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Dissipative q-Dirac Operator

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AbstractIn this paper, we investigate symmetric $q-$Dirac operator acting in $L_q^2((0,a),\mathbb{C}^2)$. We describe maximal dissipative, maximal accumulative, self-adjoint and the other extensions of such operators via the boundary conditions. We construct a self-adjoint dilation of the dissipative operator and determine the scattering matrix of dilation. Later, we construct a functional model of the diisipative operator and define its characteristic function. Finally, we prove that all root vectors of the dissipative operator are complete in the Hilbert space $L_q^2((0,a),\mathbb{C}^2)$.

1 Introduction

Studies on quantum analysis ($q-$analysis) began in the 19th century by Jackson [1]. Since then, the quantum analysis play an important role in various fields of science and engineering for example, the theory of relativity, quantum theory, basic hypergeometric functions, string theory, quantum chromodynamics. For more information, see [3], [2].

In this paper, we consider the following $q-$Dirac equations:

$$\begin{aligned} -\frac{1}{q}D_{q^{-1}}y_2+p(x)y_1 &= \lambda y_1, \\ D_q y_1+r(x)y_2 &= \lambda y_2, \end{aligned} \tag{1.1}$$

where λ is a complex parameter, and p and r are $q-$regular at zero and q is a positive number which is less than 1. This equation is the $q-$analogue of the one dimensional Dirac system

$$\begin{aligned} -y_2'+p(x)y_1 &= \lambda y_1, \\ y_1'+r(x)y_2 &= \lambda y_2. \end{aligned} \tag{1.2}$$

As is known, The equation (1.2) describe a relativistic electron in the electrostatic field (see [4]).

On the other hand, the class of dissipative operator is one of the main research areas of the operator theory. In spectral analysis of dissipative operators, the theory of dilations with applications of functional models is one of the basic methods. Specially, the characteristic function carries important information regarding the spectral properties of these operators. We know that the absence of the singular factor in the factorization of the characteristic function is guarantee the completeness of the system of root vectors of maximal dissipative operators [6].

In the present article, we work $q-$Dirac operator acting in the Hilbert space $H:=L_q^2((0,a),\mathbb{C}^2)$ $(0<a<\infty)$. In Section 2, we construct a space of boundary value for minimal symmetric $q-$Dirac operator and describe all the maximal dissipative, maximal accumulative, self-adjoint and other extensions of such operator. In Section 3, we construct a self-adjoint dilation and its incoming and outgoing spectral representations. Thus, we determine the scattering matrix of the dilation according to the Lax and Phillips scheme [5], [6]. In Section 4, using incoming spectral representations, we construct a functional model of the maximal dissipative $q-$Dirac operator. Furthermore, we determine characteristic function of this operator. Finally, we prove that all root vectors of the maximal dissipative $q-$Dirac operator are complete in the space H.

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A similar way was employed earlier in the differential/difference operator cases in [8]-[10], [15]-[17].

Now, we recall some necessary concepts of quantum analysis for convenience.

Following the standard notations in [7], [2], let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$ and $a \in A$. A *q-difference equation* is an equation that contains $q-$derivatives of a function defined on A. Let y be a complex-valued function on A. The *q-difference operator* D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{(q-1)x} \text{ for all } x \in A.$$

The $q-$derivative at zero is defined by

$$D_q y(0) = \lim_{n\to\infty} \frac{y(q^n x) - y(0)}{q^n x} \ (x \in A),$$

if the limit exists and does not depend on x. A *right-inverse* to D_q, the *Jackson* $q-$*integration* is given by

$$\int_0^x f(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x) \ (x \in A),$$

provided that the series converges, and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \ (a, b \in A).$$

A function f which is defined on A, $0 \in A$, is said to be $q-$regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0),$$

for every $x \in A$. Through the remainder of the paper, we deal only with functions $q-$regular at zero. Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $[0, a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|\, d_q x \right)^{1/2} < \infty.$$

The space $L_q^2(0, a)$ is a separable Hilbert space with the inner product

$$(f, g) := \int_0^a f(x) \overline{g(x)} d_q x, \ f, g \in L_q^2(0, a),$$

and the orthonormal basis

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{x(1-q)}}, & x = aq^n, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 0, 1, 2, ...$(see [2]).

2 Extensions of symmetric $q-$Dirac Operators

In this section, we describe all extensions (dissipative, accumulative, self-adjoint and other) of symmetric $q-$Dirac Operators. We consider the $q-$Dirac systems

$$\Gamma y := \begin{cases} -\frac{1}{q} D_{q^{-1}} y_2 + p(x) y_1 \\ D_q y_1 + r(x) y_2 \end{cases} = \lambda y = \begin{pmatrix} \lambda y_1 \\ \lambda y_2 \end{pmatrix}$$

where p and r are real-valued functions defined on $[0, a]$ and $q-$regular at zero and q is a positive number which is less that 1.

Now, using the inner product

$$(f,g) := \int_0^a (f(x), g(x))_{\mathbb{C}^2} \, d_q x,$$

we introduce convenient Hilbert space $H := L_q^2((0,a),\mathbb{C}^2)$ $(0 < a < \infty)$ of vector-valued functions.

Let us consider the set D consisting of all vector-valued functions $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H$ in which y_1 and y_2 are $q-$regular at zero and $\Gamma y \in H$. We define the maximal operator $\Upsilon_{\max}$ on the set D by the equality $\Upsilon_{\max} y := \Gamma y$. Now we have a

Lemma 2.1 (Green's formula). *Let* $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in D$. *Then, we have*

$$(\Gamma y, z) - (y, \Gamma z) = [y,z]_a - [y,z]_0,$$

where $[y,z]_x := y_1(x)\overline{z_2(q^{-1}x)} - \overline{z_1(x)} y_2(q^{-1}x)$.

Proof. Let $y, z \in D$. Then, we obtain

$$\begin{aligned}
(\Gamma y, z) - (y, \Gamma z) &= \int_0^a \left(-\frac{1}{q} D_{q^{-1}} y_2 + p(x) y_1\right) \overline{z_1} d_q x \\
&\quad + \int_0^a (D_q y_1 + r(x) y_2) \overline{z_2} d_q x \\
&\quad - \int_0^a y_1 \overline{\left(-\frac{1}{q} D_{q^{-1}} z_2 + p(x) z_1\right)} d_q x \\
&\quad - \int_0^a y_2 \overline{(D_q z_1 + r(x) z_2)} d_q x \\
&= -\int_0^a \left[\left(\frac{1}{q} D_{q^{-1}} y_2\right) \overline{z_1} + y_2 \overline{(D_q z_1)}\right] d_q x \\
&\quad + \int_0^a \left[(D_q y_1) \overline{z_2} + y_1 \overline{\left(\frac{1}{q} D_{q^{-1}} z_2\right)}\right] d_q x
\end{aligned}$$

Since

$$\begin{aligned}
D_q(\overline{z_1(x)} y_2(q^{-1}x)) &= (D_q y_2(q^{-1}x)) D_q(q^{-1}x) (\overline{z_1(x)} + y_2(x) \overline{(D_q z_1(x))} \\
&= \frac{1}{q} (D_{q^{-1}} y_2) \overline{z_1} + y_2 \overline{(D_q z_1)}
\end{aligned}$$

and

$$\begin{aligned}
D_q(\overline{z_2(q^{-1}x)} y_1(x)) &= \overline{(D_q z_2(q^{-1}x)) D_q(q^{-1}x)} y_1(x) + \overline{z_2(x)} (D_q y_1(x)) \\
&= \overline{\frac{1}{q} (D_{q^{-1}} z_2)} y_1 + \overline{z_2(x)} D_q y_1(x).
\end{aligned}$$

Hence we get

$$\begin{aligned}
(\Gamma y, z) - (y, \Gamma z) &= -\int_0^a D_q(\overline{z_1(x)} y_2(q^{-1}x)) d_q x \\
+ \int_0^a D_q(y_1(x) \overline{z_2(q^{-1}x)}) d_q x &= \int_0^a D_q \left[y_1(x) \overline{z_2(q^{-1}x)} - \overline{z_1(x)} y_2(q^{-1}x)\right] d_q x \\
&= [y,z]_a - [y,z]_0
\end{aligned}$$

□

Let $D_{\min}$ denote the linear set of all vectors $y \in D$ satisfying the conditions

$$y_1(0) = y_2(0) = y_1(a) = y_2\left(aq^{-1}\right) = 0.$$

If we restrict the operator $\Upsilon_{\max}$ to the set $D_{\min}$, then we obtain the minimal operator $\Upsilon_{\min}$. It is clear that $\Upsilon_{\min}^* = \Upsilon_{\max}$, and $\Upsilon_{\min}$ is a closed symmetric operator (see [11]). Now we recall the following.

Definition 2.2. A linear operator M (with dense domain $D(M)$) acting on some Hilbert space H is called dissipative (accumulative) if $Im(Mf, f) \geq 0$ ($Im(Mf, f) \leq 0$) for all $f \in D(M)$ and maximal dissipative (maximal accumulative) if it does not have a proper dissipative (accumulative) extension (see [8]-[10]).

Definition 2.3. A triplet $(\mathbb{H}, \Lambda_1, \Lambda_2)$ is called a space of boundary values of a closed symmetric operator M on a Hilbert space H if Λ_1 and Λ_2 are linear maps from $D(M^*)$ to H, with equal deficiency numbers and such that:

i) For every $f, g \in D(M^*)$ we have

$$(M^* f, g)_H - (f, M^* g)_H = (\Lambda_1 f, \Lambda_2 g)_{\mathbb{H}} - (\Lambda_2 f, \Lambda_1 g)_{\mathbb{H}};$$

ii) For any $F_1, F_2 \in H$ there is a vector $f \in D(A^*)$ such that $\Lambda_1 f = F_1$ and $\Lambda_2 f = F_2$ (see [12]).

Let's define by $\Lambda_1,\ \Lambda_2$ the linear maps from D to $\mathbb{C}^2$ by the formula

$$\Lambda_1 y = \begin{pmatrix} -y_1(0) \\ y_1(a) \end{pmatrix}, \ \Lambda_2 y = \begin{pmatrix} y_2(0) \\ y_2\left(aq^{-1}\right) \end{pmatrix}. \tag{2.1}$$

Now we will state and prove a theorem.

Theorem 2.4. *The triplet $\left(\mathbb{C}^2, \Lambda_1, \Lambda_2\right)$ defined by (3) is a boundary spaces of the operator $\Upsilon_{\min}$.*

Proof. Let $y, z \in D$. Then, we have

$$\begin{aligned}(\Lambda_1 y, \Lambda_2 z)_{\mathbb{C}^2} - (\Lambda_2 y, \Lambda_1 z)_{\mathbb{C}^2} &= -y_1(0)\overline{z}_2(0) + \overline{z_1}(0) y_2(0) \\ &\quad + y_1(a)\overline{z}_2\left(aq^{-1}\right) - \overline{z_1}(a) y_2\left(aq^{-1}\right).\end{aligned}$$

By Green's formula, we obtain

$$(\Lambda_1 y, \Lambda_2 z)_{\mathbb{C}^2} - (\Lambda_2 y, \Lambda_1 z)_{\mathbb{C}^2} = [y, z]_a - [y, z]_0.$$

Hence

$$(\Upsilon_{\max} y, z)_H - (y, \Upsilon_{\max} z)_H = (\Lambda_1 y, \Lambda_2 z)_{\mathbb{C}^2} - (\Lambda_2 y, \Lambda_1 z)_{\mathbb{C}^2}.$$

Thus, we obtain the first condition of the definition of a space of boundary value.

Now, we will prove the second condition. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$. Then the vector-valued function

$$y(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \alpha_1(t) u_1(t) + \alpha_2(t) v_1(t) + \beta_1(t) u_2(t) + \beta_2(t) v_2(t),$$

where $\alpha_1(t) = \begin{pmatrix} \alpha_{11}(t) \\ \alpha_{12}(t) \end{pmatrix}$, $\alpha_2(t) = \begin{pmatrix} \alpha_{21}(t) \\ \alpha_{22}(t) \end{pmatrix}$, $\beta_1(t) = \begin{pmatrix} \beta_{11}(t) \\ \beta_{12}(t) \end{pmatrix}$, $\beta_2(t) = \begin{pmatrix} \beta_{21}(t) \\ \beta_{22}(t) \end{pmatrix} \in H$ satisfy the conditions

$$\begin{aligned}\alpha_{11}(0) &= -1,\ \alpha_{12}(0) = \alpha_{11}(a) = \alpha_{12}\left(aq^{-1}\right) = 0, \\ \alpha_{22}(0) &= 1,\ \alpha_{21}(0) = \alpha_{21}(a) = \alpha_{22}\left(aq^{-1}\right) = 0, \\ \beta_{11}(a) &= 1,\ \beta_{11}(0) = \beta_{12}(0) = \beta_{12}\left(aq^{-1}\right) = 0, \\ \beta_{22}\left(aq^{-1}\right) &= 1,\ \beta_{21}(0) = \beta_{21}(a) = \beta_{22}(0) = 0,\end{aligned}$$

belongs to the set D and $\Lambda_1 y = u,\ \Lambda_2 y = v$. This finishes the proof. □

Corollary 2.5. *For any contraction K in $\mathbb{C}^2$ the restriction of the operator $\Upsilon_{\max}$ to the set of functions $y \in D$ satisfying either*

$$(K-I)\Lambda_1 y + i(K+I)\Lambda_2 y = 0 \tag{2.2}$$

or

$$(K-I)\Lambda_1 y - i(K+I)\Lambda_2 y = 0 \tag{2.3}$$

is respectively the maximal dissipative and accumulative extension of the operator $\Upsilon_{\min}$. Conversely, every maximal dissipative (accumulative) extension of the operator $\Upsilon_{\min}$ is the restriction of $\Upsilon_{\max}$ to the set of functions $y \in D$ satisfying (2.2) ((2.3)), and the extension uniquely determines the contraction K. Conditions (2.2) ((2.3)), in which K is an isometry describe the maximal symmetric extensions of $\Upsilon_{\min}$ in H. If K is unitary, these conditions define self-adjoint extensions.

In particular, the boundary conditions

$$y_2(0) + \alpha_1 y_1(0) = 0, \tag{2.4}$$

$$y_2\left(aq^{-1}\right) + \alpha_2 y_1(a) = 0, \tag{2.5}$$

with $Im\alpha_1 \geq 0$ or $\alpha_1 = \infty$, $Im\alpha_2 \geq 0$ or $\alpha_2 = \infty$, ($Im\alpha_1 = 0$ or $\alpha_1 = \infty$, $Im\alpha_2 = 0$ or $\alpha_2 = \infty$) describe the maximal dissipative (self-adjoint) extensions of $\Upsilon_{\min}$ with separated boundary conditions. Note that if $\alpha_1 = \infty$ ($\alpha_2 = \infty$), then the boundary condition (2.4) ((2.5)) should be replaced by $y_1(0) = 0$ ($y_1(a) = 0$).

From now on, we shall study the maximal dissipative operators $\Upsilon_{\alpha_1\alpha_2}$ generated by (1.1) and the boundary conditions (2.4) and (2.5) with $\Im\alpha_1 > 0$ and $Im\alpha_2 = 0$ or $\alpha_2 = \infty$.

3 Self-adjoint dilation

While we investigate the spectral analysis of the maximal dissipative operators, we will use the functional model theory of Sz.-Nagy-Foiaş (see [6]). Hence, we must construct the characteristic function of a contraction. But this is not easy. To overcome this problem, we will use the abstract scattering function of Lax-Phillips (see [5]) because it is unitary equivalent to the characteristic function of Sz.-Nagy-Foias (see [6]).

In this section, we construct a self-adjoint dilation and its incoming and outgoing spectral representations. Later, we determine the scattering matrix of the dilation according to the Lax and Phillips scheme [5], [6].

Now, let us define the *main Hilbert space of the dilation* $\mathcal{H} = \tau_- \oplus H \oplus \tau_+$ where $\tau_- = L^2(-\infty, 0)$ and $\tau_+ = L^2(0,\infty)$ are the "incoming" and "outgoing" subspaces. In the space $\mathcal{H}$, we consider the operator $\mathbf{\Gamma}$ on the set $D(\mathbf{\Gamma})$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+\rangle$, generated by the expression

$$\mathbf{\Gamma}\langle \varphi_-, y, \varphi_+\rangle = \langle i\frac{d\varphi_-}{d\xi}, \Gamma y, i\frac{d\varphi_+}{d\zeta}\rangle \tag{3.1}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty, 0)$, $\varphi_+ \in W_2^1(0,\infty)$, $y \in H$,

$$y_2(0) - \alpha_1 y_1(0) = \gamma\varphi_-(0), \; y_2(0) - \overline{\alpha_1} y_1(0) = \gamma\varphi_+(0),$$

$$y_2\left(aq^{-1}\right) - \alpha_2 y_1(a) = 0.$$

where W_2^1 are Sobolev spaces and $\gamma^2 := 2\, Im\alpha_1$, $\gamma > 0$.

Theorem 3.1. *The operator $\mathbf{\Gamma}$ is self-adjoint in $\mathcal{H}$ and it is a self-adjoint dilation of the operator $\Upsilon_{\alpha_1\alpha_2}$.*

Proof. Let $f,\ g \in D(\mathbf{\Gamma})$, $f = \langle \varphi_-, y, \varphi_+\rangle$ and $g = \langle \psi_-, z, \psi_+\rangle$. Then we have

$$(\mathbf{\Gamma} f, g)_{\mathcal{H}} - (f, \mathbf{\Gamma} g)_{\mathcal{H}} = (\mathbf{\Gamma}\langle \varphi_-, y, \varphi_+\rangle, \langle \psi_-, z, \psi_+\rangle) - (\langle \varphi_-, y, \varphi_+\rangle, \mathbf{\Gamma}\langle \psi_-, z, \psi_+\rangle)$$

$$= i\int_{-\infty}^{0}\frac{d\varphi_-}{d\xi}\overline{\psi}_-d\xi+(\Gamma y,z)_H+i\int_{0}^{\infty}\frac{d\varphi_+}{d\xi}\overline{\psi}_+d\xi$$

$$-i\int_{-\infty}^{0}\varphi_-\overline{\frac{d\psi_-}{d\xi}}d\xi-(y,\Gamma z)_H-i\int_{0}^{\infty}\varphi_+\overline{\frac{d\psi_+}{d\xi}}d\xi$$

$$= i\int_{-\infty}^{0}\frac{d\varphi_-}{d\xi}\overline{\psi}_-d\xi+[y,z]_a+i\int_{0}^{\infty}\frac{d\varphi_+}{d\xi}\overline{\psi}_+d\xi$$

$$-i\int_{-\infty}^{0}\varphi_-\overline{\frac{d\psi_-}{d\xi}}d\xi-[y,z]_0-i\int_{0}^{\infty}\varphi_+\overline{\frac{d\psi_+}{d\xi}}d\xi$$

$$= i\psi_-(0)\,\overline{\varphi}_-(0)-i\varphi_+(0)\,\overline{\psi}_+(0)+[y,z]_a-[y,z]_0.$$

By direct computation, we get

$$i\psi_-(0)\,\overline{\varphi}_-(0)-i\varphi_+(0)\,\overline{\psi}_+(0)+[y,z]_a-[y,z]_0=0.$$

Thus, $\mathbf{\Gamma}$ is a symmetric operator.

Now, we will prove that $\mathbf{\Gamma}$ is self-adjoint, i.e., $\mathbf{\Gamma}^*\subseteq\mathbf{\Gamma}$. Let $g=\langle\psi_-,z,\psi_+\rangle\in D(\mathbf{\Gamma}^*)$ and $\mathbf{\Gamma}^*g=g^*=\langle\psi_-^*,z^*,\psi_+^*\rangle\in\mathcal{H}$, such that

$$(\mathbf{\Gamma}f,g)_{\mathcal{H}}=(f,\mathbf{\Gamma}^*g)_{\mathcal{H}}=(f,g^*)_{\mathcal{H}}\,. \tag{3.2}$$

Then, it is not difficult to show that $\psi_-\in W_2^1(-\infty,0)\,,\ \psi_+\in W_2^1(0,\infty)\,,\ g\in D(\mathbf{\Gamma})$ and $g^*=\mathbf{\Gamma}g$. Using (3.2), we obtain

$$(\mathbf{\Gamma}f,g)_{\mathcal{H}}=(f,\mathbf{\Gamma}g)_{\mathcal{H}}\,,f\in D(\mathbf{\Gamma}^*)\,.$$

Furthermore, $g\in D(\mathbf{\Gamma}^*)$ satisfies the conditions

$$y_2(0)-\alpha_1y_1(0)=\gamma\varphi_-(0)\,,\ y_2(0)-\overline{\alpha_1}y_1(0)=\gamma\varphi_+(0)\,,$$

$$y_2\left(aq^{-1}\right)-\alpha_2y_1(a)=0.$$

Consequently, $D(\mathbf{\Gamma}^*)\subseteq D(\mathbf{\Gamma})\,,$ i.e., $\mathbf{\Gamma}$ is self-adjoint.

On the other hand, we know that the self-adjoint operator $\mathbf{\Gamma}$ generates on $\mathcal{H}$ a unitary group $\mathcal{U}_t=\exp(i\mathbf{\Gamma}t)$ $(t\in\mathbb{R})$. Let denote by $\mathcal{P}:\mathcal{H}\to H$ and $\mathcal{P}_1:H\to\mathcal{H}$ the mapping acting according to the formulae $\mathcal{P}:\langle\varphi_-,y,\varphi_+\rangle\to y$ and $\mathcal{P}_1:y\to\langle 0,y,0\rangle$. Let $Z_t:=\mathcal{P}\mathcal{U}_t\mathcal{P}_{1,}\,t\geq 0$. Then, the family $\{Z_t\}$ $(t\geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contraction on H. The generator of this semigroup is defined by the formula

$$By=\lim_{t\to+0}\frac{1}{it}(Z_ty-y)$$

The domain of B consists of all the vectors for which the limit exists. The operator B is maximal dissipative. The operator $\mathbf{\Gamma}$ is called the self-adjoint dilation of B (see [6], [14]). We next show that $\Upsilon_{\alpha_1\alpha_2}=B$ and therefore $\mathbf{\Gamma}$ is self-adjoint dilation of B. For this purpose, it is sufficient to verify the equality (see [6], [14])

$$\mathcal{P}(\mathbf{\Gamma}-\lambda I)^{-1}\mathcal{P}_1y=(\Upsilon_{\alpha_1\alpha_2}-\lambda I)^{-1}y,y\in H,\ text\ Im\alpha_1<0\,text. \tag{3.3}$$

Let $(\mathbf{\Gamma}-\lambda I)^{-1}\mathcal{P}_1y=g=\langle\psi_-,z,\psi_+\rangle$. Then, we have $(\mathbf{\Gamma}-\lambda I)\,g=\mathcal{P}_1y$. Conse-quently, $\Gamma z-\lambda z=y,\ \psi_-(\xi)=\psi_-(0)\,e^{-i\lambda\xi}$ and $\psi_+(\xi)=\psi_+(0)\,e^{-i\lambda\xi}$. Since $g\in D(\mathbf{\Gamma})\,,$ then $\psi_-\in W_2^1(-\infty,0)\,,$ it follows that $\psi_-(0)=0$, and consequently z satisfies the boundary condition $y_2(0)-\alpha_1y_1(0)=0$. Therefore $z\in D(\Upsilon_{\alpha_1\alpha_2})\,,$ and since point λ with $Im\lambda<0$ cannot be an eigenvalue of dissipative operator, then $z=(\Upsilon_{\alpha_1\alpha_2}-\lambda I)^{-1}y$. Thus

$$(\mathbf{\Gamma}-\lambda I)^{-1}\mathcal{P}_1y=\langle 0,\ (\Upsilon_{\alpha_1\alpha_2}-\lambda I)^{-1}y,\ \gamma^{-1}(y_2(0)-\overline{\alpha}_1y_1(0))\,e^{-i\lambda\xi}\rangle$$

for $y \in H$ and $Im\lambda < 0$. On applying the mapping $\mathcal{P}$, we obtain (3.3). Furthermore, using by (3.3), we get

$$\begin{aligned}\left(\Upsilon_{\alpha_1\alpha_2} - \lambda I\right)^{-1} &= \mathcal{P}\left(\mathbf{\Gamma} - \lambda I\right)^{-1}\mathcal{P}_1 = -i\mathcal{P}\int_0^\infty \mathcal{U}_t e^{-i\lambda t} dt \mathcal{P}_1 \\ &= -i\int_0^\infty Z_t e^{-i\lambda t} dt = \left(B - \lambda I\right)^{-1}, \ \ Im\lambda < 0,\end{aligned}$$

i.e., $\Upsilon_{\alpha_1\alpha_2} = B$. □

On the other hand, the unitary group $\{\mathcal{U}_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips (see [5]). In the following theorem, we will give its properties.

Theorem 3.2. *Let* $\tau_- = \langle L^2(-\infty, 0), 0, 0\rangle$ *and* $\tau_+ = \langle 0, 0, L^2(0,\infty)\rangle$ *be orthogonal incoming and outgoing subspaces of the unitary group* $\{\mathcal{U}_t\}, \ t \in \mathbb{R}$. *Then they have the following properties:*

(i) $\mathcal{U}_t\tau_- \subset \tau_-$, $t \leq 0$ and $\mathcal{U}_t\tau_+ \subset \tau_+$, $t \geq 0$;
(ii) $\bigcap_{t\leq 0}\mathcal{U}_t\tau_- = \bigcap_{t\geq 0}\mathcal{U}_t\tau_+ = \{0\}$;
(iii) $\tau_- \perp \tau_+$.

Proof. (i) For all λ, with $Im\lambda < 0$, we have

$$\mathcal{R}_\lambda f = \left(\mathbf{\Gamma} - \lambda I\right)^{-1} f = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\varphi_+(s)\, ds\rangle, \ f = \langle 0, 0, \varphi_+\rangle \in \tau_+,$$

i.e., $\mathcal{R}_\lambda f \in \tau_+$. Furthermore, if $g \perp \tau_+$, then

$$0 = \left(\mathcal{R}_\lambda f, g\right)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda t}\left(\mathcal{U}_t f, g\right)_{\mathcal{H}} dt, \ text\ Im\lambda < 0.$$

which implies that $(\mathcal{U}_t f, g)_{\mathcal{H}} = 0$ for all $t \geq 0$. Hence, for $t \geq 0$, $\mathcal{U}_t\tau_+ \subset \tau_+$, the proof for τ_- is similar. □

(ii) Let us define the mappings $\mathcal{P}^+ : \mathcal{H} \to L^2(0,\infty)$ and $\mathcal{P}_1^+ : L^2(0,\infty) \to \tau_+$ as follows $\mathcal{P}^+ : \langle\varphi_-, y, \varphi_+\rangle \to \varphi_+$ and $\mathcal{P}_1^+ : \varphi \to \langle 0, 0, \varphi\rangle$, respectively. We take into consider that the semigroup of isometries $\mathcal{U}_t := \mathcal{P}^+\mathcal{U}_t\mathcal{P}_1^+$ $(t \geq 0)$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2(0,\infty)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator S of the semigroup of isometries $\mathcal{U}_t$ $(t \geq 0)$ is the operator

$$S\varphi = \mathcal{P}^+\mathbf{\Gamma}\mathcal{P}_1^+\varphi = \mathcal{P}^+\mathbf{\Gamma}\langle 0, 0, \varphi\rangle = \mathcal{P}^+\langle 0, 0, i\frac{d\varphi}{d\xi}\rangle = i\frac{d\varphi}{d\xi},$$

where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $\mathcal{U}_t = V_t$, and, hence,

$$\bigcap_{t\geq 0}\mathcal{U}_t\tau_+ = \langle 0, 0, \bigcap_{t\leq 0} V_t L^2(0,\infty)\rangle = \{0\}.$$

(iii) The proof is clear.

Now, we will give a definition and three lemmas to prove the another property of incoming and outgoing subspaces of the unitary group $\{\mathcal{U}_t\}, \ t \in \mathbb{R}$.

Definition 3.3 ([8]). In the Hilbert space H, the linear operator A (with domain $D(A)$) is called *simple (or completely non-self-adjoint)* if there is no invariant subspace $N \subseteq D(A)$ $(N \neq \{0\})$ of the operator A on which the restriction A to N is self-adjoint.

Lemma 3.4. *The operator $\Upsilon_{\alpha_1\alpha_2}$ is simple.*

Proof. Suppose the assertion of the lemma is false. Then we could find a nontrivial subspace $\acute{H} \subset H$ such that $\Upsilon_{\alpha_1\alpha_2}$ induces a self-adjoint operator $\acute{\Upsilon}_{\alpha_1\alpha_2}$ with domain $D(\acute{\Upsilon}_{\alpha_1\alpha_2}) = \acute{H} \cap D\left(\Upsilon_{\alpha_1\alpha_2}\right)$. If $y \in D(\acute{\Upsilon}_{\alpha_1\alpha_2})$, then $y \in D(\acute{\Upsilon}^*_{\alpha_1\alpha_2})$ and

$$y_2\left(0\right) - \alpha_1 y_1\left(0\right) = 0,\ y_2\left(0\right) - \overline{\alpha_1} y_1\left(0\right) = 0,$$

$$y_2\left(aq^{-1}\right) - \alpha_2 y_1\left(a\right) = 0.$$

Since the eigenfunctions of the operator $\Upsilon_{\alpha_1\alpha_2}$ lie in $\acute{H}$ and are eigenfunctions of the operator $\acute{\Upsilon}_{\alpha_1\alpha_2}$, we have $y_2\left(0\right) = y_1\left(0\right) = 0$. By the uniqueness theorem of the Cauchy problem for the equation $\Gamma y = \lambda y$, we obtain $y\left(x,\lambda\right) \equiv 0$. Hence, the resolvent $R_\lambda\left(\Upsilon_{\alpha_1\alpha_2}\right)$ of the operator $\Upsilon_{\alpha_1\alpha_2}$ is a compact operator, and the spectrum of $\Upsilon_{\alpha_1\alpha_2}$ is purely discrete. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator $\acute{\Upsilon}_{\alpha_1\alpha_2}$, we obtain $\acute{H} = \{0\}$. This contradicts our assumption. □

Now, let us define $H_- = \overline{\underset{t\geq 0}{\cup} \mathcal{U}_t \tau_-}$, $H_+ = \overline{\underset{t\leq 0}{\cup} \mathcal{U}_t \tau_+}$. Then, we have a

Lemma 3.5. *The equality $H_- + H_+ = \mathcal{H}$ holds.*

Proof. From Theorem 3.2, we show that the subspace $\acute{\mathcal{H}} = \mathcal{H} \ominus \left(H_- + H_+\right)$ is invariant relative to the group $\{\mathcal{U}_t\}$ and has the form $\acute{\mathcal{H}} = \langle 0, \acute{H}, 0\rangle$, where $\acute{H}$ is a subspace in H. Therefore, if the subspace $\acute{\mathcal{H}}$ (and hence also $\acute{H}$) were nontrivial, then the unitary group $\{\mathcal{U}_t\}$ restricted to this subspace would be a unitary part of the group $\{\mathcal{U}_t\}$, and hence, the restriction $\acute{\Upsilon}_{\alpha_1\alpha_2}$ of $\Upsilon_{\alpha_1\alpha_2}$ to $\acute{H}$ would be a self-adjoint operator in $\acute{H}$. Then, it follows that $\acute{\mathcal{H}} = \{0\}$, since the operator $\Upsilon_{\alpha_1\alpha_2}$ is simple. □

Assume that $\varphi\left(x,\lambda\right) = \begin{pmatrix} \varphi_1\left(x,\lambda\right) \\ \varphi_2\left(x,\lambda\right) \end{pmatrix}$ and $\psi\left(x,\lambda\right) = \begin{pmatrix} \psi_1\left(x,\lambda\right) \\ \psi_2\left(x,\lambda\right) \end{pmatrix}$ are solutions of $\Gamma y = \lambda y$ satisfying the conditions

$$\varphi_1\left(0,\lambda\right) = 0,\ \varphi_2\left(0,\lambda\right) = -1,\ \psi_1\left(0,\lambda\right) = 1,\ \psi_2\left(0,\lambda\right) = 0.$$

The Titchmarsh-Weyl function $m_{\infty,\alpha_2}\left(\lambda\right)$ of the self-adjoint operator $\Upsilon_{\infty,\alpha_2}$ generated by the boundary conditions $y_1\left(0\right) = 0$, $y_2\left(aq^{-1}\right) - \alpha_2 y_1\left(a\right) = 0$ is determined by the condition

$$\psi_2\left(aq^{-1}\right) + m_{\infty,\alpha_2}\left(\lambda\right)\varphi_2\left(aq^{-1}\right) - \alpha_2\left[\psi_1\left(a\right) + m_{\infty,\alpha_2}\left(\lambda\right)\varphi_1\left(a\right)\right] = 0.$$

Hence, we have

$$m_{\infty,\alpha_2}\left(\lambda\right) = -\frac{\psi_2\left(aq^{-1}\right) - \alpha_2\psi_1\left(a\right)}{\varphi_2\left(aq^{-1}\right) - \alpha_2\varphi_1\left(a\right)}. \tag{3.4}$$

Note that the Weyl-Titchmarsh function $m_{\infty,\alpha_2}\left(\lambda\right)$ is a meromorphic function on $\mathbb{C}$, and is a holomorphic function with $Im\lambda \neq 0$, $Im\lambda\, Imm_{\infty,\alpha_2}\left(\lambda\right) > 0$ and $\overline{m_{\infty,\alpha_2}\left(\lambda\right)} = m_{\infty,\alpha_2}\left(\overline{\lambda}\right)$. Then $m_{\infty,\alpha_2}\left(\lambda\right)$ has a countable number of isolated poles on the real axis, these poles are the eigenvalues of the self-adjoint operator $\Upsilon_{\infty,\alpha_2}$, and the operator $\Upsilon_{\infty,\alpha_2}$ (also every self-adjoint extension of the symmetric operator $\Upsilon_{\min}$) has a purely discrete spectrum ([18], [19], [20]).

We set

$$\Omega^-_\lambda\left(x,\xi,\zeta\right) = \langle e^{-i\lambda\xi},\ \frac{1}{m_{\infty,\alpha_2}\left(\lambda\right) - \alpha_1}\gamma\varkappa\left(x,\lambda\right),\ \overline{K_{\alpha_1\alpha_2}\left(\lambda\right)} e^{-i\lambda\zeta}\rangle, \tag{3.5}$$

$$\Omega^+_\lambda\left(x,\xi,\zeta\right) = \langle K_{\alpha_1\alpha_2}\left(\lambda\right) e^{-i\lambda\xi},\ \frac{1}{m_{\infty,\alpha_2}\left(\lambda\right) - \overline{\alpha}_1}\gamma\varkappa\left(x,\lambda\right),\ e^{-i\lambda\zeta}\rangle, \tag{3.6}$$

where

$$\varkappa\left(x,\lambda\right) = \psi\left(x,\lambda\right) + m_{\infty,\alpha_2}\left(\lambda\right)\varphi\left(x,\lambda\right),$$

$$K_{\alpha_1\alpha_2}\left(\lambda\right) = \frac{m_{\infty,\alpha_2}\left(\lambda\right) - \alpha_1}{m_{\infty,\alpha_2}\left(\lambda\right) - \overline{\alpha_1}}. \tag{3.7}$$

It is clear that the vectors $\Omega_{\lambda}^{\mp}(x,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $\Omega_{\lambda}^{\mp}(x,\xi,\zeta)$ satisfies the equation $\mathbf{\Gamma}U=\lambda U$ and the corresponding boundary conditions for the operator $\mathbf{\Gamma}$.

Lemma 3.6. *Let us define the transformation $F_{\mp}: f\to \widetilde{f}_{\mp}(\lambda)$ by*

$$\begin{aligned}(F_{-}f)(\lambda)\ &:\ =\widetilde{f}_{-}(\lambda):=\frac{1}{\sqrt{2\pi}}\left(f,\Omega_{\overline{\lambda}}^{-}\right)_{\mathcal{H}},\\ (F_{+}f)(\lambda)\ &:\ =\widetilde{f}_{+}(\lambda):=\frac{1}{\sqrt{2\pi}}\left(f,\Omega_{\overline{\lambda}}^{+}\right)_{\mathcal{H}},\ f=\langle\varphi_{-},y,\varphi_{+}\rangle\end{aligned}$$

where φ_{-}, φ_{+}, y are smooth, compactly supported functions. Then the transformation $F_{\mp}$ isometrically maps $H_{\mp}$ onto $L^{2}(\mathbb{R})$. For all vectors $f,\ g\in H_{\mp}$, the Parseval equality and the inversion formulae hold:

$$(f,g)_{\mathcal{H}}=(\widetilde{f}_{-},\widetilde{g}_{-})_{L^2}=\int_{-\infty}^{\infty}\widetilde{f}_{-}(\lambda)\overline{\widetilde{g}_{-}(\lambda)}d\lambda,\ f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_{-}(\lambda)\,\Omega_{\overline{\lambda}}^{-}d\lambda, \tag{3.8}$$

$$(f,g)_{\mathcal{H}}=(\widetilde{f}_{+},\widetilde{g}_{+})_{L^2}=\int_{-\infty}^{\infty}\widetilde{f}_{+}(\lambda)\overline{\widetilde{g}_{+}(\lambda)}d\lambda,\ f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_{+}(\lambda)\,\Omega_{\overline{\lambda}}^{+}d\lambda, \tag{3.9}$$

where $\widetilde{f}_{-}(\lambda)=(F_{-}f)(\lambda)$, $\widetilde{g}_{-}(\lambda)=(F_{-}g)(\lambda)$, $\widetilde{f}_{+}(\lambda)=(F_{+}f)(\lambda)$ and $\widetilde{g}_{+}(\lambda)=(F_{+}g)(\lambda)$.

Proof. We will just prove the formula (3.8) since the proof of (3.9) is similar. By Paley-Wiener theorem, we have

$$\widetilde{f}_{-}(\lambda)=\frac{1}{\sqrt{2\pi}}\left(f,\Omega_{\overline{\lambda}}^{-}\right)_{\mathcal{H}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi_{-}(\xi)\,e^{-i\lambda\xi}d\xi\in H_{-}^{2},$$

where $f,g\in\tau_{-}, f=\langle\varphi_{-},0,0\rangle, g=\langle\psi_{-},0,0\rangle$. Using Parseval equality for Fourier integrals, we obtain

$$(f,g)_{\mathcal{H}}=\int_{-\infty}^{\infty}\varphi_{-}(\xi)\overline{\psi_{-}(\xi)}d\xi=\int_{-\infty}^{\infty}\widetilde{f}_{-}(\lambda)\overline{\widetilde{g}_{-}(\lambda)}d\lambda=(F_{-}f,F_{-}g)_{L^2},$$

where $H_{\pm}^{2}$ denote the Hardy classes in $L^{2}(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively. Now, we extend to the Parseval equality to the whole of H_{-}. We consider in H_{-} the dense set of $H_{-}^{\prime}$ of the vectors obtained as follows from the smooth, compactly supported functions in τ_{-} : $f\in H_{-}^{\prime}$ if $f=\mathcal{U}_{t}\ f_{0}$, $f_{0}=\langle\varphi_{-},0,0\rangle$, $\varphi_{-}\in C_{0}^{\infty}(-\infty,0)$, where $T=T_{f}$ is a nonnegative number depending on f. If $f,g\in H_{-}^{\prime}$, then for $T>T_{f}$ and $T>T_{g}$ we have $U_{-T}f, U_{-T}g\in\tau_{-}$, moreover, the first components of these vectors belong to $C_{0}^{\infty}(-\infty,0)$. Therefore, since the operators $\mathcal{U}_{t}$ $(t\in\mathbb{R})$ are unitary, by the equality

$$F_{-}\mathcal{U}_{t}f=(\mathcal{U}_{t}f,\Omega_{\overline{\lambda}}^{-})_{\mathcal{H}}=e^{i\lambda t}\left(f,\Omega_{\overline{\lambda}}^{-}\right)_{\mathcal{H}}=e^{i\lambda t}F_{-}f,$$

we have

$$(f,g)_{\mathcal{H}}=(\mathcal{U}_{-T}\ f,\mathcal{U}_{-T}\ g)_{\mathcal{H}}=(F_{-}\mathcal{U}_{-T}\ f,F_{-}\mathcal{U}_{-T}\ g)_{L^2}$$

and

$$(e^{i\lambda T}F_{-}f,e^{i\lambda T}F_{-}g)_{L^2}=(\widetilde{f},\widetilde{g})_{L^2}. \tag{3.10}$$

By taking the closure (3.10), we obtain the Parseval equality for the space H_{-}. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally

$$F_{-}H_{-}=\overline{\bigcup_{t\geq 0}F_{-}\mathcal{U}_{t}\tau_{-}}=\overline{\bigcup_{t\geq 0}e^{i\lambda t}H_{-}^{2}}=L^{2}(\mathbb{R}),$$

that is F_{-} maps H_{-} onto the whole of $L^{2}(\mathbb{R})$. The lemma is proved. □

It is immediate that the function $K_{\alpha_1\alpha_2}(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (3.7), $|K_{\alpha_1\alpha_2}(\lambda)| \leq 1$ for all $\lambda \in \mathbb{R}$. Hence, it explicitly follows from the formulae for the vectors Ω_λ^- and Ω_λ^+ that

$$\Omega_\lambda^+ = K_{\alpha_1\alpha_2}(\lambda)\,\Omega_\lambda^-. \tag{3.11}$$

Moreover, $H_- = H_+$. Together with Lemma 3.5, this shows that $H_- = H_+ = \mathcal{H}$.

Summarizing, we have been proved the following theorem for the incoming and outgoing subspaces (i.e., for the spaces τ_- and τ_+).

Theorem 3.7. $\overline{\bigcup_{t\geq 0} \mathcal{U}_t\tau_-} = \overline{\bigcup_{t\leq 0} \mathcal{U}_t\tau_+} = \mathcal{H}.$

Thus, the transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace τ_- mapped onto H_-^2 and the operators $\mathcal{U}_t$ are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that F_- is the incoming spectral representation for the group $\{\mathcal{U}_t\}$. Similarly, F_+ is the outgoing spectral representation for the group $\{\mathcal{U}_t\}$. It follows from (3.11) that the passage from the F_- representation of an element $f \in \mathcal{H}$ to its F_+ representation is accomplished as $\widetilde{f}_+(\lambda) = K_{\alpha_1\alpha_2}(\lambda)\,\widetilde{f}_-(\lambda)$. Consequently, according to [5], we have proved the following.

Theorem 3.8. *The scattering function of the group $\{\mathcal{U}_t\}$ is the function $\overline{K_{\alpha_1\alpha_2}}(\lambda)$ i.e., the scattering function of the self-adjoint operator $\mathbf{\Gamma}$ is the function $\overline{K_{\alpha_1\alpha_2}}(\lambda)$.*

4 Functional model of the maximal dissipative $q-$Dirac operator

In this section, we construct a functional model of the maximal dissipative $q-$Dirac operator with the help of incoming spectral representation. Furthermore, we determine characteristic function of this operator and prove that all root vectors of the maximal dissipative $q-$Dirac operator are complete.

Now, we will give some definitions.

Definition 4.1 ([6]). The analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|S(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$

Definition 4.2 ([6]). Let us define $\Psi = H_+^2 \ominus SH_+^2$, where $S(\lambda)$ be an arbitrary nonconstant inner function on the upper half-plane. It is obvious that $\Psi \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators Z_t $(t \geq 0)$ acting in Ψ according to the formula

$$Z_t\varphi = \mathcal{P}\left[e^{i\lambda t}\varphi\right], \varphi = \varphi(\lambda) \in \Psi,$$

where $\mathcal{P}$ is the orthogonal projection from H_+^2 onto Ψ. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t\to+0} (it)^{-1}(Z_t\varphi - \varphi),$$

which T is a maximal dissipative operator acting in Ψ and with the domain $D(T)$ consisting of all functions $\varphi \in \Psi$, such that the limit exists. The operator T is called *a model dissipative operator.*

Recall that this model dissipative operator, which is associated with the names of Lax-Phillips [5], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [6]. The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $V = \langle 0, H, 0\rangle$, so that $\mathcal{H} = \tau_- \oplus V \oplus \tau_+$. It follows from the explicit form of the unitary transformation F_- under the mapping F_-

$$\begin{aligned} \mathcal{H} &\rightarrow L^2(\mathbb{R}), f \rightarrow \widetilde{f}_-(\lambda) = (F_-f)(\lambda), \tau_- \rightarrow H_-^2, \tau_+ \rightarrow K_{\alpha_1\alpha_2}H_+^2, \\ V &\rightarrow H_+^2 \ominus K_{\alpha_1\alpha_2}H_+^2, \mathcal{U}_t \rightarrow (F_-\mathcal{U}_tF_-^{-1}\widetilde{f}_-)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda). \end{aligned} \tag{4.1}$$

The formulas (4.1) show that operator $\Upsilon_{\alpha_1\alpha_2}$ is a unitarily equivalent to the model dissipative operator with the characteristic function $K_{\alpha_1\alpha_2}(\lambda)$. We have thus proved following theorem.

Theorem 4.3. *The characteristic function of the maximal dissipative operator $\Upsilon_{\alpha_1\alpha_2}$ coincides with the function $K_{\alpha_1\alpha_2}(\lambda)$ defined by (3.7).*

Theorem 4.4. *For all the values of α_1 with $\Im\alpha_1 > 0$, except possibly for a single value $\alpha_1 = \alpha_1^0$ and for fixed α_2 ($Im\alpha_2 = 0$ or $\alpha_2 = \infty$), the characteristic function $K_{\alpha_1\alpha_2}(\lambda)$ of the maximal dissipative operator $\Upsilon_{\alpha_1\alpha_2}$ is a Blaschke product. The spectrum of $\Upsilon_{\alpha_1\alpha_2}$ is purely discrete and belongs to the open upper half-plane. The operator $\Upsilon_{\alpha_1\alpha_2}$ $(\alpha_1 \neq \alpha_1^0)$ has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors (or all root vectors) of the operator $\Upsilon_{\alpha_1\alpha_2}$ is complete in the space H.*

Proof. It is obvious that $K_{\alpha_1\alpha_2}(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, we can say

$$K_{\alpha_1\alpha_2}(\lambda) = e^{i\lambda c}B_{\alpha_1\alpha_2}(\lambda), \ c = c(\alpha_1) \geq 0, \tag{4.2}$$

where $B_{\alpha_1\alpha_2}(\lambda)$ is a Blaschke product. Hence, we get

$$|K_{\alpha_1\alpha_2}(\lambda)| \leq e^{-c(\alpha_1)\ Im\lambda}, \ \ Im\lambda \geq 0. \tag{4.3}$$

From (3.7), we obtain

$$m_{\infty,\alpha_2}(\lambda) = \frac{\overline{\alpha_1}K_{\alpha_1\alpha_2}(\lambda) - \alpha_1}{K_{\alpha_1\alpha_2}(\lambda) - 1}. \tag{4.4}$$

If $c(\alpha_1) > 0$, ($Im\alpha_1 > 0$), then (4.3) implies that

$$\lim_{x\to+\infty} K_{\alpha_1\alpha_2}(ix) = 0,$$

and then (4.4) gives us that

$$\lim_{x\to+\infty} m_{\infty,\alpha_2}(ix) = \alpha_1^0.$$

Since $m_{\infty,\alpha_2}(\lambda)$ does not depend on α_1, this implies that $c(\alpha_1)$ can be nonzero at not more than a single point $\alpha_1 = \alpha_1^0$. □

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