**Palestine Journal of Mathematics**

**Vol. 9(1)(2020)** , 220–230 

# The Characterization properties and Basic Hypergeometric functions of $(\mathcal{P}, \mathcal{Q})$-analogue

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Communicated by P. K. Banerji

MSC 2010 Classifications: 11B83; 33D70; 33D80; 81S40.

Keywords and phrases: $(\mathcal{P}, \mathcal{Q})$-shifted factorials, negative $(\mathcal{P}, \mathcal{Q})$-shifted factorial, $(\mathcal{P}, \mathcal{Q})$-hypergeometric functions.

**Abstract** In this paper, we acquaint some new results on $(\mathcal{P}, \mathcal{Q})$-analogue by giving definition of $(\mathcal{P}, \mathcal{Q})$-shifted factorials. And, some new relations on negative shifted factorial. Also, we prove some new relations for $(\mathcal{P}, \mathcal{Q})$-shifted factorials and $(\mathcal{P}, \mathcal{Q})$-hypergeometric functions.

## 1 Introduction

The $(\mathcal{P}, \mathcal{Q})$-analysis or post quantum calculus was discover at the last decade. Many mathematicians and physicists have widely developed the theory of $(\mathcal{P}, \mathcal{Q})$-numbers, along the traditional lines of classical and quantum calculus. Burban and Klimyk et al. [1] presented the $(\mathcal{P}, \mathcal{Q})$-Derivative , $(\mathcal{P}, \mathcal{Q})$-Anti-Derivative and $(\mathcal{P}, \mathcal{Q})$ hypergeometric functions related to quantum groups. Sadjang at al. [8, 9, 10] inspired by this produced the fundamental theorem of $(\mathcal{P}, \mathcal{Q})$-calculus, the $(\mathcal{P}, \mathcal{Q})$-Gamma and the $(\mathcal{P}, \mathcal{Q})$-Beta functions. Duran at al. [2, 3, 4] establishment a new class of Bernoulli, Euler and Genocchi polynomials founded on $(\mathcal{P}, \mathcal{Q})$-calculus and checked their many properties. The $(\mathcal{P}, \mathcal{Q})$-Anti-Derivative $[n]_{(\mathcal{P},\mathcal{Q})}$ are specified as (see [8, 6])

$$\begin{aligned} [n]_{\mathcal{P},\mathcal{Q}} &= \mathcal{P}^{n-1} + \mathcal{P}^{n-2}\mathcal{Q} + \mathcal{P}^{n-3}\mathcal{Q}^2 + \cdots + \mathcal{P}\mathcal{Q}^{n-2} + \mathcal{Q}^{n-1} \\ &= \begin{cases} \frac{\mathcal{P}^n - \mathcal{Q}^n}{\mathcal{P}-\mathcal{Q}}, & \mathcal{P} \neq \mathcal{Q} \neq 1 \\ n\mathcal{P}^{n-1}, & \mathcal{P} = \mathcal{Q} \neq 1 \\ [n]_{\mathcal{Q}}, & \mathcal{P} = 1 \\ n, & \mathcal{P} = \mathcal{Q} = 1, \end{cases} \end{aligned} \tag{1.1}$$

or

$$[n]_{\mathcal{P},\mathcal{Q}} = \frac{\mathcal{P}^n - \mathcal{Q}^n}{\mathcal{P} - \mathcal{Q}}, \qquad n = 1, 2, 3, \cdots, \qquad 0 < \mathcal{Q} < \mathcal{P} \leq 1, \tag{1.2}$$

$$[0]_{\mathcal{P},\mathcal{Q}} = 0 \qquad \text{and} \qquad [-1]_{\mathcal{P},\mathcal{Q}} = \frac{-1}{\mathcal{P}\mathcal{Q}}.$$

The $(\mathcal{P}, \mathcal{Q})$-basic number is a generalization of the $\mathcal{Q}$-number, that is

$$\lim_{\mathcal{P}\to 1} [n]_{\mathcal{P},\mathcal{Q}} = [n]_{\mathcal{Q}}.$$

The definition for $(\mathcal{P}, \mathcal{Q})$-factorial is explain in [7, 8]

$$[n]_{\mathcal{P},\mathcal{Q}}! = \prod_{m=1}^{n} [m]_{\mathcal{P},\mathcal{Q}}!, \qquad n \geq 1, \qquad [0]_{\mathcal{P},\mathcal{Q}}! = 1. \tag{1.3}$$

The $(\mathcal{P}, \mathcal{Q})$-analogues of the binomial coefficients are given as

$$\binom{n}{m}_{\mathcal{P},\mathcal{Q}} = \frac{[n]_{\mathcal{P},\mathcal{Q}}!}{[m]_{\mathcal{P},\mathcal{Q}}![n-m]_{\mathcal{P},\mathcal{Q}}!}, \qquad 0 \leq m \leq n. \tag{1.4}$$

Noted that, as $\mathcal{P} \to 1$, the $(\mathcal{P}, \mathcal{Q})$-binomial coefficients is similar to the $\mathcal{Q}$-binomial coefficients. The $(\mathcal{P}, \mathcal{Q})$-powers are introduce as

$$\left((\mathcal{A}, \mathcal{B}); (\mathcal{P}, \mathcal{Q})\right)_0 = 1$$

and

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k=\prod_{j=1}^{k}\big(\mathcal{A}\mathcal{P}^{j-1}-\mathcal{B}\mathcal{Q}^{j-1}\big),\qquad 0<\mathcal{Q}<\mathcal{P}\le 1. \tag{1.5}$$

The symbols $\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k$ are called $(\mathcal{P},\mathcal{Q})$-shifted factorials.
The $(\mathcal{P},\mathcal{Q})$-derivative operator $\mathcal{D}_{\mathcal{P},\mathcal{Q}}$ is defined by

$$\mathcal{D}_{\mathcal{P},\mathcal{Q}}f(z):=\begin{cases}\frac{f(\mathcal{P}z)-f(\mathcal{Q}z)}{(\mathcal{P}-\mathcal{Q})z}, & \text{when } z\neq 0\\ f'(0), & \text{when } z=0.\end{cases} \tag{1.6}$$

Further we define

$$\mathcal{D}^0_{\mathcal{P},\mathcal{Q}}f:=f \qquad \text{and} \qquad \mathcal{D}^n_{\mathcal{P},\mathcal{Q}}f:=\mathcal{D}_{\mathcal{P},\mathcal{Q}}\big(\mathcal{D}^{n-1}_{\mathcal{P},\mathcal{Q}}f\big),\qquad n=1,2,3,\cdots,$$

it is not very difficult to see that

$$\lim_{\mathcal{P}\to 1}\mathcal{D}_{\mathcal{P},\mathcal{Q}}f(z)=\mathcal{D}_{\mathcal{Q}}f(z),$$

if the function is differentiable at $z$. Further $\mathcal{D}_{\mathcal{P},\mathcal{Q}}$ is a linear operator and satisfies the following property

$$\mathcal{D}_{\mathcal{P},\mathcal{Q}}\big(f(z)g(z)\big)=f(\mathcal{P}z)\mathcal{D}_{\mathcal{P},\mathcal{Q}}g(z)+g(\mathcal{Q}z)\mathcal{D}_{\mathcal{P},\mathcal{Q}}f(z) \tag{1.7}$$

or

$$\mathcal{D}_{\mathcal{P},\mathcal{Q}}\big(f(z)g(z)\big)=g(\mathcal{P}z)\mathcal{D}_{\mathcal{P},\mathcal{Q}}f(z)+f(\mathcal{Q}z)\mathcal{D}_{\mathcal{P},\mathcal{Q}}g(z), \tag{1.8}$$

which is often referred to as the $(\mathcal{P},\mathcal{Q})$-product rule. This can be generalized to a $(\mathcal{P},\mathcal{Q})$-analogue of Leibniz's rule

$$\mathcal{D}^n_{\mathcal{P},\mathcal{Q}}\big(f(z)g(z)\big)=\sum_{m=0}^{n}\binom{n}{m}_{\mathcal{P},\mathcal{Q}}\Big(\mathcal{D}^{n-m}_{\mathcal{P},\mathcal{Q}}f\Big)\big(\mathcal{P}^{n-m}\mathcal{Q}^m z\big)\Big(\mathcal{D}^m_{\mathcal{P},\mathcal{Q}}g\Big)(z),\quad n=0,1,2,\cdots. \tag{1.9}$$

The following definition of $(\mathcal{P},\mathcal{Q})$-integral due to [8] is

$$\int_0^{\mathcal{A}}f(t)d_{\mathcal{P},\mathcal{Q}}t=(\mathcal{P}-\mathcal{Q})\mathcal{A}\sum_{m\ge 0}\frac{\mathcal{P}^m}{\mathcal{Q}^{m+1}}f\Big(\frac{\mathcal{P}^m}{\mathcal{Q}^{m+1}}t\Big) \tag{1.10}$$

with

$$\int_{\mathcal{A}}^{\mathcal{B}}f(t)d_{\mathcal{P},\mathcal{Q}}t=\int_0^{\mathcal{B}}f(t)d_{\mathcal{P},\mathcal{Q}}t-\int_0^{\mathcal{A}}f(t)d_{\mathcal{P},\mathcal{Q}}t$$

For detailed studies on $(\mathcal{P},\mathcal{Q})$-calculus, one can look at [8, 9, 10] and references therein.

## 2 The $(\mathcal{P},\mathcal{Q})$-Shifted Factorial

The symbols $\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k$ are called $(\mathcal{P},\mathcal{Q})$-shifted factorials for negative subscripts we define

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{-k}=\frac{1}{\prod_{j=1}^{k}\big(\mathcal{A}\mathcal{P}^{-j}-\mathcal{B}\mathcal{Q}^{-j}\big)},\quad \mathcal{A}\neq\mathcal{P},\mathcal{P}^2,\mathcal{P}^3,\cdots,\mathcal{P}^k, \tag{2.1}$$
$$\mathcal{B}\neq\mathcal{Q},\mathcal{Q}^2,\mathcal{Q}^3,\cdots,\mathcal{Q}^k,\quad k=1,2,3,\cdots.$$

**Remark 2.1.** If we take $\mathcal{A}=1$ and $\mathcal{P}=1$ we obtain $\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n=\prod_{i=1}^{k}(1-\mathcal{B}\mathcal{Q}^{i-1})=(\mathcal{B};\mathcal{Q})_k$ that is $\mathcal{Q}$-analogue.

**Proposition 2.2.** *The $(\mathcal{P},\mathcal{Q})$-shifted factorials for negative subscripts satisfies the relation*

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{-n}=\Big[\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_n\Big]^{-1}=\frac{(\mathcal{P}\mathcal{Q})^{\binom{n+1}{2}}}{\big((\mathcal{A}\mathcal{Q},\mathcal{B}\mathcal{P});(\mathcal{Q},\mathcal{P})\big)_n},\quad n=0,1,2\cdots. \tag{2.2}$$

*Proof.* From the definition of $(\mathcal{P},\mathcal{Q})$-shifted factorials for negative subscripts, we have

$$\begin{aligned}
\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{-n} &= \Big(\prod_{j=1}^{n}\big(\mathcal{A}\mathcal{P}^{-j}-b\mathcal{Q}^{-j}\big)\Big)^{-1}\\
&= \Big((\mathcal{A}\mathcal{P}^{-1}-\mathcal{B}\mathcal{Q}^{-1})(\mathcal{A}\mathcal{P}^{-2}-\mathcal{B}\mathcal{Q}^{-2})\cdots(\mathcal{A}\mathcal{P}^{-n}-\mathcal{B}\mathcal{Q}^{-n})\Big)^{-1}\\
&= \Big[\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_{n}\Big]^{-1}\\
&= \frac{1}{(\frac{\mathcal{A}}{\mathcal{P}}-\frac{\mathcal{B}}{\mathcal{Q}})(\frac{\mathcal{A}}{\mathcal{P}^2}-\frac{\mathcal{B}}{\mathcal{Q}^2})\cdots(\frac{\mathcal{A}}{\mathcal{P}^n}-\frac{\mathcal{B}}{\mathcal{Q}^n})}\\
&= \frac{(\mathcal{P}\mathcal{Q})^{\binom{n+1}{2}}}{(\mathcal{A}\mathcal{Q}-\mathcal{B}\mathcal{P})(\mathcal{A}\mathcal{Q}^2-\mathcal{B}\mathcal{P}^2)\cdots(\mathcal{A}\mathcal{Q}^n-\mathcal{B}\mathcal{P}^n)}\\
&= \frac{(\mathcal{P}\mathcal{Q})^{\binom{n+1}{2}}}{\big((\mathcal{A}\mathcal{Q},\mathcal{B}\mathcal{P});(\mathcal{Q},\mathcal{P})\big)_{n}}.
\end{aligned}$$

This proves (2.2). □

**Proposition 2.3.** *If we replace $\mathcal{P}$ by $\mathcal{P}^{-1}$ and $\mathcal{Q}$ by $\mathcal{Q}^{-1}$ the $(\mathcal{P},\mathcal{Q})$-shifted factorials satisfies the relation*

$$\Big((\mathcal{A},\mathcal{B});\Big(\frac{1}{\mathcal{P}},\frac{1}{\mathcal{Q}}\Big)\Big)_{n} = \frac{\big((\mathcal{A},\mathcal{B});(\mathcal{Q},\mathcal{P})\big)_{n}}{(\mathcal{P}\mathcal{Q})^{\binom{n}{2}}}. \tag{2.3}$$

*Proof.* By using the definition (1.5) we obtain

$$\begin{aligned}
\Big((\mathcal{A},\mathcal{B});\Big(\frac{1}{\mathcal{P}},\frac{1}{\mathcal{Q}}\Big)\Big)_{n} &= \prod_{i=1}^{n}\big(\mathcal{A}\mathcal{P}^{-i+1}-\mathcal{B}\mathcal{Q}^{-i+1}\big)\\
&= \frac{(\mathcal{A}-\mathcal{B})(\mathcal{A}\mathcal{Q}-\mathcal{B}\mathcal{P})(\mathcal{A}\mathcal{Q}^2-\mathcal{B}\mathcal{P}^2)\cdots(\mathcal{A}\mathcal{Q}^{n-1}-\mathcal{B}\mathcal{P}^{n-1})}{(\mathcal{P}\mathcal{Q})^{\binom{n}{2}}}\\
&= \frac{\big((\mathcal{A},\mathcal{B});(\mathcal{Q},\mathcal{P})\big)_{n}}{(\mathcal{P}\mathcal{Q})^{\binom{n}{2}}}. \qquad (2.4)
\end{aligned}$$

Hence the proof of proposition (2.3). □

**Proposition 2.4.** *The $(\mathcal{P},\mathcal{Q})$-shifted factorials satisfies the relation*

$$\big((r\mathcal{A},r\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n} = \prod_{i=1}^{n}\big(r\mathcal{A}\mathcal{P}^{i-1}-r\mathcal{B}\mathcal{Q}^{i-1}\big) = r^{n}\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n} \tag{2.5}$$

*Proof.* The proof of above proposition is simple. □

**Proposition 2.5.** *The relationship between $\mathcal{Q}$ and $(\mathcal{P},\mathcal{Q})$-analogue, are given by the*

$$\Big(\frac{\mathcal{B}}{\mathcal{A}};\frac{\mathcal{Q}}{\mathcal{P}}\Big)_{n} = \frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n}}{\mathcal{A}^{n}\mathcal{P}^{\binom{n}{2}}} \tag{2.6}$$

*Proof.* By using the definition of $\mathcal{Q}$-analogue [5, page 6]

$$\begin{aligned}
\Big(\frac{\mathcal{B}}{\mathcal{A}};\frac{\mathcal{Q}}{\mathcal{P}}\Big)_{n} &= \prod_{i=1}^{n}\Big(1-\frac{\mathcal{B}}{\mathcal{A}}\Big(\frac{\mathcal{Q}}{\mathcal{P}}\Big)^{i-1}\Big)\\
&= \prod_{i=1}^{n}\frac{(\mathcal{A}\mathcal{P}^{i-1}-\mathcal{B}\mathcal{Q}^{i-1})}{\mathcal{A}\mathcal{P}^{i-1}}\\
&= \frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n}}{\mathcal{A}^{n}\mathcal{P}^{\binom{n}{2}}}.
\end{aligned}$$

□

**Definition 2.6.** Here, we can also define

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_\infty=\prod_{j=1}^{\infty}\big(\mathcal{A}\mathcal{P}^{j-1}-\mathcal{B}\mathcal{Q}^{j-1}\big),\qquad 0<\mathcal{Q}<\mathcal{P}\le 1. \tag{2.7}$$

This implies that

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n=\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{A}\mathcal{P}^n,\mathcal{B}\mathcal{Q}^n);(\mathcal{P},\mathcal{Q})\big)_\infty},\qquad 0<\mathcal{Q}<\mathcal{P}\le 1, \tag{2.8}$$

and for any complex number $\lambda$

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_\lambda=\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{A}\mathcal{P}^\lambda,\mathcal{B}\mathcal{Q}^\lambda);(\mathcal{P},\mathcal{Q})\big)_\infty},\qquad 0<\mathcal{Q}<\mathcal{P}\le 1. \tag{2.9}$$

We list a number of transformation formulas for the $(\mathcal{P},\mathcal{Q})$-shifted factorials, where $k$ and $n$ are non-negative integers:

**Proposition 2.7.**

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n+k}=\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A}\mathcal{P}^n,\mathcal{B}\mathcal{Q}^n);(\mathcal{P},\mathcal{Q})\big)_k. \tag{2.10}$$

*Proof.* The proof is easy. □

**Proposition 2.8.**

$$\big((\mathcal{A}\mathcal{P}^n,\mathcal{B}\mathcal{Q}^n);(\mathcal{P},\mathcal{Q})\big)_k\Big[\big((\mathcal{A}\mathcal{P}^k,\mathcal{B}\mathcal{Q}^k);(\mathcal{P},\mathcal{Q})\big)_n\Big]^{-1}=\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k\Big[\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\Big]^{-1} \tag{2.11}$$

*Proof.*

$$\begin{aligned}\frac{\big((\mathcal{A}\mathcal{P}^n,\mathcal{B}\mathcal{Q}^n);(\mathcal{P},\mathcal{Q})\big)_k}{\big((\mathcal{A}\mathcal{P}^k,\mathcal{B}\mathcal{Q}^k);(\mathcal{P},\mathcal{Q})\big)_n}&=\frac{\prod_{j=1}^{k}\big(\mathcal{A}\mathcal{P}^n\mathcal{P}^{j-1}-\mathcal{B}\mathcal{Q}^n\mathcal{Q}^{j-1}\big)}{\prod_{j=1}^{n}\big(\mathcal{A}\mathcal{P}^k\mathcal{P}^{j-1}-\mathcal{B}\mathcal{Q}^k\mathcal{Q}^{j-1}\big)}\\&=\frac{(\mathcal{A}\mathcal{P}^n-\mathcal{B}\mathcal{Q}^n)(\mathcal{A}\mathcal{P}^{n+1}-\mathcal{B}\mathcal{Q}^{n+1})\cdots(\mathcal{A}\mathcal{P}^{n+k-1}-\mathcal{B}\mathcal{Q}^{n+k-1})}{(\mathcal{A}\mathcal{P}^k-\mathcal{B}\mathcal{Q}^k)(\mathcal{A}\mathcal{P}^{k+1}-\mathcal{B}\mathcal{Q}^{k+1})\cdots(\mathcal{A}\mathcal{P}^{k+n-1}-\mathcal{B}\mathcal{Q}^{k+n-1})}\\&=\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k}{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n}.\end{aligned}$$

Therefore, (2.11) is true for any non-negative integers $n$ and $k$. □

**Proposition 2.9.**

$$\big((\mathcal{A}\mathcal{P}^k,\mathcal{B}\mathcal{Q}^k);(\mathcal{P},\mathcal{Q})\big)_{n-k}=\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\Big[\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k\Big]^{-1}. \tag{2.12}$$

*Proof.* The proof follows easily by definition of $(\mathcal{P},\mathcal{Q})$-shifted factorials. □

**Proposition 2.10.**

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n=(-\mathcal{A}\mathcal{B})^n(\mathcal{P}\mathcal{Q})^{\binom{n}{2}}\big((\mathcal{A}^{-1}\mathcal{P}^{1-n},\mathcal{B}^{-1}\mathcal{Q}^{1-n});(\mathcal{P},\mathcal{Q})\big)_n. \tag{2.13}$$

*Proof.*

$$\begin{aligned}\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n&=\prod_{j=1}^{n}\big(\mathcal{A}\mathcal{P}^{j-1}-\mathcal{B}\mathcal{Q}^{j-1}\big)\\&=(\mathcal{A}-\mathcal{B})(\mathcal{A}\mathcal{P}-\mathcal{B}\mathcal{Q})\cdots(\mathcal{A}\mathcal{P}^{n-1}-\mathcal{B}\mathcal{Q}^{n-1})\\&=(-\mathcal{A}\mathcal{B})^n(\mathcal{P}\mathcal{Q})^{\binom{n}{2}}(\mathcal{A}^{-1}-\mathcal{B}^{-1})(\mathcal{A}^{-1}\mathcal{P}^{-1}-\mathcal{B}^{-1}\mathcal{Q}^{-1})\cdots(\mathcal{A}^{-1}\mathcal{P}^{1-n}-\mathcal{B}^{-1}\mathcal{Q}^{1-n})\\&=(-\mathcal{A}\mathcal{B})^n(\mathcal{P}\mathcal{Q})^{\binom{n}{2}}\big((\mathcal{A}^{-1}\mathcal{P}^{1-n},\mathcal{B}^{-1}\mathcal{Q}^{1-n});(\mathcal{P},\mathcal{Q})\big)_n.\end{aligned}$$

□

**Proposition 2.11.**

$$\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_n=(-\mathcal{A}\mathcal{B})^n(\mathcal{P}\mathcal{Q})^{-n-\binom{n}{2}}\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n,\qquad \mathcal{A}\neq 0,\mathcal{B}\neq 0. \tag{2.14}$$

*Proof.*

$$\begin{aligned}\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_n&=\prod_{j=1}^{n}\big(\mathcal{A}\mathcal{P}^{-n}\mathcal{P}^{j-1}-\mathcal{B}\mathcal{Q}^{-n}\mathcal{Q}^{j-1}\big)\\&=(\mathcal{A}\mathcal{P}^{-n}-\mathcal{B}\mathcal{Q}^{-n})(\mathcal{A}\mathcal{P}^{-n+1}-\mathcal{B}\mathcal{Q}^{-n+1})\cdots(\mathcal{A}\mathcal{P}^{-1}-\mathcal{B}\mathcal{Q}^{-1})\\&=(-\mathcal{A}\mathcal{B})^n(\mathcal{P}\mathcal{Q})^{-n-\binom{n}{2}}(\mathcal{A}^{-1}\mathcal{P}-\mathcal{B}^{-1}\mathcal{Q})\cdots(\mathcal{A}^{-1}\mathcal{P}^n-\mathcal{B}^{-1}\mathcal{Q}^n)\\&=(-\mathcal{A}\mathcal{B})^n(\mathcal{P}\mathcal{Q})^{-n-\binom{n}{2}}\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n,\qquad \mathcal{A}\neq 0,\mathcal{B}\neq 0.\end{aligned}$$

□

**Proposition 2.12.**

$$\frac{\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{C}\mathcal{P}^{-n},\mathcal{D}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_n}=\frac{\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{C}^{-1}\mathcal{P},\mathcal{D}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}\left(\frac{\mathcal{A}\mathcal{B}}{\mathcal{C}\mathcal{D}}\right)^n,\quad \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\neq 0 \tag{2.15}$$

*Proof.* The proof follows easily by taking ratio of two $(\mathcal{P},\mathcal{Q})$-shifted factorials. □

**Proposition 2.13.**

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n-k}=\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A}^{-1}\mathcal{P}^{1-n},\mathcal{B}^{-1}\mathcal{Q}^{1-n});(\mathcal{P},\mathcal{Q})\big)_k}\left(\frac{-\mathcal{P}\mathcal{Q}}{\mathcal{A}\mathcal{B}}\right)^k(\mathcal{P}\mathcal{Q})^{\binom{k}{2}-nk}, \tag{2.16}$$

$$\mathcal{A},\mathcal{B}\neq 0,\quad k=0,1,2,\cdots n.$$

*Proof.*

$$\begin{aligned}\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{n-k}&=(\mathcal{A}-\mathcal{B})(\mathcal{A}\mathcal{P}-\mathcal{B}\mathcal{Q})(\mathcal{A}\mathcal{P}^2-\mathcal{B}\mathcal{Q}^2)\cdots(\mathcal{A}\mathcal{P}^{n-k-1}-\mathcal{B}\mathcal{Q}^{n-k-1})\\&=\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n}{(\mathcal{A}\mathcal{P}^{n-k}-\mathcal{B}\mathcal{Q}^{n-k})(\mathcal{A}\mathcal{P}^{n-k+1}-\mathcal{B}\mathcal{Q}^{n-k+1})\cdots(\mathcal{A}\mathcal{P}^{n-1}-\mathcal{B}\mathcal{Q}^{n-1})}\\&=\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A}^{-1}\mathcal{P}^{1-n},\mathcal{B}^{-1}\mathcal{Q}^{1-n});(\mathcal{P},\mathcal{Q})\big)_k}\left(\frac{-\mathcal{P}\mathcal{Q}}{\mathcal{A}\mathcal{B}}\right)^k(\mathcal{P}\mathcal{Q})^{\binom{k}{2}-nk}.\end{aligned}$$

□

**Proposition 2.14.**

$$\big((\mathcal{P}^{-n},\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_k=\frac{(-1)^k(\mathcal{P}\mathcal{Q})^{\binom{k}{2}-nk}\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_{n-k}}. \tag{2.17}$$

*Proof.*

$$\begin{aligned}\big((\mathcal{P}^{-n},\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_k&=(\mathcal{P}^{-n},\mathcal{Q}^{-n})(\mathcal{P}^{-n+1},\mathcal{Q}^{-n+1})\cdots(\mathcal{P}^{-n+k-1}-\mathcal{Q}^{-n+k-1})\\&=(-1)^k(\mathcal{P}\mathcal{Q})^{-nk}(\mathcal{P}\mathcal{Q})^{\sum(k-1)}(\mathcal{P}^{n-k+1},\mathcal{Q}^{n-k+1})\cdots(\mathcal{P}^n-\mathcal{Q}^n)\\&=\frac{(-1)^k(\mathcal{P}\mathcal{Q})^{\binom{k}{2}-nk}\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_{n-k}}.\end{aligned}$$

□

**Proposition 2.15.**

$$\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_k=\frac{\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A}^{-1}\mathcal{P}^{1-k},\mathcal{B}^{-1}\mathcal{Q}^{1-k});(\mathcal{P},\mathcal{Q})\big)_n}\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k(\mathcal{P}\mathcal{Q})^{-nk},\quad \mathcal{A},\mathcal{B}\neq 0. \tag{2.18}$$

*Proof.*

$$\begin{aligned}\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_k&=(\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n})(\mathcal{A}\mathcal{P}^{-n+1},\mathcal{B}\mathcal{Q}^{-n+1})\cdots(\mathcal{A}\mathcal{P}^{-n+k-1}-\mathcal{B}\mathcal{Q}^{-n+k-1})\\&=(-\mathcal{A}\mathcal{B})^k(\mathcal{P}\mathcal{Q})^{-nk}(\mathcal{P}\mathcal{Q})^{\sum(k-1)}(\mathcal{A}^{-1}\mathcal{P}^{n-k+1},\mathcal{B}^{-1}\mathcal{Q}^{n-k+1})\cdots(\mathcal{A}^{-1}\mathcal{P}^{n}-\mathcal{B}^{-1}\mathcal{Q}^{n})\\&=\frac{(-\mathcal{A}\mathcal{B})^k(\mathcal{P}\mathcal{Q})^{-nk}(\mathcal{P}\mathcal{Q})^{\sum(k-1)}\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_{n-k}}\\&=\frac{(-\mathcal{A}\mathcal{B})^k(\mathcal{P}\mathcal{Q})^{\sum(k-1)}\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n(\mathcal{A}^{-1},\mathcal{B}^{-1})\cdots(\mathcal{A}^{-1}\mathcal{P}^{1-k}-\mathcal{B}^{-1}\mathcal{Q}^{1-k})}{(\mathcal{P}\mathcal{Q})^{nk}\big((\mathcal{A}^{-1}\mathcal{P}^{1-k},\mathcal{B}^{-1}\mathcal{Q}^{1-k});(\mathcal{P},\mathcal{Q})\big)_n}\\&=\frac{\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A}^{-1}\mathcal{P}^{1-k},\mathcal{B}^{-1}\mathcal{Q}^{1-k});(\mathcal{P},\mathcal{Q})\big)_n}\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_k(\mathcal{P}\mathcal{Q})^{-nk},\quad \mathcal{A},\mathcal{B}\neq 0.\end{aligned}$$

□

**Remark 2.16.** When $k$ is replace by $n-k$ in above proposition then, we get

$$\big((\mathcal{A}\mathcal{P}^{-n},\mathcal{B}\mathcal{Q}^{-n});(\mathcal{P},\mathcal{Q})\big)_{n-k}=(-\mathcal{A}\mathcal{B})^{n-k}(\mathcal{P}\mathcal{Q})^{\sum k-\sum n}\frac{\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A}^{-1}\mathcal{P},\mathcal{B}^{-1}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_k},\quad \mathcal{A},\mathcal{B}\neq 0. \tag{2.19}$$

**Proposition 2.17.**

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{2n}=\big((\mathcal{A},\mathcal{B});(\mathcal{P}^2,\mathcal{Q}^2)\big)_n\big((\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q});(\mathcal{P}^2,\mathcal{Q}^2)\big)_n. \tag{2.20}$$

*Proof.* The proof follows easily by definition of $(\mathcal{P},\mathcal{Q})$-shifted factorials. □

**Remark 2.18.** Similarly, we can prove that

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{3n}=\big((\mathcal{A},\mathcal{B}),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}),(\mathcal{A}\mathcal{P}^2,\mathcal{B}\mathcal{Q}^2);(\mathcal{P}^3,\mathcal{Q}^3)\big)_n, \tag{2.21}$$

where

$$\begin{aligned}&\big((\mathcal{A},\mathcal{B}),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}),(\mathcal{A}\mathcal{P}^2,\mathcal{B}\mathcal{Q}^2);(\mathcal{P}^3,\mathcal{Q}^3)\big)_n\\&\quad=\big((\mathcal{A},\mathcal{B});(\mathcal{P}^3,\mathcal{Q}^3)\big)_n\big((\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q});(\mathcal{P}^3,\mathcal{Q}^3)\big)_n\big((\mathcal{A}\mathcal{P}^2,\mathcal{B}\mathcal{Q}^2);(\mathcal{P}^3,\mathcal{Q}^3)\big)_n.\end{aligned}$$

**Remark 2.19.** So, the generalization of above two equations are can be prove by the principal of mathematical induction

$$\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_{kn}=\big((\mathcal{A},\mathcal{B}),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}),\cdots(\mathcal{A}\mathcal{P}^{k-1},\mathcal{B}\mathcal{Q}^{k-1});(\mathcal{P}^k,\mathcal{Q}^k)\big)_n. \tag{2.22}$$

where

$$\begin{aligned}&\big((\mathcal{A},\mathcal{B}),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}),\cdots(\mathcal{A}\mathcal{P}^{k-1},\mathcal{B}\mathcal{Q}^{k-1});(\mathcal{P}^k,\mathcal{Q}^k)\big)_n\\&\quad=\big((\mathcal{A},\mathcal{B});(\mathcal{P}^k,\mathcal{Q}^k)\big)_n\big((\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q});(\mathcal{P}^k,\mathcal{Q}^k)\big)_n\cdots\big((\mathcal{A}\mathcal{P}^{k-1},\mathcal{B}\mathcal{Q}^{k-1});(\mathcal{P}^k,\mathcal{Q}^k)\big)_n.\end{aligned}$$

**Proposition 2.20.**

$$\big((\mathcal{A}^2,\mathcal{B}^2);(\mathcal{P}^2,\mathcal{Q}^2)\big)_n=\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A},-\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n. \tag{2.23}$$

*Proof.* The proof follows easily by definition of $(\mathcal{P},\mathcal{Q})$-shifted factorials and using the formula $(\mathcal{A}^2-\mathcal{B}^2)=(\mathcal{A}-\mathcal{B})(\mathcal{A}+\mathcal{B})$, we get the result. □

**Remark 2.21.** Also, we can prove that

$$\big((\mathcal{A}^3,\mathcal{B}^3);(\mathcal{P}^3,\mathcal{Q}^3)\big)_n=\big((\mathcal{A},\mathcal{B}),(\mathcal{A},\mathcal{B}\omega),(\mathcal{A},\mathcal{B}\omega^2);(\mathcal{P},\mathcal{Q})\big)_n, \tag{2.24}$$

where

$$\begin{aligned}&\big((\mathcal{A},\mathcal{B}),(\mathcal{A},\mathcal{B}\omega),(\mathcal{A},\mathcal{B}\omega^2);(\mathcal{P},\mathcal{Q})\big)_n\\&\qquad=\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A},\mathcal{B}\omega);(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A},\mathcal{B}\omega^2);(\mathcal{P},\mathcal{Q})\big)_n,\end{aligned}$$

and $\omega=e^{2\pi\iota/3}$

**Remark 2.22.** So, the generalization of above two equations are can be prove by the principal of mathematical induction

$$\big((\mathcal{A}^k,\mathcal{B}^k);(\mathcal{P}^k,\mathcal{Q}^k)\big)_n=\big((\mathcal{A},\mathcal{B}),(\mathcal{A},\omega_k\mathcal{B}),\cdots(\mathcal{A},\omega_k^{k-1}\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n, \tag{2.25}$$

where

$$\begin{aligned}&\big((\mathcal{A},\mathcal{B}),(\mathcal{A},\omega_k\mathcal{B}),\cdots(\mathcal{A},\omega_k^{k-1}\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\\&\qquad=\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A},\omega_k\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\cdots\big((\mathcal{A},\omega_k^{k-1}\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n,\end{aligned}$$

and $\omega_k=e^{2\pi\iota/k}$

**Proposition 2.23.** *By using above proposition we have*

$$\frac{(\mathcal{A}^2\mathcal{P}^{2n}-\mathcal{B}^2\mathcal{Q}^{2n})}{(\mathcal{A}^2-\mathcal{B}^2)}=\frac{\big((\mathcal{A}^2\mathcal{P}^2,\mathcal{B}^2\mathcal{Q}^2);(\mathcal{P}^2,\mathcal{Q}^2)\big)_n}{\big((\mathcal{A}^2,\mathcal{B}^2);(\mathcal{P}^2,\mathcal{Q}^2)\big)_n}=\frac{\big((\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A}\mathcal{P},-\mathcal{B}\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{A},-\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n}. \tag{2.26}$$

*Proof.* The proof follows easily by definition. □

**Remark 2.24.** By the similarity of above result, we can prove by using proposition

$$\frac{(\mathcal{A}^3\mathcal{P}^{3n}-\mathcal{B}^3\mathcal{Q}^{3n})}{(\mathcal{A}^3-\mathcal{B}^3)}=\frac{\big((\mathcal{A}^3\mathcal{P}^3,\mathcal{B}^3\mathcal{Q}^3);(\mathcal{P}^3,\mathcal{Q}^3)\big)_n}{\big((\mathcal{A}^3,\mathcal{B}^3);(\mathcal{P}^3,\mathcal{Q}^3)\big)_n}=\frac{\big((\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}\omega),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}\omega^2);(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A},\mathcal{B}),(\mathcal{A},\mathcal{B}\omega),(\mathcal{A},\mathcal{B}\omega^2);(\mathcal{P},\mathcal{Q})\big)_n}, \tag{2.27}$$

where $\omega=e^{2\pi\iota/3}$

**Remark 2.25.** So, the generalization of above two equations are can be written as

$$\frac{(\mathcal{A}^k\mathcal{P}^{kn}-\mathcal{B}^k\mathcal{Q}^{kn})}{(\mathcal{A}^k-\mathcal{B}^k)}=\frac{\big((\mathcal{A}^k\mathcal{P}^k,\mathcal{B}^k\mathcal{Q}^k);(\mathcal{P}^k,\mathcal{Q}^k)\big)_n}{\big((\mathcal{A}^k,\mathcal{B}^k);(\mathcal{P}^k,\mathcal{Q}^k)\big)_n}=\frac{\big((\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}),(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}\omega_k),\cdots,(\mathcal{A}\mathcal{P},\mathcal{B}\mathcal{Q}\omega_k^{k-1});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{A},\mathcal{B}),(\mathcal{A},\omega_k\mathcal{B}),\cdots(\mathcal{A},\omega_k^{k-1}\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n}, \tag{2.28}$$

where $\omega_k=e^{2\pi\iota/k}$

## 3 $(\mathcal{P},\mathcal{Q})$-Hypergeometric Functions

First of all, we have needed to announce a $(\mathcal{P},\mathcal{Q})$-series containing several parameters. For that purpose, we recall that the Gauss [1813] hypergeometric series is defined by

$$\mathcal{F}(\mathcal{A},\mathcal{B};\mathcal{C};z)={}_2\mathcal{F}_1(\mathcal{A},\mathcal{B};\mathcal{C};z)={}_2\mathcal{F}_1\left[\begin{array}{c}\mathcal{A},\mathcal{B}\\ \mathcal{C}\end{array};z\right]=\sum_{n\geq0}\frac{(\mathcal{A})_n(\mathcal{B})_n}{(\mathcal{C})_n n!}z^n, \tag{3.1}$$

where $\mathcal{C}\neq0,-1,-2,\cdots$ so that no zero factors appear in the Eq. (3.1). Gauss' series or Eq. (3.1) converges absolutely for $|z|<1$ and for $|z|=1$ when $Re(\mathcal{C}-\mathcal{A}-\mathcal{B})>0$. Heine in [1847, 1878] introduced the series

$$\Phi(\lambda,\mu,\delta,\mathcal{Q},z)={}_2\Phi_1(\mathcal{Q}^\lambda,\mathcal{Q}^\mu;\mathcal{Q}^\delta;\mathcal{Q},z)$$

with

$${}_2\Phi_1(\mathcal{A},\mathcal{B};\mathcal{C};\mathcal{Q},z)={}_2\Phi_1\left[\begin{array}{c}\mathcal{A},\mathcal{B}\\ \mathcal{C}\end{array};\mathcal{Q},z\right]=\sum_{n\geq0}\frac{(\mathcal{A};\mathcal{Q})_n(\mathcal{B};\mathcal{Q})_n}{(\mathcal{C};\mathcal{Q})_n(\mathcal{Q};\mathcal{Q})_n}z^n, \tag{3.2}$$

where $\delta \neq -m$ and $\mathcal{C} \neq \mathcal{Q}^{-m}$ for $m = 0, 1, 2, \cdots$. Heine's series or Eq. (3.2) converges absolutely for $|z| < 1$ when $|\mathcal{Q}| < 1$, and it is a $\mathcal{Q}$-analogue of Gauss' series or Eq. (3.1) because, by taking a formal term-wise limit,

$$\lim_{\mathcal{Q}\to 1^-} {}_2\Phi_1(\mathcal{Q}^\lambda, \mathcal{Q}^\mu; \mathcal{Q}^\delta; \mathcal{Q}, z) = {}_2\mathcal{F}_1(\lambda, \mu; \delta; z).$$

Here, we introduce a $(\mathcal{P},\mathcal{Q})$-series as

$${}_2\Phi_1\left(\begin{array}{c}(\mathcal{P}^\lambda,\mathcal{Q}^\lambda),(\mathcal{P}^\mu,\mathcal{Q}^\mu)\\(\mathcal{P}^\delta,\mathcal{Q}^\delta)\end{array}\middle|(\mathcal{P},\mathcal{Q}),z\right),$$

with

$${}_2\Phi_1\left(\begin{array}{c}(\mathcal{A},\mathcal{B}),(\mathcal{C},\mathcal{D})\\(\mathcal{E},\mathcal{F})\end{array}\middle|(\mathcal{P},\mathcal{Q}),z\right) = \sum_{n\geq 0}\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{C},\mathcal{D});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{E},\mathcal{F});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}z^n, \quad (3.3)$$

where $\delta \neq -m$ and $(\mathcal{E},\mathcal{F}) \neq (\mathcal{P}^{-m},\mathcal{Q}^{-m})$ for $m = 0, 1, \cdots$, the Eq. (3.3) converges absolutely for $|z| < 1$ when $\left|\frac{\mathcal{Q}}{\mathcal{P}}\right| < 1$ and it is a $(\mathcal{P},\mathcal{Q})$-analogue of Gauss' series or Eq. (3.1) because, by taking a formal termwise limit,

$$\lim_{\mathcal{P}\to 1^-} {}_2\Phi_1\left(\begin{array}{c}(\mathcal{P}^\lambda,\mathcal{Q}^\lambda),(\mathcal{P}^\mu,\mathcal{Q}^\mu)\\(\mathcal{P}^\delta,\mathcal{Q}^\delta)\end{array}\middle|(\mathcal{P},\mathcal{Q}),z\right) = {}_2\Phi_1(\mathcal{Q}^\lambda,\mathcal{Q}^\mu;\mathcal{Q}^\delta;\mathcal{Q},z) \quad (3.4)$$

and

$$\lim_{\mathcal{Q}\to 1^-} {}_2\Phi_1(\mathcal{Q}^\lambda, \mathcal{Q}^\mu; \mathcal{Q}^\delta; \mathcal{Q}, z) = {}_2\mathcal{F}_1(\lambda, \mu; \delta; z).$$

The generalized hypergeometric series with $i$ numerator parameters $(\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}}),\cdots,(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}})$ and $j$ denominator parameters $(\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}}),\cdots,(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}})$ is defined by

$$\begin{aligned}&{}_i\Phi_j\left(\begin{array}{c}(\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}})\\(\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}})\end{array}\middle|(\mathcal{P},\mathcal{Q});z\right)\\&=\sum_{n\geq 0}\frac{\big((\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}}),\cdots,(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}}),\cdots,(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}\left[(-1)^n\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^{\binom{n}{2}}\right]^{1+j-i}z^n,\end{aligned} \quad (3.5)$$

where $\big((\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}}),\cdots,(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n = \big((\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n\cdots\big((\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n$. Also, for any $j$, $\big((\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n \neq 0$, if one of $r$ is such that $(\mathcal{A}_{r\mathcal{P}},\mathcal{A}_{r\mathcal{Q}}) = (\mathcal{P}^{-n},\mathcal{Q}^{-n})$ where $n$ is a non-negative integer, this $(\mathcal{P},\mathcal{Q})$-hypergeometric function is a polynomials in $z$, otherwise the radius of convergence $l$ of the $(\mathcal{P},\mathcal{Q})$-hypergeometric series is given by

$$l = \begin{cases}\infty, & i < j+1\\ 1, & i = j+1\\ 0, & i > j+1.\end{cases}$$

**Remark 3.1.** The special case when $i = j+1$ we get

$$\begin{aligned}&{}_{j+1}\Phi_j\left(\begin{array}{c}(\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(j+1)\mathcal{P}},\mathcal{A}_{(j+1)\mathcal{Q}})\\(\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}})\end{array}\middle|(\mathcal{P},\mathcal{Q});z\right)\\&\qquad=\sum_{n\geq 0}\frac{\big((\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}}),\cdots,(\mathcal{A}_{(j+1)\mathcal{P}},\mathcal{A}_{(j+1)\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}}),\cdots,(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}z^n.\end{aligned} \quad (3.6)$$

**Remark 3.2.** If $\mathcal{A}_{1\mathcal{P}} = \mathcal{A}_{2\mathcal{P}} = \cdots = \mathcal{A}_{i\mathcal{P}} = \mathcal{B}_{1\mathcal{P}} = \cdots = \mathcal{B}_{j\mathcal{P}} = 1$ and $\mathcal{A}_{1\mathcal{Q}} = \mathcal{A}_1, \cdots, \mathcal{A}_{i\mathcal{Q}} = \mathcal{A}_i, \mathcal{B}_{1\mathcal{Q}} = \mathcal{B}_1, \cdots, \mathcal{B}_{j\mathcal{Q}} = \mathcal{B}_j$ then

$$\lim_{\mathcal{P}\to 1} {}_i\Phi_j \left( \begin{array}{c} (1,\mathcal{A}_1);\cdots;(1,\mathcal{A}_i) \\ (1,\mathcal{B}_1);\cdots;(1,\mathcal{B}_j) \end{array} \middle| (\mathcal{P},\mathcal{Q}); z \right) = {}_i\Phi_j \left( \begin{array}{c} \mathcal{A}_1;\cdots;\mathcal{A}_i \\ \mathcal{B}_1;\cdots;\mathcal{B}_j \end{array} \middle| \mathcal{Q}; z \right). \tag{3.7}$$

**Remark 3.3.** We assume that each $(\mathcal{P},\mathcal{Q})$-hypergeometric function is in fact a polynomials when

$$\lim_{(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}})\to\infty} {}_i\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); \frac{z}{(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}})} \right)$$
$$= {}_{i-1}\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); z \right). \tag{3.8}$$

Many limit relations between $(\mathcal{P},\mathcal{Q})$-hypergeometric orthogonal polynomials are based on the observations that

$${}_i\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}});(\lambda,\mu) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{(j-1)\mathcal{P}},\mathcal{B}_{(j-1)\mathcal{Q}});(\lambda,\mu) \end{array} \middle| (\mathcal{P},\mathcal{Q}); z \right)$$
$$= {}_{i-1}\Phi_{j-1} \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{(j-1)\mathcal{P}},\mathcal{B}_{(j-1)\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); z \right). \tag{3.9}$$

$$\lim_{\lambda\to\infty} {}_i\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}});\lambda(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); \frac{z}{\lambda} \right)$$
$$= {}_{i-1}\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); (\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}})z \right). \tag{3.10}$$

$$\lim_{\lambda\to\infty} {}_i\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{(j-1)\mathcal{P}},\mathcal{B}_{(j-1)\mathcal{Q}});\lambda(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); \lambda z \right)$$
$$= {}_i\Phi_{j-1} \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{(j-1)\mathcal{P}},\mathcal{B}_{(j-1)\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); \frac{z}{(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}})} \right), \tag{3.11}$$

and

$$\lim_{\lambda\to\infty} {}_i\Phi_j \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}});\lambda(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{(j-1)\mathcal{P}},\mathcal{B}_{(j-1)\mathcal{Q}});\lambda(\mathcal{B}_{s\mathcal{P}},\mathcal{B}_{s\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); z \right)$$
$$= {}_{i-1}\Phi_{j-1} \left( \begin{array}{c} (\mathcal{A}_{1\mathcal{P}},\mathcal{A}_{1\mathcal{Q}});\cdots;(\mathcal{A}_{(i-1)\mathcal{P}},\mathcal{A}_{(i-1)\mathcal{Q}}) \\ (\mathcal{B}_{1\mathcal{P}},\mathcal{B}_{1\mathcal{Q}});\cdots;(\mathcal{B}_{(j-1)\mathcal{P}},\mathcal{B}_{(j-1)\mathcal{Q}}) \end{array} \middle| (\mathcal{P},\mathcal{Q}); \frac{(\mathcal{A}_{i\mathcal{P}},\mathcal{A}_{i\mathcal{Q}})z}{(\mathcal{B}_{j\mathcal{P}},\mathcal{B}_{j\mathcal{Q}})} \right). \tag{3.12}$$

Here, we also introduce some transformation formulas for ${}_2\Phi_1$ series

**Theorem 3.4.**

$$ {}_2\Phi_1\left(\begin{array}{c}(\mathcal{A},\mathcal{B}),(\mathcal{C},\mathcal{D})\\(\mathcal{E},\mathcal{F})\end{array}\bigg|(\mathcal{P},\mathcal{Q}),z\right) = \frac{\big((\mathcal{C},\mathcal{D}),(\mathcal{E}\mathcal{P},\mathcal{B}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{E},\mathcal{F}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{}_2\Phi_1\left(\begin{array}{c}(\mathcal{D}\mathcal{E},\mathcal{C}\mathcal{F}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z)\\(\mathcal{E}\mathcal{P},\mathcal{B}\mathcal{C}z)\end{array}\bigg|(\mathcal{P},\mathcal{Q}),\frac{\mathcal{P}}{\mathcal{E}\mathcal{C}}\right) \tag{3.13} $$

$$ = \frac{\big((\mathcal{E}\mathcal{D},\mathcal{F}\mathcal{C}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{D}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{E},\mathcal{F}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{}_2\Phi_1\left(\begin{array}{c}(\mathcal{F}\mathcal{P},\mathcal{B}\mathcal{D}z),(\mathcal{C},\mathcal{D})\\(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{D}z)\end{array}\bigg|(\mathcal{P},\mathcal{Q}),\frac{\mathcal{F}\mathcal{C}}{\mathcal{E}\mathcal{D}}\right) \tag{3.14} $$

$$ = \frac{\big((\mathcal{F}\mathcal{P},\mathcal{B}\mathcal{D}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{}_2\Phi_1\left(\begin{array}{c}(\mathcal{B}\mathcal{E},\mathcal{A}\mathcal{F}),(\mathcal{D}\mathcal{E},\mathcal{C}\mathcal{F})\\(\mathcal{E},\mathcal{F})\end{array}\bigg|(\mathcal{P},\mathcal{Q}),\frac{\mathcal{B}\mathcal{D}z}{\mathcal{F}\mathcal{P}}\right) \tag{3.15} $$

*Proof.*

$$
\begin{aligned}
{}_2\Phi_1\left(\begin{array}{c}(\mathcal{A},\mathcal{B}),(\mathcal{C},\mathcal{D})\\(\mathcal{E},\mathcal{F})\end{array}\bigg|(\mathcal{P},\mathcal{Q}),z\right) &= \sum_{n\geq 0}\frac{\big((\mathcal{A},\mathcal{B});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{C},\mathcal{D});(\mathcal{P},\mathcal{Q})\big)_n}{\big((\mathcal{E},\mathcal{F});(\mathcal{P},\mathcal{Q})\big)_n\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_n}z^n\\
&= \sum_{n\geq 0}\frac{\mathcal{A}^n\left(\frac{\mathcal{B}}{\mathcal{A}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n\mathcal{C}^n\left(\frac{\mathcal{D}}{\mathcal{C}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}{\mathcal{E}^n\left(\frac{\mathcal{F}}{\mathcal{E}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n\mathcal{P}^n\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}z^n \qquad (\because Proposition\quad 2.4)\\
&= \frac{\left(\frac{\mathcal{D}}{\mathcal{C}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{F}}{\mathcal{E}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}\sum_{n\geq 0}\frac{\left(\frac{\mathcal{B}}{\mathcal{A}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n\left(\frac{\mathcal{F}}{\mathcal{E}}\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^n;\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{D}}{\mathcal{C}}\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^n;\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}\left(\frac{\mathcal{A}\mathcal{C}z}{\mathcal{E}\mathcal{P}}\right)^n \qquad (\because [5, page6])\\
&= \frac{\left(\frac{\mathcal{D}}{\mathcal{C}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{F}}{\mathcal{E}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}\sum_{n\geq 0}\frac{\left(\frac{\mathcal{B}}{\mathcal{A}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}{\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}\left(\frac{\mathcal{A}\mathcal{C}z}{\mathcal{E}\mathcal{P}}\right)^n\sum_{m\geq 0}\frac{\left(\frac{\mathcal{F}\mathcal{C}}{\mathcal{E}\mathcal{D}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}{\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}\left(\frac{\mathcal{D}}{\mathcal{C}}\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^n\right)^m\\
&= \frac{\left(\frac{\mathcal{D}}{\mathcal{C}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{F}}{\mathcal{E}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}\sum_{m\geq 0}\frac{\left(\frac{\mathcal{F}\mathcal{C}}{\mathcal{E}\mathcal{D}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}{\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}\left(\frac{\mathcal{D}}{\mathcal{C}}\right)^m\sum_{n\geq 0}\frac{\left(\frac{\mathcal{B}}{\mathcal{A}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}{\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_n}\left(\frac{\mathcal{A}\mathcal{C}z}{\mathcal{E}\mathcal{P}}\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^m\right)^n\\
&= \frac{\left(\frac{\mathcal{D}}{\mathcal{C}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{F}}{\mathcal{E}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}\sum_{m\geq 0}\frac{\left(\frac{\mathcal{F}\mathcal{C}}{\mathcal{E}\mathcal{D}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}{\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}\frac{\left(\frac{\mathcal{B}\mathcal{C}z}{\mathcal{E}\mathcal{P}}\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^m;\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{A}\mathcal{C}z}{\mathcal{E}\mathcal{P}}\left(\frac{\mathcal{Q}}{\mathcal{P}}\right)^m;\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}\\
&= \frac{\left(\frac{\mathcal{D}}{\mathcal{C}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty\left(\frac{\mathcal{B}\mathcal{C}z}{\mathcal{E}\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}{\left(\frac{\mathcal{F}}{\mathcal{E}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty\left(\frac{\mathcal{A}\mathcal{C}z}{\mathcal{E}\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_\infty}\sum_{m\geq 0}\frac{\left(\frac{\mathcal{F}\mathcal{C}}{\mathcal{E}\mathcal{D}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m\left(\frac{\mathcal{A}\mathcal{C}z}{\mathcal{E}\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}{\left(\frac{\mathcal{Q}}{\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m\left(\frac{\mathcal{B}\mathcal{C}z}{\mathcal{E}\mathcal{P}};\frac{\mathcal{Q}}{\mathcal{P}}\right)_m}\\
&= \frac{\big((\mathcal{C},\mathcal{D}),(\mathcal{E}\mathcal{P},\mathcal{B}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{E},\mathcal{F}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}\\
&\qquad\times\sum_{m\geq 0}\frac{\big((\mathcal{D}\mathcal{E},\mathcal{C}\mathcal{F});(\mathcal{P},\mathcal{Q})\big)_m\big((\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_m}{\big((\mathcal{E}\mathcal{P},\mathcal{B}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_m\big((\mathcal{P},\mathcal{Q});(\mathcal{P},\mathcal{Q})\big)_m}\left(\frac{\mathcal{P}}{\mathcal{C}\mathcal{E}}\right)^n\\
&= \frac{\big((\mathcal{C},\mathcal{D}),(\mathcal{E}\mathcal{P},\mathcal{B}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{\big((\mathcal{E},\mathcal{F}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z);(\mathcal{P},\mathcal{Q})\big)_\infty}{}_2\Phi_1\left(\begin{array}{c}(\mathcal{D}\mathcal{E},\mathcal{C}\mathcal{F}),(\mathcal{E}\mathcal{P},\mathcal{A}\mathcal{C}z)\\(\mathcal{E}\mathcal{P},\mathcal{B}\mathcal{C}z)\end{array}\bigg|(\mathcal{P},\mathcal{Q}),\frac{\mathcal{P}}{\mathcal{C}\mathcal{E}}\right).
\end{aligned}
$$

□

A short way to prove rest of the two equality is just to iterate first equality. The latter formula is a $(\mathcal{P},\mathcal{Q})$-analogue of Euler's transformation formula:

$$ {}_2\mathcal{F}_1\left[\begin{array}{c}\mathcal{A},\mathcal{B}\\\mathcal{C}\end{array};z\right] = (1-z)^{\mathcal{C}-\mathcal{A}-\mathcal{B}}{}_2F_1\left[\begin{array}{c}\mathcal{C}-\mathcal{A},\mathcal{C}-\mathcal{B}\\\mathcal{C}\end{array};z\right] \tag{3.16} $$

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Received: March 15, 2018.
Accepted: August 24, 2018.