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CERTAIN GEOMETRIC PROPERTIES OF η-RICCI SOLITON ON η-EISTEIN PARA-KENMOTSU MANIFOLDS

S.K. Yadav, S.K. Chaubey and D.L. Suthar

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Abstract In paracontact geometry, we consider η-Ricci soliton on η-Einstein para-Kenmotsu manifolds $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ and prove that on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$, if ζ is a recurrent torse forming η-Ricci soliton then ζ is concurrent as well as Killing vector field. Further we prove that if the torse forming η-Ricci soliton on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ is recurrent, then there exist no any parallel symmetric $(0, 2)$-tensor field which is constant multiple of the metric. Moreover, we found the condition for Ricci soliton on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ to be shrinking, steady and expanding. Finally an example of η-Ricci soliton on $(M, \varphi, g, \zeta, \lambda, \mu)$ has been constructed.

1 Introduction

In 1976, Sato [25] introduced the notion of almost paracontact manifolds. Before Sato, Takahashi [28], defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric. Further in 1985, Kaneyuki and Williams [20] defined the notion of almost paracontact structure on pseudo-Riemannian manifold of dimension $n = (2m + 1)$. Later Zamkovoy [35] showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n + 1, n)$. The notion of para-Kenmotsu manifold was introduced by Welyczko [30]. This structure is an analogous of Kenmotsu manifold [12] in para-contact geometry. Para-Kenmostu (briefly p-Kenmostu) and special para-Kenmotsu (briefly sp-Kenmotsu) manifolds was studied by Sinha and SaiPrasad [27], Blaga [4] and Prasad and Satyanarayan [22] and also $(LCS)_{2n+1}$-manifolds was studied by Yadav et al. [31, 32, 33].

In 1982, Hamilton [18] made the fundamental observation that Ricci flow is an excellent tool for simplifying the structure of the manifold. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. Moreover the Ricci soliton represent a natural generalization of Einstein metrics on a Riemannian manifold, being generalized fixed points of Hamilton's Ricci flow

$$\frac{\partial}{\partial \mathrm{t}} g = -2S.$$

where g is a Riemannian metric, S is the Ricci curvature tensor, t is time.

Let $\varphi_t : M \to M, t \in R$ be a family of diffeomorphism which is one parameter group of transformations then it gives rise to a vector field called infinitesimal generator and integral curves. Ricci soliton move under the Ricci flow simply by diffeomorphism of the initial metric that is they are stationary points of the Ricci flow in space of metrics of $\varphi_t : M \to M$. Here the metric $g(t)$ is the pull back of the initial metric $g(0)$ of φ_t. The evolution equation defining the Ricci flow is a kind of nonlinear diffusion equation, an analogue of heat equation for metrics. Under Ricci flow, a metric can be improved to evolve into more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold: it will expand in the directions of negative Ricci curvature and shrink in the positive case. Ricci soliton have been studied in many contexts: on Kahler manifolds [12], on contact and Lorentzian manifolds [7, 12, 19, 26, 29], on Sasakian [15, 16], α-Sasakian [19] and K-contact manifolds [26], on Kenmotsu [5, 24], on f-Kenmotsu manifolds [12], on $(LCS)_n$-manifolds [9], on LP-Sasakian manifolds [23], on para-Kenmotsu [1] etc. In paracontcat geometry, Ricci soliton firstly appeared in the paper of Calvoruso and Perrone [11]. Bejan and Crasmarean [8] studied Ricci soliton on 3-dimensional normal paraconcat manifolds. In the context of general relativity, the Ricci solitons have been studied by Ali and Ahsan [2].

A more general notion is that of η-Ricci soliton introduced by Cho and Kimura [10], which was treated by Calin and Crasmareanu [13] on Hopf hypersurfaces in complex space forms. It was further classified by many authors in various context: LP-Sasakian manifolds [3], para-Kenmotsu manifolds [4] and η-Einstein $(LCS)_n$-manifolds [17] etc.

2 Preliminaries

Let (M^n, g) be n–dimensional smooth manifold equipped with an almost paracontact metric structure $(\varphi, \zeta, \eta, g)$ that is φ is a tensor field of type $(1, 1)$, ζ is a vector field, η is a 1-form and g is a pseudo-Riemannian metric such that

$$\varphi^2 X = X - \eta(X)\zeta, \quad \eta(\zeta) = 1, \quad \varphi\zeta = 0, \quad \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \zeta) = \eta(X), \tag{2.3}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.4}$$

for all vector fields $X, Y \in TM^n$. Then (M^n, g) is called almost paracontact metric manifold. If an almost paracontact metric manifold satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\zeta - \eta(Y)\varphi X, \tag{2.5}$$

for all vector fields $X, Y \in TM^n$, then (M^n, g) is called almost para-Kenmotsu manifold.

A normal almost para-Kenmotsu manifold is called a para-Kenmotsu manifold. The para-Kenmotsu structure for 3-dimensional normal paracontact metric structure was introduced by Welyczko [30]. From the above equation it follows that

$$\nabla_X \zeta = X - \eta(Y)\zeta, \tag{2.6}$$

Moreover, on such manifold the following relations hold:

$$R(X, Y, Z, W) = g(X, Z)(Y, W) - g(Y, Z)g(X, W), \tag{2.7}$$

$$R(\xi, X)Y = \{-g(X, Y)\xi + \eta(Y)X\}, \tag{2.8}$$

$$R(\xi, X)\xi = \{\, -\eta(Y)\xi + Y\}, \tag{2.9}$$

$$S(X, Y) = -(n-1)g(X, Y), \tag{2.10}$$

$$S(\zeta, \xi) = -(n-1), \tag{2.11}$$

$$QX = -(n-1)X, \tag{2.12}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor defined as $g(QX, Y) = S(X, Y)$.

Definition 2.1. A vector field ζ is called torse forming if it obey the following properties

$$\nabla_X \zeta = fX + \gamma(X)\zeta, \tag{2.13}$$

for a smooth function $f \in C^\infty(M)$ and λ is an 1-form, for all vector field X on M. A torse forming vector field ζ is called recurrent if $f = 0$.

Definition 2.2. A vector field ϑ is called concurrent if

$$\nabla_X \vartheta = 0, \tag{2.14}$$

for any vector field X on M.

Definition 2.3. A tensor h of second order is said to be a parallel tensor if $\nabla h = 0$.

Definition 2.4. A para-Kenmotsu manifold (M^n, g) is said to be an η-Einstein if its Ricci tensor S of type $(0, 2)$ is of the form

$$S = ag + b\eta \otimes \eta, \tag{2.15}$$

where a and b are smooth functions on (M^n, g).

Definition 2.5. A vector field X on a para-Kenmotsu manifold is said to be conformal Killing vector field [34] if

$$L_X g = \rho\, g, \tag{2.16}$$

where ρ is a function on the manifold. If $\rho = 0$, then the vector field ρ is said to be a Killing vector field.

Proposition 2.6. *In an η–Einstein para-Kenmotsu manifold, the following relations hold*

i) $S(\varphi X, Y) = S(X, \varphi Y) = -ag(\varphi X, Y)$,
ii) $S(X, \zeta) = (a + b)\eta(X)$,
iii) $S(\zeta, \zeta) = (a + b)$,
iv) $S(X, \varphi^2 Y) = -S(\varphi X, \varphi Y) = S(X, Y) - (a + b)\eta(X)\eta(Y)$.

Proof. In view of (2.1) and (2.4), from (2.15), we get the results i), ii) and iii). Also substituting $Y = \varphi Y$ in i), using (2.1) and iii), we get the result iv). Our preposition is proved. □

3 $\eta-$Ricci solitons on $(M^n, \varphi, \zeta, \eta, g)$

Let $(M, \phi, \zeta, \eta, g)$ be an almost paracontact metric manifold. We follow the equation

$$(\mathrm{L}_\zeta g) + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where L_ζ is the Lie-derivative operator along the vector field ζ, S is the Ricci tensor field of the metric g, and λ and μ are real constants. We write $L_\zeta g$ in term of the Levi-Civita connection ∇, and we have

$$2\mathrm{S}(X, Y) = -g(\nabla_Y \zeta, Y) - g(Y, \nabla_X \zeta) - 2\lambda \mathrm{g}(\mathrm{X}, \mathrm{Y}) - 2\mu\eta(\mathrm{X})\eta(\mathrm{Y}), \tag{3.2}$$

for any $X, Y \in \chi(M)$.
The structure (g, ζ, λ, μ) that follow the equation (3.1) is said to be an η-Ricci soliton on $(M.g)$. [10]; in particular, if $\mu = 0$, (g, ζ, λ) is a Ricci soliton [18] and it is called shrinking, steady, or expanding according as λ is negative, zero or positive, respectively.
For para-Kenmotsu manifold, equation (3.2) becomes

$$\mathrm{S}(\mathrm{X}, \mathrm{Y}) = -(1 + \lambda)\mathrm{g}(\mathrm{X}, \mathrm{Y}) - (\mu - 1)\eta(\mathrm{X})\eta(\mathrm{Y}), \tag{3.3}$$

$$\mathrm{S}(\mathrm{X},\zeta) = \mathrm{S}(\zeta, \mathrm{X}) = -(\mu + \lambda)\eta(\mathrm{X}). \tag{3.4}$$

From [30] on a $(2n + 1)$ -dimensional paracontact manifold $S(X, \zeta) = -(\dim(M) - 1)\, \eta(X) = -2n\, \eta(X)$, therefore from (3.4), we get

$$\lambda + \mu = 2n. \tag{3.5}$$

For this, the Ricci operator defined by $g(QX, Y) = S(X, Y)$, i.e.,

$$\mathrm{QX} = -(2n + 1 - \mu)\mathrm{X} - (\mu - 1)\eta(\mathrm{X})\zeta. \tag{3.6}$$

We now study η-Ricci soliton on an η-Einstein para-Kenmotsu manifolds $(M, \varphi, g, \ \zeta, \lambda, \mu, a, b)$, and prove the following results.

Theorem 3.1. *If $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ is η-Ricci soliton on an $\eta-$ Einstein para-Kenmotsu manifold, then following relations hold*
$i)$ $a+b+\lambda+\mu=0$, $ii)$ ζ *is a geodesic vector field,*

$iii)$ $(\nabla_\zeta\varphi)\zeta=0$, $iv)$ $\nabla_\zeta\eta=0$, $v)$ $\nabla_\zeta S=0$, $vi)$ $\nabla_\zeta Q=0$.

Proof. From (2.15) and (3.1), we get

$$g(\nabla_X\zeta, Y)+g(X,\nabla_Y\zeta)+2(a+\lambda)g(X,Y)+2(b+\mu)\eta(X)\eta(Y)=0, \tag{3.7}$$

Taking $X=Y=\zeta$ in (3.7), using (2.1) and (2.6), we get

$$g(\nabla_\zeta\zeta,\zeta)=-(a+b)-(\lambda+\mu), \tag{3.8}$$

But it is well know that $g(\nabla_\zeta\zeta,\zeta)=0$, for any vector field X on M. It follow that
$i)$ $a+b+\lambda+\mu=0$. This implies that $b+\mu=-(a+\lambda)$ using this result in (3.7), we have

$$g(\nabla_X\zeta, Y)+g(X,\nabla_Y\zeta)+2(a+\lambda)\{g(X,Y)-\eta(X)\eta(Y)\}=0, \tag{3.9}$$

Replacing $Y=\zeta$in (3.9), we get $g(X,\nabla_\zeta\zeta)=0$, for any vector field X on M.
It follow that $\nabla_\zeta\zeta=0$, i.e., $ii)$ ζ is a geodesic vector field.
The proof of other part, i.e., iii) and iv) is clear from ii). For general concept of ∇S and ∇Q, it is clear from (2.15)

$$(\nabla_X S)(Y,Z)=b\{\,\eta(Y)\,g(Z,\nabla_X\zeta)+\eta(Z)\,g(Y,\nabla_X\zeta)\},$$

and

$$(\nabla_X Q)Y=b\{\,\eta(Y)\,\nabla_X\zeta+\,g(Y,\nabla_X\zeta)\zeta\}.$$

For $X=Y=Z=\zeta$ from the above the result $v)$ $\nabla_\zeta S=0$, $vi)$ $\nabla_\zeta Q=0$ are verified. Our theorem is proved. □

Theorem 3.2. *If ζ is a torse forming η–Ricci soliton on an η–Einstein para-Kenmotsu manifold $(M,\varphi,g,\zeta,\lambda,\mu,a,b)$ then following relations hold.*

$i)$ $f=-(a+\lambda)$ $ii)$ $d\eta=0$

$iii)$ $a+b=(n+1)(a+\lambda)^2$ $iv)$ $\mu=-2a+(n+1)(a+\lambda)^2-\lambda$.

Proof. Since ζ is a torse forming η–Ricci soliton on $(M,\varphi,g,\zeta,\lambda,\mu,a,b)$. From (2.13), we get $g(\nabla_{X)}\zeta,\zeta)=f\eta(X)-\gamma(X)$, this implies that $\gamma=-f\eta$. Therefore from (2.13), we obtain

$$\nabla_X\zeta=f\,[X-\eta(X)]\,\zeta, \tag{3.10}$$

In view of (3.10), equation (3.9) becomes

$$(f+a+\lambda)\{g(X,Y)-\eta(X)\eta(Y)\}=0, \tag{3.11}$$

for any vector field X, Y on M. It follows that $i)$ $f=-(a+\lambda)$.
Also from (i), equation (3.10) reduces to

$$\nabla_X\zeta=-(a+\lambda)\,\varphi^2X. \tag{3.12}$$

It means $\nabla_X\zeta$ is collinear to φ^2X, for any vector field X. That follows $ii)$ $d\eta=0$, i.e., η is closed.
Again, we know that

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z. \tag{3.13}$$

In view of (3.10) , for $Z=\zeta$, equation (3.13) reduces to

$$R(X,Y)\zeta=(a+\lambda)^2\{\,\eta(Y)X+\eta(X)\,Y\,\}, \tag{3.14}$$

It follows that

$$S(X,\,\zeta)=(n+1)(a+\lambda)^2\eta(X)\,, \tag{3.15}$$

By Proposition 2.6 and equation (3.15), we yield $a+b=(n+1)(a+\lambda)^2$, $\mu=-2a+(n+1)(a+\lambda)^2-\lambda$. Our theorem is proved. □

Corollary 3.3. *If ζ is a recurrent torse forming η-Ricci soliton on an η–Einstein para-Kenmotsu manifold $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$, then ζ is i) concurrent ii) Killing vector field.*

Proof. Let ζ is a recurrent, then $f = 0$, this implies that $\lambda = -a$. Thus from (3.12), we get $\nabla_X \zeta = 0$ for all X on M., i.e., ζ is a concurrent vector field. And the result ii) is obvious. □

Theorem 3.4. *If ζ is a torse forming Ricci soliton on an η–Einstein para-Kenmotsu manifold $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ then the Ricci soliton is shrinking, steady and expanding in pursuance of $a > b, a = b$ and $a < b$ respectively.*

Proof. In particular, $\mu = 0$ therefore from Theorem 3.2, we yield $-2a+(n+1)(a+\lambda)^2-\lambda = 0$. It follows that $a + b = 2a + \lambda, i.e., \lambda = -(a - b)$. Hence the proof is complete. □

Theorem 3.5. *If the torse forming η-Ricci soliton on η-Einstein para-Kenmotsu manifold $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ is regular, then any parallel symmetric $(0, 2)-$tensor field is a constant multiple of the metric.*

Proof. Let h be a $(0, 2)-$ symmetric tensor field on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$,

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \tag{3.16}$$

It follows that

$$h(R(X, Y)Z, W) + h\,(Z, R(X, Y)W) = 0, \tag{3.17}$$

for any vector fields X, Y, Z, W on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$.
Put $X = Z = W = \zeta$ in (3.17) and using (3.14) , we get

$$(a + \lambda)^2\{\,h\,(Y, \zeta) + \eta(Y)\,h\,(\zeta, \zeta)\} = 0. \tag{3.18}$$

Since $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ is regular , i.e., $(a + \lambda) \neq 0$. Therefore , from (3.18), it is clear that

$$h\,(Y, \zeta) + \eta(Y)\,h\,(\zeta, \zeta) = 0, \tag{3.19}$$

In view of (3.12), differentiating (3.19) covariantly, we get.

$$h\ (Y, X) = \ h\,(\zeta, \zeta)\,g(X, Y) = 0,$$

for any vector fields X, Y on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$. As ∇ -parallel, it follows $h\,(\zeta, \zeta)$ is constant. Hence, the proof is complete. □

Corollary 3.6. *If the torse forming η-Ricci soliton on an η-Einstein para-Kenmotsu manifold $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$ is recurrent ,then theredo not exist any parallel symmetric $(0, 2)-$ tensor field which are constant multiple of the metric.*

Theorem 3.7. *If a conformal Killing vector field X on an η-Einstein para-Kenmotsu manifold is orthogonal to ζ, then X is Killing.*

Proof. Let the vector field X be a conformal Killing vector field on $(M, \varphi, g, \zeta, \lambda, \mu, a, b)$. Then for a function ρ, we have

$$(L_X g)\,(Y, Z) = \rho g(Y, Z) = \frac{\rho}{a}\{S(Y, Z) - b\eta(Y)\eta(Z)\}. \tag{3.20}$$

From (2.6), we get $\nabla_\zeta \zeta = 0$. So the integral curves are geodesics. Then from (3.20) for $Y = Z = \zeta$, we get

$$\rho = (L_X g)\ (\zeta, \zeta).$$

On the other hand

$$(L_X g)\ (\zeta, \zeta) = 2g(\nabla_\zeta X, \zeta),$$

and

$$2\nabla_\zeta\,(g(X, \zeta)) = 2g(\nabla_\zeta X, \zeta),$$

Therefore, we have

$$\rho = (L_X g)\,(\zeta, \zeta) = 2\nabla_\zeta\,(g(X, \zeta)) = 2g(\nabla_\zeta X, \zeta). \tag{3.21}$$

If X is orthogonal to ζ, $\rho = 0$ and hence $(L_X g) = 0$; i.e., X is a Killing vector field. Hence the proof is complete. □

Theorem 3.8. *If a vector field U on an η-Einstein para- Kenmotsu manifold $(M,\varphi,g,\zeta,\lambda,\mu,a,b)$ leaves the curvature tensor invariant, then U is Killing vector field.*

Proof. Since U be a vector field on $(M,\varphi,g,\zeta,\lambda,\mu,a,b)$, such that $L_U R = 0$. Then, we have

$$\begin{aligned}&(L_U g)\,(R(W,X)Y,Z) + (L_U g)\,(R(W,X)Z,Y)\\ &= 4\{(L_U g)(Y,Z)g(\phi W,X) + g(Y,Z)(L_U g)(\phi W,X)\},\end{aligned} \tag{3.22}$$

Taking $W = Y = Z = \zeta$ in (3.22) and using (2.9), we get

$$(L_U g)(X,\zeta) = \eta(X)(L_U g)(\zeta,\zeta) + (L._U g)(\phi X,\zeta). \tag{3.23}$$

Again putting $W = Y = \zeta$ in (3.22), we have

$$\begin{aligned}&(L_U g)(X,Z) - \eta(X)(L_U g)(\zeta,Z) + (L_U g)(\phi X,Z) + (L_U g)(X,\zeta)\eta(Z)\\ &-g(X,Z)(L_U g)(\zeta,\zeta) + \eta(Z)(L_U g)(X,\zeta) = 0,\end{aligned} \tag{3.24}$$

In view of equations (3.23) and (3.24), we get

$$(L_U g)(X,Z) = g(X,Z))(L_U g)(\zeta,\zeta). \tag{3.25}$$

Since $S(\zeta,\zeta) = (a+b)$. Applying Lie-derivative on it and keeping in mind that $L_U R = 0$ implies that $L_U S = 0$, we get $S\,(L_U\zeta,\zeta) = 0$. But $S(\zeta,\zeta) = (a+b)$. So $L_U\zeta = 0$. Hence $g\,(L_U\zeta,\zeta) = 0$, thus $(L_U g)(\zeta,\zeta) = 0$. So, in view of (3.21), we get $\rho = 0$ this implies that vector field U is Killing vector field. Hence the proof is complete. □

4 Example of η-Ricci soliton on para-Kenmotsu manifold

Let $M = \Re^3$ and (x,y,z) be the standard coordinates in $\Re^3$. Setting

$$\varphi = \frac{\partial}{\partial y}\otimes dx + \frac{\partial}{\partial x}\otimes dy, \qquad \zeta = -\frac{\partial}{\partial z}, \eta = -dz, \qquad g = dx\otimes dx - dy\otimes dy + dz\otimes dz.$$

Thus the data (φ,ζ,η,g) is para-Kenmotsu structure on $\Re^3$. To verify the conditions in the definition on a linearly independent system, we consider

$$E_1 = \frac{\partial}{\partial x}, \qquad E_2 = \frac{\partial}{\partial y}, \qquad E_3 = -\frac{\partial}{\partial z}.$$

This follows

$$\varphi E_1 = E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0, \quad \eta(E_3) = 1, \quad \eta(E_1) = \eta(E_2) = 0$$
$$[E_1,E_2] = 0\,, \qquad [E_3,E_1] = 0, \qquad [E_2,E_3] = 0.$$

The Levi-Civita connecion ∇ is calculated by using Koszul's formula

$$\begin{aligned}&\nabla_{E_1}E_3 = E_1, && \nabla_{E_1}E_1 = -E_3, && \nabla_{E_1}E_2 = 0,\\ &\nabla_{E_2}E_2 = E_3, && \nabla_{E_2}E_3 = E_2, && \nabla_{E_2}E_1 = 0,\\ &\nabla_{E_3}E_3 = 0, && \nabla_{E_3}E_2 = E_2, && \nabla_{E_3}E_1 = E_1.\end{aligned}$$

Using this results we can easily calculate the Riemann R and the Ricci curvature tensor field S as follows

$$\begin{aligned}&R(E_2,E_1)E_1 = -E_2, && R(E_1,E_3)E_3 = -E_1, && R(E_1,E_2)E_2 = E_1,\\ &R(E_2,E_3)E_3 = -E_2, && R(E_3,E_1)E_1 = E_3, && R(E_3,E_2)E_2 = -E_3.\\ &S(E_1,E_1) = 0, && S(E_2,E_2) = 0, && S(E_3,E_3) = -2.\end{aligned}$$

From (3.3), we came to that point

$$S(E_1,E_1) = -(1+\lambda), \qquad S(E_2,E_2) = -(1+\lambda), \qquad S(E_3,E_3) = -(\lambda+\mu).$$

Thus, we conclude that for $\lambda = -1$ and $\mu = 3$, the data (g,ξ,λ,μ) admits η–Ricci soliton on $M^3(\phi,\xi,\eta,g)$.

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Author information

S.K. Yadav, Department of Mathematics, Poornima College of Engineering, ISI-6, RIICO Institutional Area, Sitapura, Jaipur, 302022, INDIA.
E-mail: `prof_sky16@yahoo.com`

S.K. Chaubey, Section of Mathematics, Department of Information Technology, Shinas College of Technology,, SULTANATE OF OMAN.
E-mail: `sk22_math@yahoo.co.in`

D.L. Suthar, Department of Mathematics, Wollo University, P.O. Box: 1145, Dessie, South Wollo, Amhara Region,, ETHIOPIA.
E-mail: `dlsuthar@gmail.com`

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