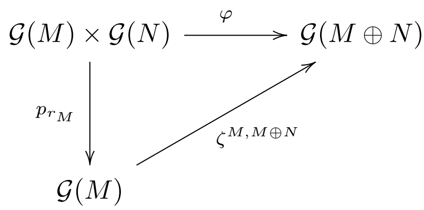
From Lemma 4.5, it follows that there is a surjective function

$$\text{genus}(M) \times \text{genus}(N) \to \text{genus}(M \oplus N)$$

The following proposition describe the crossing map $\mathcal{G}(M) \to \mathcal{G}(M \oplus N)$.

Proposition 4.6. *Let $\nu : \mathcal{G}(M) \to \mathcal{G}(M \oplus N)$ be a function defined such that $\zeta([X]) = [X \oplus N]$. Then ζ is an epimorphism. Moreover ζ is isomorphism if N is stably free.*

Proof.



where p_{r_M} is a projection and $\varphi = \zeta^{M,M\oplus N} \circ p_{r_M}$ an epimorphism by Lemma 4.5. Thus $\zeta^{M,M\oplus N}$ is an epimorphism.

Let $[X]$ be in Ker(ζ). Then $[X \oplus N] = [M \oplus M \oplus N]$, that is, $X \oplus N \oplus \mathcal{O}^{(m)} = M \oplus M \oplus N \oplus \mathcal{O}^{(m)}$. Hence $X \oplus \mathcal{O}^{(n)} = M^{(2)} \oplus \mathcal{O}^{(n)}$, since N is stably free. Therefore $[X] = [M^{(2)}]$ and since $\mathcal{G}(M) \cong \mathcal{G}(M^{(2)})$, we thus have $[M^{(2)}] = [M]$. □

In closing, as Jacobinski in [9] proved that there are mutual embeddings for lattices belonging to the same genus, we have formulated his result in terms of module index as presented in Theorem 2.4, which allows us to define an algebraic structure on the genus of lattices. Inspired by Witbooi's result [18] using the property of mutual embeddings, we generalized his result to $\mathcal{O}$-lattices. This has enabled us to define an abelian monoid structure on the genus set of such modules. Some homomorphisms between genera class groups have been constructed.
The next step will be looking for more richer structure such as group on the genus of lattices, which will enable us to make some computations.

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