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THE METRICS FOR RHOMBICUBOCTAHEDRON AND RHOMBICOSIDODECAHEDRON

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Abstract Polyhedrons have been studied by mathematicians and geometers during many years, because of their symmetries. The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. There are some relations between metrics and polyhedra. For example, it has been shown that cube, octahedron, deltoidal icositetrahedron are maximum, taxicab, Chinese Checker's unit sphere, respectively. In this study, I introduce two new metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are rhombicuboctahedron and rhombicosidodecahedron. Also some properties about these metrics are given.

1 Introduction

The history of man's interest in symmetry goes back many centuries. Symmetry is the primary matter of aesthetic thus it has been worked on, in various fields, for example in physics, chemistry, biology, art, architecture and of course in mathematics. Polyhedra have attracted the attention because of their symmetries. Consequently, polyhedra take place in many studies with respect to different fields. A polyhedron is a three-dimensional figure made up of polygons. When discussing polyhedra one will use the terms faces, edges and vertices. Each polygonal part of the polyhedron is called a face. A line segment along which two faces come together is called an edge. A point where several edges and faces come together is called a vertex. That is, a polyhedron is a solid in three dimensions with flat faces, straight edges and vertices. In the early days of the study, the polyhedra involved to only convex polyhedra. If the line segment joining any two points in the set is also in the set, the set is called a convex set. There are many thinkers that have worked on convex polyhedra since the ancient Greeks. The Greek scientist defined two classes of convex equilateral polyhedron with polyhedral symmetry, the Platonic and the Archimedean. Johannes Kepler found a third class, the rhombic polyhedra and Eugène Catalan discovered a fourth class. The Archimedean solids and their duals the Catalan solids are less well known than the Platonic solids. Whereas the Platonic solids are composed of one shape, these forms that Archimedes wrote about are made of at least two different shapes, all forming identical vertices. They are thirteen polyhedra in this type. Since each solid has a 'dual' there are also thirteen Catalan solids which is named after Belgian mathematician Eugène Catalan in 1865, these are made by placing a point in the middle of the faces of the Archimedean Solids and joining the points together with straight lines. The Catalan solids are all convex.

As it is stated in [3] and [6], polyhedra have been used for explaining the world around us in philosophical and scientific way. There are only five regular convex polyhedra known as the Platonic solids. These regular polyhedra were known by the Ancient Greeks. They are generally known as the "Platonic" or "cosmic" solids because Plato mentioned them in his dialogue Timeous, where each is associated with one of the five elements - the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron with universe (or with ether, the material of the heavens). The story of the rediscovery of the

Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result. For more detailed knowledge, see [3] and [6].

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [13].

Some mathematicians have studied and improved metric space geometry. According to mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. In [1, 2, 4, 5, 7, 8, 9, 10, 11] the authors give some metrics which the spheres of the 3-dimensional analytical space furnished by these metrics are some of Platonic solids, Archimedian solids and Catalan solids. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. In this study, two new metrics are introduced, and showed that the spheres of the 3-dimensional analytical space furnished by these metrics are rhombicuboctahedron and rhombicosidodecahedron. Also some properties about these metrics are given.

2 Rhombicuboctahedron Metric and Some Properties

It has been stated in [14], an Archimedean solid is a symmetric, semiregular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. A polyhedron is called semiregular if its faces are all regular polygons and its corners are alike. And, identical vertices are usually means that for two taken vertices there must be an isometry of the entire solid that transforms one vertex to the other.

The Archimedean solids are the only 13 polyhedra that are convex, have identical vertices, and their faces are regular polygons (although not equal as in the Platonic solids).

Five Archimedean solids are derived from the Platonic solids by truncating (cutting off the corners) a percentage less than 1/2.

Two special Archimedean solids can be obtained by full truncating (percentage 1/2) either of two dual Platonic solids: the Cuboctahedron, which comes from trucating either a Cube, or its dual an Octahedron. And the Icosidodecahedron, which comes from truncating either an Icosahedron, or its dual a Dodecahedron. Hence their "double name".

The next two solids, the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) apparently seem to be derived from truncating the two preceding ones. However, it is apparent from the above discussion on the percentage of truncation that one cannot truncate a solid with unequally shaped faces and end up with regular polygons as faces. Therefore, these two solids need be constructed with another technique. Actually, the can be built from the original platonic solids by a process called expansion. It consists on separating apart the faces of the original polyhedron with spherical symmetry, up to a point where they can be linked through new faces which are regular polygons. The name of the Truncated Cuboctahedron (also called Great Rhombicuboctahedron) and of the Truncated Icosidodecahedron (also called Great Rhombicosidodecahedron) again seem to indicate that they can be derived from truncating the Cuboctahedron and the Icosidodecahedron. But, as reasoned above, this is not possible.

Finally, there are two special solids which have two chiral (specular symmetric) variations: the Snub Cube and the Snub Dodecahedron. These solids can be constructed as an alternation of another Archimedean solid. This process consists on deleting alternated vertices and creating new triangles at the deleted vertices.

One of the Archimedean solids is the rhombicuboctahedron. It has 8 triangular and 18 square faces, 24 vertices and 48 edges (See figure 1(a)). The large polyhedron in the 1495 portrait of Luca Pacioli, traditionally though controversially attributed to Jacopo de' Barbari, is a glass rhombicuboctahedron half-filled with water. The first printed version of the rhombicuboctahedron was by Leonardo da Vinci and appeared in his 1509 Divina Proportione. [15] (See figure 1(c),(d)). In figure 1(b) it is seen that the progresing of the expansion of cube and octahedron.

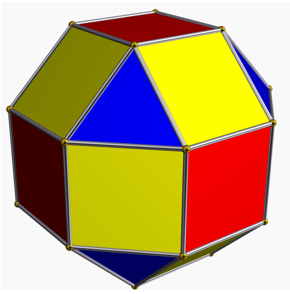

Figure 1(a) Rhombicuboctahedron

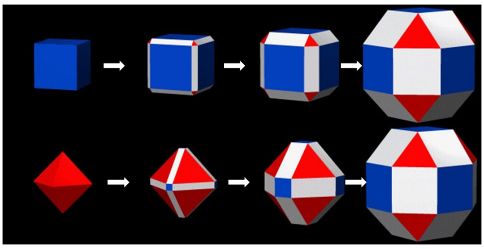

Figure 1(b) Expansion of cube and octahedron

Figure 1(c) Portrait of Luca Pacioli

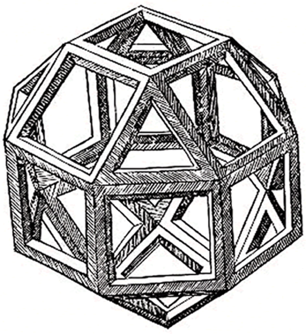

Figure 1(d) Divina Proportione

The metric that unit sphere is rhombicuboctahedron is described as following:

Definition 2.1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.The distance function $d_{RC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ rhombicuboctahedron distance between P_1 and P_2 is defined by
$d_{RC}(P_1, P_2)$=

$$\frac{2\sqrt{2}+1}{7} \max \left\{ \begin{array}{l} X_{12} + \frac{2-\sqrt{2}}{2} \max \left\{ \begin{array}{c} 2\sqrt{2}X_{12}, \left(2+\sqrt{2}\right)(Y_{12}+Z_{12}), \\ X_{12} + \frac{3+\sqrt{2}}{2} Y_{12}, X_{12} + \frac{3+\sqrt{2}}{2} Z_{12} \end{array} \right\}, \\ Y_{12} + \frac{2-\sqrt{2}}{2} \max \left\{ \begin{array}{c} 2\sqrt{2}Y_{12}, \left(2+\sqrt{2}\right)(X_{12}+Z_{12}), \\ Y_{12} + \frac{3+\sqrt{2}}{2} Z_{12}, Y_{12} + \frac{3+\sqrt{2}}{2} X_{12} \end{array} \right\}, \\ Z_{12} + \frac{2-\sqrt{2}}{2} \max \left\{ \begin{array}{c} 2\sqrt{2}Z_{12}, \left(2+\sqrt{2}\right)(X_{12}+Y_{12}), \\ Z_{12} + \frac{3+\sqrt{2}}{2} X_{12}, Z_{12} + \frac{3+\sqrt{2}}{2} Y_{12} \end{array} \right\} \end{array} \right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$.

According to rhombicuboctahedron distance, there are three different paths from P_1 to P_2. These paths are

$i)$ a line segment which is parallel to a coordinate axis.

$ii)$ union of three line segments each of which is parallel to a coordinate axis.

$iii)$ union of two line segments each of which is parallel to a coordinate axis.

Thus rhombicuboctahedron distance between P_1 and P_2 is for (i) Euclidean lengths of line segment, for (ii) $\frac{\sqrt{2}+1}{7}$ times the sum of Euclidean lengths of mentioned three line segments, and for (iii) $\frac{\sqrt{2}}{2}$ times the sum of Euclidean lengths of mentioned two line segments.

Figure 2 illustrates rhombicuboctahedron way from P_1 to P_2 if maximum value is $|y_1 - y_2|$, $\frac{\sqrt{2}+1}{7}(|x_1 - x_2|+|y_1 - y_2|+|z_1 - z_2|)$, $\frac{\sqrt{2}}{2}(|x_1 - x_2| + |y_1 - y_2|)$ or $\frac{\sqrt{2}}{2}(|y_1 - y_2| + |z_1 - z_2|)$.

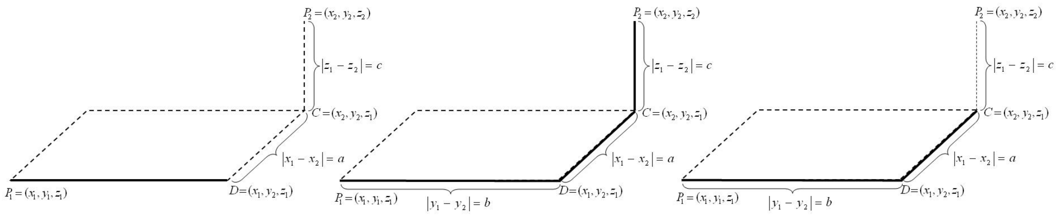

Figure 2: RC way from P_1 to P_2

Lemma 2.2. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$, respectively. Then*

$$d_{RC}(P_1,P_2) \geq \frac{2\sqrt{2}+1}{7}\left(X_{12} + \frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}X_{12}, \left(2+\sqrt{2}\right)(Y_{12}+Z_{12}), \\ X_{12} + \frac{3+\sqrt{2}}{2}Y_{12}, X_{12} + \frac{3+\sqrt{2}}{2}Z_{12} \end{array}\right\}\right),$$

$$d_{RC}(P_1,P_2) \geq \frac{2\sqrt{2}+1}{7}\left(Y_{12} + \frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}Y_{12}, \left(2+\sqrt{2}\right)(X_{12}+Z_{12}), \\ Y_{12} + \frac{3+\sqrt{2}}{2}Z_{12}, Y_{12} + \frac{3+\sqrt{2}}{2}X_{12} \end{array}\right\}\right),$$

$$d_{RC}(P_1,P_2) \geq \frac{2\sqrt{2}+1}{7}\left(Z_{12} + \frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}Z_{12}, \left(2+\sqrt{2}\right)(X_{12}+Y_{12}), \\ Z_{12} + \frac{3+\sqrt{2}}{2}X_{12}, Z_{12} + \frac{3+\sqrt{2}}{2}Y_{12} \end{array}\right\}\right).$$

Proof. Proof is trivial by the definition of maximum function. □

Theorem 2.3. *The distance function d_{RC} is a metric. Also according to d_{RC}, the unit sphere is an rhombicuboctahedron in $\mathbb{R}^3$.*

Proof. Let $d_{RC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ be the rhombicuboctahedron distance function and $P_1=(x_1, y_1, z_1)$, $P_2=(x_2, y_2, z_2)$ and $P_3=(x_3, y_3, z_3)$ are distinct three points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$, respectively. To show that d_{RC} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d_{RC}(P_1,P_2) \geq 0$ and $d_{RC}(P_1,P_2) = 0$ iff $P_1 = P_2$

M2) $d_{RC}(P_1,P_2) = d_{RC}(P_2,P_1)$

M3) $d_{RC}(P_1,P_3) \leq d_{RC}(P_1,P_2) + d_{RC}(P_2,P_3)$.

Since absolute values is always nonnegative value $d_{RC}(P_1,P_2) \geq 0$. If $d_{RC}(P_1,P_2) = 0$ then there are possible three cases. These cases are

1) $$d_{RC}(P_1,P_2) = \frac{2\sqrt{2}+1}{7}\left(X_{12} + \frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}X_{12}, \left(2+\sqrt{2}\right)(Y_{12}+Z_{12}), \\ X_{12} + \frac{3+\sqrt{2}}{2}Y_{12}, X_{12} + \frac{3+\sqrt{2}}{2}Z_{12} \end{array}\right\}\right)$$

2) $$d_{RC}(P_1,P_2) = \frac{2\sqrt{2}+1}{7}\left(Y_{12} + \frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}Y_{12}, \left(2+\sqrt{2}\right)(X_{12}+Z_{12}), \\ Y_{12} + \frac{3+\sqrt{2}}{2}Z_{12}, Y_{12} + \frac{3+\sqrt{2}}{2}X_{12} \end{array}\right\}\right)$$

3) $$d_{RC}(P_1,P_2) = \frac{2\sqrt{2}+1}{7}\left(Z_{12} + \frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}Z_{12}, \left(2+\sqrt{2}\right)(X_{12}+Y_{12}), \\ Z_{12} + \frac{3+\sqrt{2}}{2}X_{12}, Z_{12} + \frac{3+\sqrt{2}}{2}Y_{12} \end{array}\right\}\right).$$

Case I: If

$$d_{RC}(P_1,P_2)=\tfrac{2\sqrt{2}+1}{7}\left(X_{12}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c}2\sqrt{2}X_{12},\left(2+\sqrt{2}\right)(Y_{12}+Z_{12}),\\ X_{12}+\frac{3+\sqrt{2}}{2}Y_{12},X_{12}+\frac{3+\sqrt{2}}{2}Z_{12}\end{array}\right\}\right),$$

then

$$\begin{array}{l}\frac{2\sqrt{2}+1}{7}\left(X_{12}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c}2\sqrt{2}X_{12},\left(2+\sqrt{2}\right)(Y_{12}+Z_{12}),\\ X_{12}+\frac{3+\sqrt{2}}{2}Y_{12},X_{12}+\frac{3+\sqrt{2}}{2}Z_{12}\end{array}\right\}\right)=0\\ \Leftrightarrow X_{12}=0\text{ and }\max\left\{2\sqrt{2}X_{12},\left(2+\sqrt{2}\right)(Y_{12}+Z_{12}),X_{12}+\frac{3+\sqrt{2}}{2}Y_{12},X_{12}+\frac{3+\sqrt{2}}{2}Z_{12}\right\}=0\\ \Leftrightarrow x_1=x_2,\,y_1=y_2,\,z_1=z_2\\ \Leftrightarrow (x_1,y_1,z_1)=(x_2,y_2,z_2)\\ \Leftrightarrow P_1=P_2\end{array}$$

The other cases can be shown by similar way in Case I. Thus it is obtained that $d_{RC}(P_1,P_2)=0$ iff $P_1=P_2$.

Since $|x_1-x_2|=|x_2-x_1|$, $|y_1-y_2|$=$|y_2-y_1|$ and $|z_1-z_2|=|z_2-z_1|$, obviously $d_{RC}(P_1,P_2)=d_{RC}(P_2,P_1)$. That is, d_{RC} is symmetric.

$X_{13},\ Y_{13},\ Z_{13},\ X_{23},\ Y_{23},\ Z_{23}$ denote $|x_1-x_3|,\ |y_1-y_3|,\ |z_1-z_3|,\ |x_2-x_3|,\ |y_2-y_3|,\ |z_2-z_3|$, respectively.

$$\begin{array}{l}d_{RC}(P_1,P_3)\\ =\frac{2\sqrt{2}+1}{7}\max\left\{\begin{array}{l}X_{13}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c}2\sqrt{2}X_{13},\left(2+\sqrt{2}\right)(Y_{13}+Z_{13}),\\ X_{13}+\frac{3+\sqrt{2}}{2}Y_{13},X_{13}+\frac{3+\sqrt{2}}{2}Z_{13}\end{array}\right\},\\ Y_{13}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c}2\sqrt{2}Y_{13},\left(2+\sqrt{2}\right)(X_{13}+Z_{13}),\\ Y_{13}+\frac{3+\sqrt{2}}{2}Z_{13},Y_{13}+\frac{3+\sqrt{2}}{2}X_{13}\end{array}\right\},\\ Z_{12}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c}2\sqrt{2}Z_{13},\left(2+\sqrt{2}\right)(X_{13}+Y_{13}),\\ Z_{13}+\frac{3+\sqrt{2}}{2}X_{13},Z_{13}+\frac{3+\sqrt{2}}{2}Y_{13}\end{array}\right\}\end{array}\right\}\\ \leq\frac{2\sqrt{2}+1}{7}\max\left\{\begin{array}{l}X_{12}+X_{23}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{l}2\sqrt{2}(X_{12}+X_{23}),\\ \left(2+\sqrt{2}\right)(Y_{12}+Y_{23}+Z_{12}+Z_{23}),\\ X_{12}+X_{23}+\frac{3+\sqrt{2}}{2}(Y_{12}+Y_{23}),\\ X_{12}+X_{23}+\frac{3+\sqrt{2}}{2}(Z_{12}+Z_{23})\end{array}\right\},\\ Y_{12}+Y_{23}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{l}2\sqrt{2}(Y_{12}+Y_{23}),\\ \left(2+\sqrt{2}\right)(X_{12}+X_{23}+Z_{12}+Z_{23}),\\ Y_{12}+Y_{23}+\frac{3+\sqrt{2}}{2}(Z_{12}+Z_{23}),\\ Y_{12}+Y_{23}+\frac{3+\sqrt{2}}{2}(X_{12}+X_{23})\end{array}\right\},\\ Z_{12}+Z_{23}+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{l}2\sqrt{2}(Z_{12}+Z_{23}),\\ \left(2+\sqrt{2}\right)(X_{12}+X_{23}+Y_{12}+Y_{23}),\\ Z_{12}+Z_{23}+\frac{3+\sqrt{2}}{2}(X_{12}+X_{23}),\\ Z_{12}+Z_{23}+\frac{3+\sqrt{2}}{2}(Y_{12}+Y_{23})\end{array}\right\}\end{array}\right\}\\ =I.\end{array}$$

Therefore one can easily find that $I\leq d_{RC}(P_1,P_2)+d_{RC}(P_2,P_3)$ from Lemma 2.2. So $d_{RC}(P_1,P_3)\leq d_{RC}(P_1,P_2)+d_{RC}(P_2,P_3)$. Consequently, rhombicuboctahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that rhombicuboctahedron distance is 1 from

$O=(0,0,0)$ is $S_{RC}=$

$$\left\{(x,y,z):\frac{2\sqrt{2}+1}{7}\max\left\{\begin{array}{c} |x|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|x|\,,\left(2+\sqrt{2}\right)(|y|+|z|)\,, \\ |x|+\frac{3+\sqrt{2}}{2}\,|y|\,,|x|+\frac{3+\sqrt{2}}{2}\,|z| \end{array}\right\}, \\ |y|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|y|\,,\left(2+\sqrt{2}\right)(|x|+|z|)\,, \\ |y|+\frac{3+\sqrt{2}}{2}\,|z|\,,|y|+\frac{3+\sqrt{2}}{2}\,|x| \end{array}\right\}, \\ |z|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|z|\,,\left(2+\sqrt{2}\right)(|x|+|y|)\,, \\ |z|+\frac{3+\sqrt{2}}{2}\,|x|\,,|z|+\frac{3+\sqrt{2}}{2}\,|y| \end{array}\right\} \end{array}\right\}=1\right\}.$$

Thus the graph of S_{RC} is as in the figure 3:

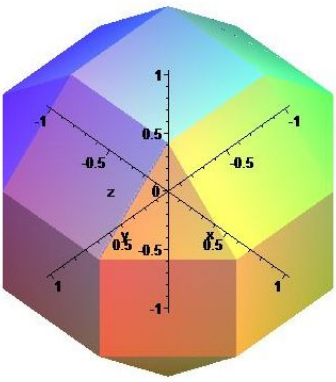

Figure 3 The unit sphere in terms of d_{RC}: Rhombicuboctahedron

□

Corollary 2.4. *The equation of the rhombicuboctahedron with center* (x_0,y_0,z_0) *and radius* r *is*

$$\frac{2\sqrt{2}+1}{7}\max\left\{\begin{array}{c} |x-x_0|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|x-x_0|\,,\left(2+\sqrt{2}\right)(|y-y_0|+|z-z_0|)\,, \\ |x-x_0|+\frac{3+\sqrt{2}}{2}\,|y-y_0|\,,|x-x_0|+\frac{3+\sqrt{2}}{2}\,|z-z_0| \end{array}\right\}, \\ |y-y_0|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|y-y_0|\,,\left(2+\sqrt{2}\right)(|x-x_0|+|z-z_0|)\,, \\ |y-y_0|+\frac{3+\sqrt{2}}{2}\,|z-z_0|\,,|y-y_0|+\frac{3+\sqrt{2}}{2}\,|x-x_0| \end{array}\right\}, \\ |z-z_0|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|z-z_0|\,,\left(2+\sqrt{2}\right)(|x-x_0|+|y-y_0|)\,, \\ |z-z_0|+\frac{3+\sqrt{2}}{2}\,|x-x_0|\,,|z-z_0|+\frac{3+\sqrt{2}}{2}\,|y-y_0| \end{array}\right\} \end{array}\right\}=r$$

which is a polyhedron which has 26 faces and 24 vertices. Coordinates of the vertices are translation to (x_0,y_0,z_0) *all permutations of the three axis components and all possible +/- sign changes of each axis component of* $\left(\left(\sqrt{2}-1\right)r,\left(\sqrt{2}-1\right)r,r\right)$.

Lemma 2.5. *Let* l *be the line through the points* $P_1=(x_1,y_1,z_1)$ *and* $P_2=(x_2,y_2,z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector* (p,q,r)*, then*

$$d_{RC}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where

$$\mu(P_1P_2)=\frac{\frac{2\sqrt{2}+1}{7}\max\left\{\begin{array}{c} |p|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|p|\,,\left(2+\sqrt{2}\right)(|q|+|r|)\,, \\ |p|+\frac{3+\sqrt{2}}{2}\,|q|\,,|p|+\frac{3+\sqrt{2}}{2}\,|r| \end{array}\right\}, \\ |q|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|q|\,,\left(2+\sqrt{2}\right)(|p|+|r|)\,, \\ |q|+\frac{3+\sqrt{2}}{2}\,|r|\,,|q|+\frac{3+\sqrt{2}}{2}\,|p| \end{array}\right\}, \\ |r|+\frac{2-\sqrt{2}}{2}\max\left\{\begin{array}{c} 2\sqrt{2}\,|r|\,,\left(2+\sqrt{2}\right)(|p|+|q|)\,, \\ |r|+\frac{3+\sqrt{2}}{2}\,|p|\,,|r|+\frac{3+\sqrt{2}}{2}\,|q| \end{array}\right\} \end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $r \in \mathbb{R}$. Thus, $d_{RC}(P_1, P_2)$ is equal to

$$|\lambda| \left(\frac{2\sqrt{2}+1}{7} \max \left\{ \begin{array}{l} |p| + \frac{2-\sqrt{2}}{2} \max \left\{ \begin{array}{c} 2\sqrt{2}\,|p|\,, \left(2+\sqrt{2}\right)(|q|+|r|)\,, \\ |p| + \frac{3+\sqrt{2}}{2}\,|q|\,, |p| + \frac{3+\sqrt{2}}{2}\,|r| \end{array} \right\}, \\ |q| + \frac{2-\sqrt{2}}{2} \max \left\{ \begin{array}{c} 2\sqrt{2}\,|q|\,, \left(2+\sqrt{2}\right)(|p|+|r|)\,, \\ |q| + \frac{3+\sqrt{2}}{2}\,|r|\,, |q| + \frac{3+\sqrt{2}}{2}\,|p| \end{array} \right\}, \\ |r| + \frac{2-\sqrt{2}}{2} \max \left\{ \begin{array}{c} 2\sqrt{2}\,|r|\,, \left(2+\sqrt{2}\right)(|p|+|q|)\,, \\ |r| + \frac{3+\sqrt{2}}{2}\,|p|\,, |r| + \frac{3+\sqrt{2}}{2}\,|q| \end{array} \right\} \end{array} \right\} \right)$$

and $d_E(A, B) = |\lambda| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. □

The above lemma says that d_{RC}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 2.6. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{RC}(P_1, X) = d_{RC}(P_2, X)$.*

Corollary 2.7. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{RC}(X, P_1) \,/\, d_{RC}(X, P_2) = d_E(X, P_1) \,/\, d_E(X, P_2)\,.$$

That is, the ratios of the Euclidean and d_{RC}−distances along a line are the same.

3 Rhombicosadodecahedron Metric and Some Properties

The rhombicosidodecahedron is an Archimedean solid. It has 20 regular triangular faces, 30 square faces, 12 regular pentagonal faces, 60 vertices and 120 edges [16] (See Figure 4(a)). It is seen that progresing of expansion of dodecahedron and icosahedron in figure 4(b).

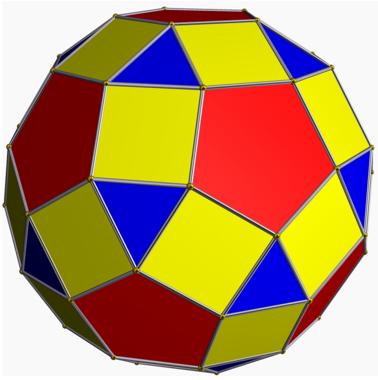

Figure 4(a)rhombicosidodecahedron

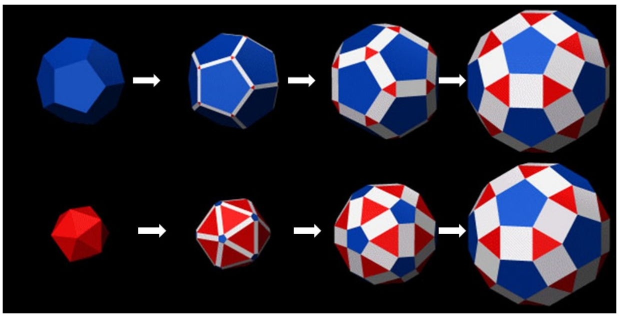

Figure 4(b) Expansion of dodecahedron and icosahedron

The metric that unit sphere is rhombicosidodecahedron is described as following:

Definition 3.1. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.The distance function $d_{RI} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ rhombicosidodecahedron distance between P_1 and P_2 is defined by

$d_{RI}(P_1, P_2) =$

$$\frac{4+\sqrt{5}}{11} \max \left\{ \begin{array}{l} X_{12} + \frac{3\sqrt{5}-5}{4} \max \left\{ \begin{array}{l} \frac{4\sqrt{5}}{5} X_{12}, \frac{\sqrt{5}+5}{5}(X_{12}+Z_{12})\,, \frac{3\sqrt{5}+5}{5}(Y_{12}+Z_{12})\,, \\ \frac{4\sqrt{5}+10}{15} X_{12} + \frac{6\sqrt{5}+20}{15} Y_{12}, X_{12} + \frac{7\sqrt{5}+5}{10} Z_{12} + \frac{15-\sqrt{5}}{10} Y_{12} \end{array} \right\}, \\ Y_{12} + \frac{3\sqrt{5}-5}{4} \max \left\{ \begin{array}{l} \frac{4\sqrt{5}}{5} Y_{12}, \frac{\sqrt{5}+5}{5}(X_{12}+Y_{12})\,, \frac{3\sqrt{5}+5}{5}(X_{12}+Z_{12})\,, \\ \frac{4\sqrt{5}+10}{15} Y_{12} + \frac{6\sqrt{5}+20}{15} Z_{12}, Y_{12} + \frac{7\sqrt{5}+5}{10} X_{12} + \frac{15-\sqrt{5}}{10} Z_{12} \end{array} \right\}, \\ Z_{12} + \frac{3\sqrt{5}-5}{4} \max \left\{ \begin{array}{l} \frac{4\sqrt{5}}{5} Z_{12}, \frac{\sqrt{5}+5}{5}(Y_{12}+Z_{12})\,, \frac{3\sqrt{5}+5}{5}(X_{12}+Y_{12})\,, \\ \frac{4\sqrt{5}+10}{15} Z_{12} + \frac{6\sqrt{5}+20}{15} X_{12}, Z_{12} + \frac{7\sqrt{5}+5}{10} Y_{12} + \frac{15-\sqrt{5}}{10} X_{12} \end{array} \right\} \end{array} \right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$.

According to rhombicosidodecahedron distance, there are five different paths from P_1 to P_2. These paths are

$i)$ a line segment which is parallel to a coordinate axis,

$ii)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{\sqrt{5}}{2})$ angle with another coordinate axis,

$iii)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{133-45\sqrt{5}}{341})$ angle with another coordinate axis,

$iv)$ union of three line segments which one is parallel to a coordinate axis and other line segments makes $\arctan(\frac{1}{2})$ and $\arctan(\frac{\sqrt{5}}{2})$ angle with one of other coordinate axis,

$v)$ union of three line segments each of which is parallel to a coordinate axis.

Thus rhombicuboctahedron distance between P_1 and P_2 is for (i) Euclidean length of line segment, for (ii) $\frac{5\sqrt{5}+9}{22}$ times the sum of Euclidean lengths of mentioned two line segments, for (iii) $\frac{11\sqrt{5}+27}{66}$ times the sum of Euclidean lengths of two line segments, for (iv) $\frac{\sqrt{5}+1}{4}$ times the sum of Euclidean lengths of three line segments, and for (v) $\frac{\sqrt{5}+4}{11}$ times the sum of Euclidean lengths of three line segments. Figure 5 shows that the path between P_1 and P_2 in case of the maximum is $|y_1 - y_2|$, $\frac{5\sqrt{5}+9}{22}\left(|y_1 - y_2| + \frac{3-\sqrt{5}}{2}|x_1 - x_2|\right)$, $\frac{\sqrt{5}+1}{4}\left(|y_1 - y_2| + \frac{\sqrt{5}-1}{2}|x_1 - x_2| + \frac{3-\sqrt{5}}{2}|z_1 - z_2|\right)$, $\frac{11\sqrt{5}+27}{66}\left(|y_1 - y_2| + \frac{44\sqrt{5}-77}{31}|z_1 - z_2|\right)$ or $\frac{4+\sqrt{5}}{11}\left(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|\right)$.

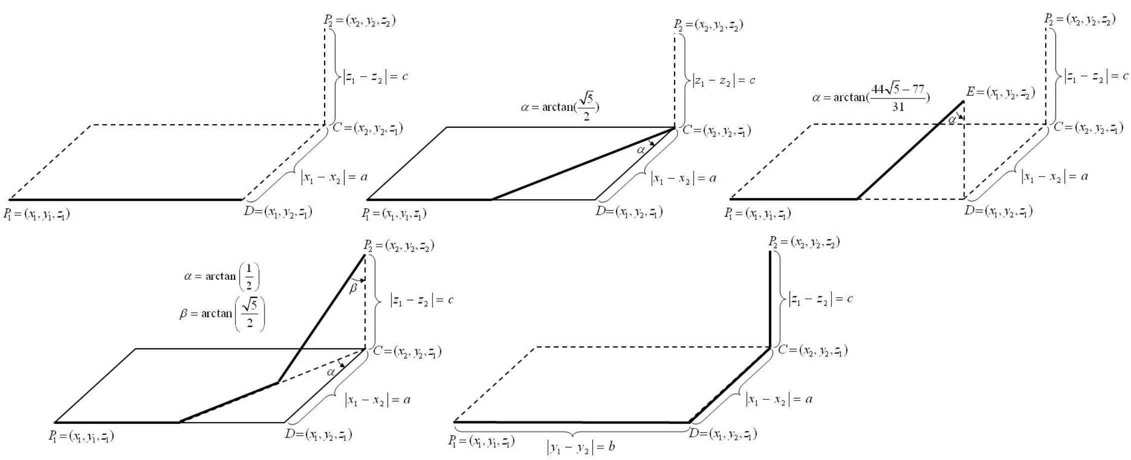

Figure 5: RI way from P_1 to P_2

Lemma 3.2. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. Then*

$$d_{RI}(P_1,P_2) \geq \frac{4+\sqrt{5}}{11}\left(X_{12} + \frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{4\sqrt{5}}{5}X_{12},\ \frac{\sqrt{5}+5}{5}(X_{12}+Z_{12}),\ \frac{3\sqrt{5}+5}{5}(Y_{12}+Z_{12}),\\ \frac{4\sqrt{5}+10}{15}X_{12} + \frac{6\sqrt{5}+20}{15}Y_{12},\ X_{12} + \frac{7\sqrt{5}+5}{10}Z_{12} + \frac{15-\sqrt{5}}{10}Y_{12}\end{array}\right\}\right)$$

$$d_{RI}(P_1,P_2) \geq \frac{4+\sqrt{5}}{11}\left(Y_{12} + \frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{4\sqrt{5}}{5}Y_{12},\ \frac{\sqrt{5}+5}{5}(X_{12}+Y_{12}),\ \frac{3\sqrt{5}+5}{5}(X_{12}+Z_{12}),\\ \frac{4\sqrt{5}+10}{15}Y_{12} + \frac{6\sqrt{5}+20}{15}Z_{12},\ Y_{12} + \frac{7\sqrt{5}+5}{10}X_{12} + \frac{15-\sqrt{5}}{10}Z_{12}\end{array}\right\}\right)$$

$$d_{RI}(P_1,P_2) \geq \frac{4+\sqrt{5}}{11}\left(Z_{12} + \frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{4\sqrt{5}}{5}Z_{12},\ \frac{\sqrt{5}+5}{5}(Y_{12}+Z_{12}),\ \frac{3\sqrt{5}+5}{5}(X_{12}+Y_{12}),\\ \frac{4\sqrt{5}+10}{15}Z_{12} + \frac{6\sqrt{5}+20}{15}X_{12},\ Z_{12} + \frac{7\sqrt{5}+5}{10}Y_{12} + \frac{15-\sqrt{5}}{10}X_{12}\end{array}\right\}\right).$$

where $X_{12}=|x_1 - x_2|$, $Y_{12}=|y_1 - y_2|$, $Z_{12}=|z_1 - z_2|$.

Proof. Proof is trivial by the definition of maximum function. □

Theorem 3.3. *The distance function d_{RI} is a metric. Also according to d_{RI}, unit sphere is a rhombicosidodecahedron in $\mathbb{R}^3$.*

Proof. One can easily show that the rhombicosidodecahedron distance function satisfies the metric axioms by similar way in Theorem 2.3.

Consequently, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that rhombicosidodecahedron distance is 1 from $O=(0,0,0)$ is $S_{RI}=$

$$\left\{(x,y,z):\frac{4+\sqrt{5}}{11}\max\left\{\begin{array}{l}|x|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|x|+|z|\right),\frac{3\sqrt{5}+5}{5}\left(|y|+|z|\right),\\ \frac{4\sqrt{5}}{5}|x|,\frac{4\sqrt{5}+10}{15}|x|+\frac{6\sqrt{5}+20}{15}|y|,\\ |x|+\frac{7\sqrt{5}+5}{10}|z|+\frac{15-\sqrt{5}}{10}|y|\end{array}\right\},\\ |y|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|x|+|y|\right),\frac{3\sqrt{5}+5}{5}\left(|x|+|z|\right),\\ \frac{4\sqrt{5}}{5}|y|,\frac{4\sqrt{5}+10}{15}|y|+\frac{6\sqrt{5}+20}{15}|z|,\\ |y|+\frac{7\sqrt{5}+5}{10}|x|+\frac{15-\sqrt{5}}{10}|z|\end{array}\right\},\\ |z|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|y|+|z|\right),\frac{3\sqrt{5}+5}{5}\left(|x|+|y|\right),\\ \frac{4\sqrt{5}}{5}|z|,\frac{4\sqrt{5}+10}{15}|z|+\frac{6\sqrt{5}+20}{15}|x|,\\ |z|+\frac{7\sqrt{5}+5}{10}|y|+\frac{15-\sqrt{5}}{10}|x|\end{array}\right\}\end{array}\right\}=1\right\}.$$

Thus the graph of S_{RI} is as in the figure 6:

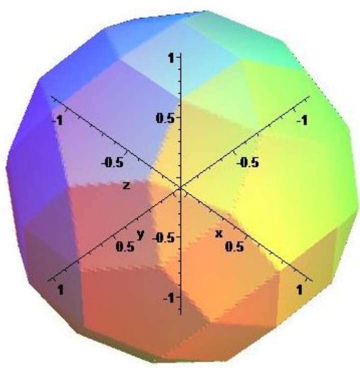

Figure 6 The unit sphere in terms of d_{RI}: Rhombicosidodecahedron

□

Corollary 3.4. *The equation of the rhombicosidodecahedron with center* (x_0,y_0,z_0) *and radius* r *is*

$$\frac{4+\sqrt{5}}{11}\max\left\{\begin{array}{l}|x-x_0|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|x-x_0|+|z-z_0|\right),\frac{3\sqrt{5}+5}{5}\left(|y-y_0|+|z-z_0|\right),\\ \frac{4\sqrt{5}}{5}|x-x_0|,\frac{4\sqrt{5}+10}{15}|x-x_0|+\frac{6\sqrt{5}+20}{15}|y-y_0|,\\ |x-x_0|+\frac{7\sqrt{5}+5}{10}|z-z_0|+\frac{15-\sqrt{5}}{10}|y-y_0|\end{array}\right\},\\ |y-y_0|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|x-x_0|+|y-y_0|\right),\frac{3\sqrt{5}+5}{5}\left(|x-x_0|+|z-z_0|\right),\\ \frac{4\sqrt{5}}{5}|y-y_0|,\frac{4\sqrt{5}+10}{15}|y-y_0|+\frac{6\sqrt{5}+20}{15}|z-z_0|,\\ |y-y_0|+\frac{7\sqrt{5}+5}{10}|x-x_0|+\frac{15-\sqrt{5}}{10}|z-z_0|\end{array}\right\},\\ |z-z_0|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|y-y_0|+|z-z_0|\right),\frac{3\sqrt{5}+5}{5}\left(|x-x_0|+|y-y_0|\right),\\ \frac{4\sqrt{5}}{5}|z-z_0|,\frac{4\sqrt{5}+10}{15}|z-z_0|+\frac{6\sqrt{5}+20}{15}|x-x_0|,\\ |z-z_0|+\frac{7\sqrt{5}+5}{10}|y-y_0|+\frac{15-\sqrt{5}}{10}|x-x_0|\end{array}\right\}\end{array}\right\}=r.$$

which is a polyhedron which has 62 faces and 60 vertices. Coordinates of the vertices are translation to (x_0,y_0,z_0) *all posible +/- sign components of the points* $\left(\left(\sqrt{5}-2\right)r,\left(\sqrt{5}-2\right)r,r\right)$, $\left(r,\left(\sqrt{5}-2\right)r,\left(\sqrt{5}-2\right)r\right)$, $\left(\left(\sqrt{5}-2\right)r,r,\left(\sqrt{5}-2\right)r\right)$, $\left(0,\frac{\sqrt{5}-1}{2}r,\frac{3\sqrt{5}-5}{2}r\right)$, $\left(\frac{3\sqrt{5}-5}{2}r,0,\frac{\sqrt{5}-1}{2}r\right)$, $\left(\frac{\sqrt{5}-1}{2}r,\frac{3\sqrt{5}-5}{2}r,0\right)$, $\left(\frac{\sqrt{5}-1}{2}r,\frac{3-\sqrt{5}}{2}r,\left(3-\sqrt{5}\right)r\right)$, $\left(\left(3-\sqrt{5}\right)r,\frac{\sqrt{5}-1}{2}r,\frac{3-\sqrt{5}}{2}r\right)$ *and* $\left(\frac{3-\sqrt{5}}{2}r,\left(3-\sqrt{5}\right)r,\right.$

Lemma 3.5. *Let* l *be the line through the points* $P_1=(x_1,y_1,z_1)$ *and* $P_2=(x_2,y_2,z_2)$ *in the analytical 3-dimensional space and* d_E *denote the Euclidean metric. If* l *has direction vector*

(p,q,r), *then*

$$d_{RI}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where

$$\mu(P_1P_2)=\frac{\frac{4+\sqrt{5}}{11}\max\left\{\begin{array}{c}|p|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|p|+|r|\right),\frac{3\sqrt{5}+5}{5}\left(|q|+|r|\right),\\ \frac{4\sqrt{5}}{5}|p|,\frac{4\sqrt{5}+10}{15}|p|+\frac{6\sqrt{5}+20}{15}|q|,\\ |p|+\frac{7\sqrt{5}+5}{10}|r|+\frac{15-\sqrt{5}}{10}|q|\end{array}\right\},\\ |q|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|p|+|q|\right),\frac{3\sqrt{5}+5}{5}\left(|p|+|r|\right),\\ \frac{4\sqrt{5}}{5}|q|,\frac{4\sqrt{5}+10}{15}|q|+\frac{6\sqrt{5}+20}{15}|r|,\\ |q|+\frac{7\sqrt{5}+5}{10}|p|+\frac{15-\sqrt{5}}{10}|r|\end{array}\right\},\\ |r|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|q|+|r|\right),\frac{3\sqrt{5}+5}{5}\left(|p|+|q|\right),\\ \frac{4\sqrt{5}}{5}|r|,\frac{4\sqrt{5}+10}{15}|r|+\frac{6\sqrt{5}+20}{15}|p|,\\ |r|+\frac{7\sqrt{5}+5}{10}|q|+\frac{15-\sqrt{5}}{10}|p|\end{array}\right\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1-x_2=\lambda p$, $y_1-y_2=\lambda q$, $z_1-z_2=\lambda r$, $r\in\mathbb{R}$. Thus,

$$d_{RI}(P_1,P_2)=|\lambda|\,\frac{4+\sqrt{5}}{11}\max\left\{\begin{array}{c}|p|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|p|+|r|\right),\frac{3\sqrt{5}+5}{5}\left(|q|+|r|\right),\\ \frac{4\sqrt{5}}{5}|p|,\frac{4\sqrt{5}+10}{15}|p|+\frac{6\sqrt{5}+20}{15}|q|,\\ |p|+\frac{7\sqrt{5}+5}{10}|r|+\frac{15-\sqrt{5}}{10}|q|\end{array}\right\},\\ |q|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|p|+|q|\right),\frac{3\sqrt{5}+5}{5}\left(|p|+|r|\right),\\ \frac{4\sqrt{5}}{5}|q|,\frac{4\sqrt{5}+10}{15}|q|+\frac{6\sqrt{5}+20}{15}|r|,\\ |q|+\frac{7\sqrt{5}+5}{10}|p|+\frac{15-\sqrt{5}}{10}|r|\end{array}\right\},\\ |r|+\frac{3\sqrt{5}-5}{4}\max\left\{\begin{array}{l}\frac{\sqrt{5}+5}{5}\left(|q|+|r|\right),\frac{3\sqrt{5}+5}{5}\left(|p|+|q|\right),\\ \frac{4\sqrt{5}}{5}|r|,\frac{4\sqrt{5}+10}{15}|r|+\frac{6\sqrt{5}+20}{15}|p|,\\ |r|+\frac{7\sqrt{5}+5}{10}|q|+\frac{15-\sqrt{5}}{10}|p|\end{array}\right\}\end{array}\right\}$$

and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result. □

The above lemma says that d_{RI}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 3.6. *If* P_1, P_2 *and* X *are any three collinear points in* $\mathbb{R}^3$, *then* $d_E(P_1,X)=d_E(P_2,X)$ *if and only if* $d_{RI}(P_1,X)=d_{RI}(P_2,X)$.

Corollary 3.7. *If* P_1, P_2 *and* X *are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{RI}(X,P_1)\ /\ d_{RI}(X,P_2)=d_E(X,P_1)\ /\ d_E(X,P_2)\ .$$

That is, the ratios of the Euclidean and $d_{RI}-$*distances along a line are the same.*

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