Palestine Journal of Mathematics

Vol. 9(1)(2020) , 212–219

EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON LOCAL CONDITIONS

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Communicated by S.P. Goyal

MSC 2010 Classifications: Primary 26A33; Secondary 34A08.

Keywords and phrases: Caputo's fractional derivative, Implicit fractional differential equations, Fractional integral, Non local conditions, Krasnoselskii's theorem.

Abstract The aim of this paper is to establish the existence and uniqueness of solution for nonlinear implicit fractional differential equations with non local conditions. The arguments are based upon the Krasnoselskii's fixed point theorem and contraction mapping principle. Finally, one illustrative example is given to demonstrate the theoretical results.

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. In recent years, fractional differential equations have been proved to be valuable tools for the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, electromagnetics, optics, medicine, economics, astrophysics, chemical engineering, chaotic dynamics, statistical physics and so on. see for example ([1],[3],[5],[6],[11],[12],[13],[14],[16],[18]), and the references therein. Due to its importance in different fields, it is receiving increasing attention and has held a central place in attention researchers and mathematicians.

In recent years, many authors have studied the existence of solutions of fractional differential equations with Caputo fractional derivative. See for example ([2], [4], [8], [10], [17], [19], [20]) and references therein.

In [7], using Banach contraction principle, M. Benchohraa and S. Bouriaha discussed existence and stability results for nonlinear boundary value problem for implicit fractional differential equations (IFDES) of the type:

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)),\ t \in J = [0, T],\ T > 0,\ 0 < \alpha \leq 1$$

$$ay(0) + by(T) = c$$

where $^{c}D^{\alpha}$ is the fractional derivative of caputo, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and a, b, c are real constants with $a + b \neq 0$, and

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)),\ t \in J = [0, T],\ T > 0,\ 0 < \alpha \leq 1$$

$$y(0) + g(y) = y_0$$

where $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and y_0 a real constant.

In [9], using Banach contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type, M. Benchohra and J.E. Lazreg investigated the existence and uniqueness results for nonlinear implicit fractional differential equations (NIFDEs) with boundary conditions of the type:

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)),\ t \in J = [0, T],\ T > 0,\ 1 < \alpha \leq 2$$

$$y(0) = y_0,\ y(T) = y_1$$

where ${}^cD^\alpha$ is the fractional derivative of caputo, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0, y_1 \in \mathbb{R}$.

In [15], using Krasnoselskii's fixed point theorem, G. M. N'Guérékata investigated the existence and uniqueness of solutions to the Cauchy problem for the fractional differential equation with non local conditions of the type :

$${}^cD^\alpha y(t) = f(t, y(t)),\ t \in I = [0, T],\ 0 < \alpha < 1$$

$$y(0) + g(y) = y_0$$

Motivated by the above cited works, in this paper, we investigate the existence and uniqueness results to the following nonlinear implicit fractional differential equation (NIFDE) with nonlocal conditions:

$${}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)),\ t \in J = [0, T],\ T > 0,\ 0 < \alpha < 1 \tag{1.1}$$

$$y(0) + g(y) = y_0 \tag{1.2}$$

where ${}^cD^\alpha$ is the fractional derivative of caputo, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and y_0 a real constant. We take an example of nonlocal conditions as follows:

$$g(y) = \sum_{i=1}^{p} c_i y(t_i) \tag{1.3}$$

where c_i, i=1,.....,p are constants and $0 < t_1 < \ldots\ldots < t_p \leq T$.

2 Preliminaries

In this section, we collect some definitions, notations and results from ([13], [16],[20]) which are used throughout this paper. By C(J, $\mathbb{R}$) we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$||y||_\infty := sup\{|y(t)| : t \in J\}.$$

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L^1([0, T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s)\, ds.$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\, dt$, $\alpha > 0$.

Definition 2.2. For a function h given on the interval [0,T], the caputo fractional-order α of h, is defined by

$${}^cD^\alpha h(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s)\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Lemma 2.4. *Let $\alpha \geq 0$. Then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has a solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots\ldots + c_{n-1} t^{n-1}$, where c_i, i=0,1 ,2 ,......,n are constants and n=[α]+1.

Lemma 2.5. *Let $0 < \alpha \leq 1$ and let $h : [0,T] \to \mathbb{R}$ is a continuous function. Then the linear problem*

$$^cD^{\alpha}y(t) = h(t), t \in J$$

$$y(0) + g(y) = y_0$$

has a unique solution which is given by:

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds.$$

Lemma 2.6. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, then the problem (1.1)–(1.2) is equivalent to the followimg problem*

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(t, y(s), {}^cD^{\alpha}y(s))\,ds.$$

Theorem 2.7. *(Krasnoselskii's fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be two operators such that*
(i) $Ax + By \in M$ whenever $x, y \in M$;
(ii) A is compact and continuous;
(iii) B is a contraction mapping.
Then there exists $z \in M$ such that $z = Az + Bz$.

3 Existence of Solutions

We investigate in our paper the NIFDE (1.1)-(1.2) with the following assumptions:

(H1): $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

(H2): $|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq p(t)\,|x - \bar{x}| + N\,|y - \bar{y}|$, $t \in J$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}$, where $p(t) \in C(J, \mathbb{R}_+)$, $0 < N < 1$.

(H3): $g : C(J,\mathbb{R}) \to \mathbb{R}$ is a continuous function and $||g(x) - g(y)|| \leq b\,||x - y||$, for all $x, y \in C(J,\mathbb{R})$.

(H4): $|f(t,x,y)| \leq q(t)\,|x| + L\,|y|$, $t \in J$ and $x, y \in \mathbb{R}$, where $q(t) \in C(J,\mathbb{R}_+)$ and $0 < L < 1$.

Theorem 3.1. *Under assumptions (H1)-(H3), if $b < \frac{1}{2}$ and $P^* \leq \frac{(1-N)\Gamma(\alpha+1)}{2T^{\alpha}}$, where $P^* = Sup\{p(t) : t \in J\}$, then Eq. (1.1)–(1.2) has a unique solution.*

Proof. Define $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$Ty(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(t, y(s), {}^cD^{\alpha}y(s))\,ds. \tag{3.1}$$

Choose

$$R \geq 2\left(|y_0| + G + \frac{MNT^{\alpha}}{(1-N)\Gamma(\alpha+1)} + \frac{MT^{\alpha}}{\Gamma(\alpha+1)}\right) \tag{3.2}$$

and let

$$M = Sup_{t\in J}\,||f(t,0,0)||\,. \tag{3.3}$$

Then we can show that $T(B_R) \subset B_R$, where $B_R := \{y \in C(J,\mathbb{R}) : ||y||_{\infty} \leq R\}$. So let $y \in B_R$

and set $G = Sup_{y\in C(J,\mathbb{R})} ||g(y)||$. Thus we get

$$\begin{aligned}
|Ty(t)| &\leq |y_0| + G + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(t,y(s),{}^cD^\alpha y(s))\right| ds \\
&\leq |y_0| + G \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(t,y(s),{}^cD^\alpha y(s)) - f(t,0,0)\right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(t,0,0)\right| ds \\
&\leq |y_0| + G + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{p(s)\,|y(s)| + N\,|{}^cD^\alpha y(s)|\}\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} M\,ds
\end{aligned}$$

Note that, for any $t \in J$

$$\begin{aligned}
|{}^cD^\alpha y(t)| &\leq |f(t,y(t),{}^cD^\alpha y(t)) - f(t,0,0)| + |f(t,0,0)| \\
&\leq p(t)\,|y(t)| + N\,|{}^cD^\alpha y(t)| + M
\end{aligned}$$

This gives

$$|{}^cD^\alpha y(t)| \leq \frac{p(t)}{(1-N)}\,|y(t)| + \frac{M}{(1-N)} \tag{3.4}$$

Using (3.4), we have
$|Ty(t)|$

$$\begin{aligned}
&\leq |y_0| + G + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{p(s)\,|y(s)| + \frac{Np(s)}{(1-N)}\,|y(s)| \\
&\quad + \frac{MN}{(1-N)}\}ds + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq |y_0| + G + \frac{P^*R}{(1-N)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\quad + \frac{MN}{(1-N)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \left(|y_0| + G + \frac{MNT^\alpha}{(1-N)\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)}\right) + \frac{P^*RT^\alpha}{(1-N)\Gamma(\alpha+1)} \\
&\leq R
\end{aligned}$$

by the choice of P^* and R.
Now take $x, y \in C(J,\mathbb{R})$ and for any $t \in J$, then we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq b\,||x-y|| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{p(t)\,|x(s)-y(s)| \\
&\quad + N\,|{}^cD^\alpha x(s) - {}^cD^\alpha y(s)|\}\,ds
\end{aligned}$$

Note that, for any $t \in J$

$$\begin{aligned}
|{}^cD^\alpha x(t) - {}^cD^\alpha y(t))| &\leq |f(t,x(t),{}^cD^\alpha x(t)) - f(t,y(t),{}^cD^\alpha y(t)))| \\
&\leq p(t)\,|x(t)-y(t)| + N\,|{}^cD^\alpha x(t) - {}^cD^\alpha x(t)|
\end{aligned}$$

This gives

$$|{}^cD^\alpha x(t) - {}^cD^\alpha y(t)| \leq \frac{p(t)}{(1-N)}\,|x(t)-y(t)| \tag{3.5}$$

Using (3.5), we have

$$||Tx(t) - Ty(t)||_\infty \leq \Omega_{P^*.T.\alpha} ||x - y||_\infty ,$$

where $\Omega_{P^*.T.\alpha} := b + \frac{P^* T^\alpha}{(1-N)\Gamma(\alpha+1)}$ depends only on parameters of problem. And since $\Omega_{P^*.T.\alpha} < 1$, by contraction mapping principle, T has a unique fixed point which is a unique solution of the problem (1.1)–(1.2). □

Our next result is based on Krasnoselskii's fixed point theorem.

Theorem 3.2. *Assume (H1),(H3) with $b < 1$, (H4) and $Q^* \leq \frac{(1-L)\Gamma(\alpha+1)}{2T^\alpha}$, where $Q^* = Sup\{q(t) : t \in J\}$. Then Eq. (1.1)-(1.2) has at least one solution on J.*

Proof. Choose $R \geq 2(|y_0| + G)$ and consider $B_R := \{y \in C(J, R) : ||y||_\infty \leq R\}$. Now we define operators A, B on B_R as follows:

$$Ax(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t, x(s), {}^cD^\alpha x(s))\, ds,$$

and

$$Bx(t) = y_0 - g(x).$$

Let $x, y, \in B_R, t \in J$ and set $G = Sup_{y \in C(J,\mathbb{R})} ||g(y)||$, then we have

$$\begin{aligned} |Ax(t) + By(t)| &\leq |y_0| + G + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(t, x(s), {}^cD^\alpha x(s))|\, ds \\ &\leq |y_0| + G + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \{q(s)\,|x(s)| + L\,|{}^cD^\alpha x(s)|\}\, ds \end{aligned}$$

Note that, for any $t \in J$

$$\begin{aligned} |{}^cD^\alpha x(t)| &\leq |f(t, x(t), {}^cD^\alpha x(t))| \\ &\leq q(t)\,|x(t)| + L\,|{}^cD^\alpha x(t)| \end{aligned}$$

This gives

$$|{}^cD^\alpha x(t)| \leq \frac{q(t)}{(1-L)} |x(t)| \tag{3.6}$$

Using (3.6), we have

$$\begin{aligned} |Ax(t) + By(t)| &\leq |y_0| + G + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \{\frac{q(s)}{(1-L)} |x(s)|\}\, ds \\ &\leq |y_0| + G + \frac{Q^* R T^\alpha}{(1-L)\Gamma(\alpha+1)} \\ &\leq R \end{aligned}$$

Thus

$$||Ax(t) + By(t)||_\infty \leq R.$$

This gives $Ax + By \in B_R$. It is clear that from (H3), B is a contration mapping for $b < 1$. Since x is continuous, then $Ax(t)$ is continuous in view of (H1). Note that A is uniformly bounded on B_R. This follows from the inequality

$$||Ax(t)||_\infty \leq \frac{Q^* R T^\alpha}{(1-L)\Gamma(\alpha+1)}$$

Now let's prove $Ax(t)$ is equicontinuous.
Let $t_1, t_2 \in J$ and $x \in B_R$

$$
\begin{aligned}
&|Ax(t_1) - Ax(t_2)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} \left|f(t, x(s), {}^cD^\alpha x(s))\right| ds \\
&\quad -\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left|f(t, x(s), {}^cD^\alpha x(s))\right| ds \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\}\{q(s)\,|x(s)| + L\,|{}^cD^\alpha x(s)|\}ds \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\{q(s)\,|x(s)| + L\,|{}^cD^\alpha x(s)|\}ds
\end{aligned}
$$

Using (3.6), we get

$$
\begin{aligned}
|Ax(t_1) - Ax(t_2)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} \frac{q(s)}{(1-L)} \,|x(s)|\, ds \\
&\quad +\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \frac{q(s)}{(1-L)} \,|x(s)|\; ds \\
&\leq \frac{Q^* R}{(1-L)\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\}\, ds \\
&\quad +\frac{Q^* R}{(1-L)\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds \\
&\leq \frac{Q^* R}{(1-L)\Gamma(\alpha+1)} \{2(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha\}.
\end{aligned}
$$

which does not depend on x. So $A(B_R)$ is relatively compact. By Arzela-Ascoil Theorem, A is compact. As a consequence of Krasnoselskii's theorem, We conclude that Eq. (1.1)–(1.2) has at least one solution. □

4 Illustrative Example

Consider the boundary value problem:

$$
{}^cD^{1/2}y(t) = \frac{e^{-t}}{(9+e^t)}\left[\frac{|y(t)|}{1+|y(t)|}\right] - \frac{1}{2}\left[\frac{\left|{}^cD^{1/2}y(t)\right|}{1+\left|{}^cD^{1/2}y(t)\right|}\right], t \in J = [0,1] \tag{4.1}
$$

$$
y(0) + \sum_{i=1}^{n} c_i y(t_i) = 1 \tag{4.2}
$$

where $0 < t_1 <$ .... $< t_n < 1$ and c_i, i=1,.....,n are positive constants with

$$
\sum_{i=1}^{n} c_i \leq \frac{1}{3} \tag{4.3}
$$

Set

$$
f(t, x, y) = \frac{e^{-t}}{(9+e^t)}\left[\frac{x}{1+x}\right] - \frac{1}{2}\left[\frac{y}{1+y}\right],\ t \in [0,1], x, y \in [0, +\infty).
$$

Clearly f is continuous. For each $x, \bar{x}, y, \bar{y} \in R$ and $t \in [0,1]$:

$$
\begin{aligned}
|f(t,x,y) - f(t,\bar{x},\bar{y})| &\leq \frac{e^{-t}}{(9+e^t)} |x - \bar{x}| + \frac{1}{2} |y - \bar{y}| \\
&\leq \frac{1}{10} |x - \bar{x}| + \frac{1}{2} |y - \bar{y}|\,.
\end{aligned}
$$

we see that $p(t) = \frac{e^{-t}}{(9+e^t)} : [0, 1] \to (0, \infty)$ is continuous function.

Also, we have

$$\begin{aligned} |g(x) - g(\bar{x})| &\leq \left| \sum_{i=1}^{n} c_i x - \sum_{i=1}^{n} c_i \bar{x} \right| \\ &\leq \sum_{i=1}^{n} c_i \, |x - \bar{x}| \\ &\leq \frac{1}{3} \, |x - \bar{x}| \, . \end{aligned}$$

Hence condition (H2) and (H3) is satisfied with $P^* = \frac{1}{10}$, $N = \frac{1}{2}$ and $b = \frac{1}{3}$. We have

$$\frac{(1-N)\Gamma(\alpha+1)}{2T^\alpha} = \frac{\sqrt{\pi}}{8} > P^* = \frac{1}{10}$$

and

$$b < \frac{1}{2}.$$

Also for each $x, y \in R$ and $t \in [0, 1]$:

$$\begin{aligned} |f(t, x, y)| &\leq \frac{e^{-t}}{(9 + e^t)} \, |x| + \frac{1}{2} \, |y| \\ &\leq \frac{1}{10} \, |x| + \frac{1}{2} \, |y| \, . \end{aligned}$$

we see that $q(t) = \frac{e^{-t}}{(9+e^t)} : [0, 1] \to (0, \infty)$ is continuous function.
Hence the condition (H3), (H4) are satisfied with $Q^* = \frac{1}{10}, L = \frac{1}{2}$ and $b = \frac{1}{3}$.
We have

$$\frac{(1-L)\Gamma(\alpha+1)}{2T^\alpha} = \frac{\sqrt{\pi}}{8} > Q^* = \frac{1}{10}$$

and

$$b < 1.$$

It follows from Theorem (3.1) and (3.2) the problem (4.1)–(4.2) has unique solution on [0,1].

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Received: October 18, 2017.
Accepted: March 27, 2018