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SOME EXISTENCE AND STABILITY RESULTS FOR INTEGRO-DIFFERENTIAL EQUATION BY HILFER-KATUGAMPOLA FRACTIONAL DERIVATIVE

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Abstract. In this paper, we discuss the existence, uniqueness and stability of integro-differential equation with Hilfer-Katugampola fractional derivative. The arguments are based upon Schauder fixed point theorem, Banach contraction principle and ulam type stability.

1 Introduction

In this paper, we discuss this paper is to establish existence results by Schauder fixed-point theorem and four types of Ulam stability, namely Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability for integro-differential equation involving Hilfer-Katugampola fractional derivative of the form

$$\begin{cases} {}^{\rho}D_{a_+}^{\alpha,\beta}x(t) = f(t, x(t), \int_a^t h(t,s,x(s))ds), \ \ t \in J := (a,b], \\ {}^{\rho}I_{a_+}^{1-\gamma}x(a) = x_0, \ \ \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where ${}^{\rho}D_{a_+}^{\alpha,\beta}$ is Hilfer-Katugampola fractional derivative of order α and type β and ${}^{\rho}I_{a_+}^{1-\gamma}$ is generalized fractional integral of order $1-\gamma$, $\rho > 0$ where $f : J \times R \times R \to R,\ h : \Delta \times R \to R$ are continuous. Here, $\Delta = \{(t,s) : a \leq s \leq t \leq b\}$. For brevity let us take

$$Hx(t) = \int_a^t h(t,s,x(s))ds.$$

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer). In recent years, fractional differential equations(FDEs) arise naturally in various fields such as rheology, fractals, chaotic dynamics, modelling and control theory, signal processing, bioengineering and biomedical applications, etc. Theory of FDEs has been extensively studied by many authors [4, 7, 11, 15, 16]. Recentely, much attention has been paid to existence results for the integro-differential equation see [2, 3, 6]

The stability problem of functional equations (of group homomorphisms) was raised by Ulam in 1940 in a talk given at Wisconsin University [17]. The question posed by Ulam was "Under what conditions does there exist an additive mapping near an approximately additive mapping?" In 1941, Hyers [5] gave the first answer to the question of Ulam (for the additive mapping) in the case Banach spaces. In 1978, Rassias established the Hyers-Ulam stability of linear and nonlinear mapping. He was the first to prove the stability of the linear mapping. This result of Rassias attracted several mathematicians worldwide who began to be stimulated to investigate the stability problems of differential equations [1, 12, 13, 20, 21].

U. N. Katugampola [8] introduced generalized fractional derivative and it has been studied extensively by some researchers [9, 10, 18, 19]. Further a new fractional derivative which is known as Hilfer-Katugampola fractional derivative was introduced in [14], which is the interpolation of Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, generalized and Caputo-type fractional derivatives, as well as Weyl and Liouville fractional derivatives for particular cases of integration extremes.

The paper is organized as follows. In section 2, we present notations and definition used throughout the paper. In Section 3, we discuss the existence and uniqueness results for integro-differential equations. In Section 4, stability results is analyzed.

2 Preliminary

In this section, we recall some definitions and results from fractional calculus. The following observations are taken from [7, 9, 14]. Throughout this paper, let $C[a,b]$ a space of continuous functions from J into R with the norm

$$\|x\|_C = \sup\{|x(t)| : t \in J\}.$$

The weighted space $C_{\gamma,\rho}[a,b]$ of functions f on $(a,b]$ is defined by

$$C_{\gamma,\rho}[a,b] = \left\{ f : (a,b] \to R : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(x) \in C[a,b] \right\}, 0 \le \gamma < 1,$$

with the norm

$$\|f\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(x) \right\|_C = \max_{t \in J} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma f(x) \right|, C_{0,\rho}[a,b] = C[a,b].$$

Let $\delta_\rho = \left(t^\rho \frac{d}{dt}\right)$. For $n \in N$ we denote by $C^n_{\delta_{\rho,\gamma}}[a,b]$ the Banach space of functions f which are continuously differentiable, with the operator δ_ρ, on $[a,b]$ up to $(n-1)$ order and the derivative $\delta^n_\rho f$ of order n on $(a,b]$ such that $\delta^n_\rho f \in C_{\gamma,\rho}[a,b]$, this is

$$C^n_{\delta_{\rho,\gamma}}[a,b] = \left\{ \delta^k_\rho f \in C[a,b], k = 0,1,...,n-1, \delta^n_\rho f \in C_{\gamma,\rho}[a,b] \right\}$$

with the norm

$$\|f\|_{C^n_{\delta_{\rho,\gamma}}} = \sum_{k=0}^{n-1} \|\delta^k_\rho f\|_C + \|\delta^n_\rho f\|_{C_{\gamma,\rho}}, \quad \|f\|_{C^n_{\delta_\rho}} = \sum_{k=0}^{n} \max_{x \in R} |\delta^k_\rho f(x)|.$$

For $n = 0$, we have

$$C^0_{\delta_{\rho,\gamma}}[a,b] = C_{\gamma,\rho}[a,b].$$

Definition 2.1. The generalized left-sided fractional integral ${}^\rho I^\alpha_{a^+} f$ of order $\alpha \in C(\Re(\alpha))$ is defined by

$$\left({}^\rho I^\alpha_{a^+}\right) f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds, \ t > a, \tag{2.1}$$

if the integral exists.

The generalized fractional derivative, corresponding to the generalised fractional integral (2.1), is defined for $0 \le a < t$, by

$$\left({}^\rho D^\alpha_{a^+} f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t (t^\rho - s^\rho)^{n-\alpha+1} s^{\rho-1} f(s) ds, \tag{2.2}$$

if the integral exists.

Definition 2.2. The Hilfer-Katugampola fractional derivative with respect to t, with $\rho > 0$, is defined by

$$\begin{aligned} \left({}^\rho D^{\alpha,\beta}_{a^\pm} f\right)(t) &= \left(\pm {}^\rho I^{\beta(1-\alpha)}_{a^\pm} \left(t^{\rho-1} \frac{d}{dt}\right) {}^\rho I^{(1-\beta)(1-\alpha)}_{a^\pm} f\right)(t) \\ &= \left(\pm {}^\rho I^{\beta(1-\alpha)}_{a^\pm} \delta_\rho {}^\rho I^{(1-\beta)(1-\alpha)}_{a^\pm} f\right)(t). \end{aligned} \tag{2.3}$$

- The operator ${}^\rho D^{\alpha,\beta}_{a^+}$ can be written as

$${}^\rho D^{\alpha,\beta}_{a^+} = {}^\rho I^{\beta(1-\alpha)}_{a^+} \delta_\rho {}^\rho I^{1-\gamma}_{a^+} = {}^\rho I^{\beta(1-\alpha)}_{a^+} {}^\rho D^\gamma_{a^+}, \ \ \gamma = \alpha + \beta - \alpha\beta.$$

- The fractional derivative ${}^\rho D^{\alpha,\beta}_{a^+}$ is considered as interpolator, with the convenient parameters, of the following fractional derivatives

(i) Hilfer fractional derivative when $\rho \to 1$.
(ii) Hilfer-Hadamard fractional derivative when $\rho \to 0$.
(iii) Generalized fractional derivative when $\beta = 0$.
(iv) Caputo-type fractional derivative when $\beta = 1$.
(v) Riemann-Liouville fractional derivative when $\beta = 0, \rho \to 1$.
(vi) Hadamard fractional derivative when $\beta = 0, \rho \to 0$.
(vii) Caputo fractional derivative when $\beta = 1, \rho \to 1$.
(viii) Caputo-Hadamard fractional derivative when $\beta = 1, \rho \to 0$.
(ix) Liouville fractional derivative when $\beta = 0, \rho \to 1, a = 0$.
(x) Hadamard fractional derivative when $\beta = 0, \rho \to 1, a = -\infty$.

- We consider the following parameters $\alpha, \beta, \gamma, \mu$ satisfying

$$\gamma = \alpha + \beta - \alpha\beta,\ 0 \le \gamma < 1,\ 0 \le \mu < 1,\ \alpha > 0,\ \beta < 1.$$

Lemma 2.3. *Let $\alpha, \beta > 0$, $1 \le p \le \infty$, $0 < a < b < \infty$ and $\rho, c \in R$, and $\rho \ge c$. Then, for $f \in X_c^p(a,b)$ the semigroup property is valid. This is,*

$$({}^\rho I_{a^+}^\alpha\, {}^\rho I_{a^+}^\beta f)(x) = ({}^\rho I_{a^+}^{\alpha+\beta})(x),$$

and

$$({}^\rho D_{a^+}^\alpha\, {}^\rho I_{a^+}^\alpha f)(x) = f(x).$$

Lemma 2.4. *Let $x > a$, ${}^\rho I_{a^+}^\alpha$ and ${}^\rho D_{a^+}^\alpha$, according to Eq.(2.1) and (2.2), respectively. Then*

$${}^\rho I_{a^+}^\alpha \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1}, \alpha \ge 0, \beta > 0.$$

$${}^\rho D_{a^+}^\alpha \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}(x) = 0, 0 < \alpha < 1.$$

Lemma 2.5. *Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $f \in C_\gamma$ and ${}^\rho I_{a+}^{1-\alpha} f \in C_\gamma^1[a,b]$, then*

$$\left({}^\rho I_{a^+}^\alpha\, {}^\rho D_{a^+}^\alpha\right)(x) = f(x) - \frac{\left({}^\rho I_{a+}^{1-\alpha} f\right)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1},$$

for all $x \in (a,b]$.

Lemma 2.6. *Let $0 < a < b < \infty$, $\alpha > 0$, $0 \le \gamma < 1$ and $f \in C_{\gamma,\rho}[a,b]$. If $\alpha > \gamma$, then ${}^\rho I_{a+}^\alpha f$ is continuous on $[a,b]$ and*

$$\left({}^\rho I_{a^+}^\alpha f\right)(a) = \lim_{t \to a^+}\left({}^\rho I_{a^+}^\alpha\right) f(t) = 0.$$

In order to solve our problem, the following spaces are presented.

$$C_{1-\gamma,\rho}^{\alpha,\beta}[a,b] = \left\{f \in C_{1-\gamma,\rho}[a,b], {}^\rho D_{a^+}^{\alpha,\beta} f \in C_{\mu,\rho}[a,b]\right\}$$

and

$$C_{1-\gamma,\rho}^{\gamma}[a,b] = \left\{f \in C_{1-\gamma,\rho}[a,b], {}^\rho D_{a^+}^{\gamma} f \in C_{1-\gamma,\rho}[a,b]\right\}.$$

It is obvious that

$$C_{1-\gamma,\rho}^{\gamma}[a,b] \subset C_{1-\gamma,\rho}^{\alpha,\beta}[a,b].$$

Lemma 2.7. *Let $0 < \alpha < 1$, $0 \le \beta \le 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $C_{1-\gamma}^\gamma[a,b]$, then*

$${}^\rho I_{a^+}^\gamma\, {}^\rho D_{a^+}^\gamma f = {}^\rho I_{a^+}^\alpha\, {}^\rho D_{a^+}^{\alpha,\beta} f \tag{2.4}$$

and

$${}^\rho D_{a^+}^\gamma\, {}^\rho I_{a^+}^\alpha f = {}^\rho D_{a^+}^{\beta(1-\alpha)} f. \tag{2.5}$$

Lemma 2.8. *Let $f \in L^1(a,b)$. If ${}^\rho D_{a^+}^{\beta(1-\alpha)} f$ exists on $L^1(a,b)$, then*

$$ {}^\rho D_{a^+}^{\alpha,\beta} {}^\rho I_{a^+}^{\alpha} f = {}^\rho I_{a^+}^{\beta(1-\alpha)} {}^\rho D_{a^+}^{\beta(1-\alpha)} f. $$

Lemma 2.9. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}[a,b]$ and ${}^\rho I_{a^+}^{1-\beta(1-\alpha)} \in C_{1-\gamma}^1[a,b]$, then ${}^\rho D_{a^+}^{\alpha,\beta} f {}^\rho I_{a^+}^{\alpha}$ exists on $(a,b]$ and*

$$ {}^\rho D_{a^+}^{\alpha,\beta} {}^\rho I_{a^+}^{\alpha} f = f. $$

Lemma 2.10. *Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $a \leq t < b$ (some $b \leq \infty$), and let $g(t)$ be a nonnegative, nondecreasing continuous function defined on $a \leq t < b$, such that $g(t) \leq K$ for some constant K. Further let $x(t)$ be a nonnegative locally integrable on $a \leq t < b$ function with*

$$ |x(t)| \leq a(t) + g(t) \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} x(s) ds, \quad t \in J $$

with some $\alpha > 0$. Then

$$ |x(t)| \leq a(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{n\alpha-1} s^{\rho-1} \right] x(s) ds, \quad a \leq t < b. $$

The proof of above lemma is similar to Theorem 1 in [22].

Lemma 2.11. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. If $f : J \times R \times R \to R$ is a function such that $f(\cdot, x(\cdot), Hx(\cdot)) \in C_{1-\gamma}[a,b]$ for all $x \in C_{1-\gamma}[a,b]$. A function $x \in C_{1-\gamma}^{\gamma}[a,b]$ is the solution of fractional initial value problem*

$$ \begin{cases} {}^\rho D_{a_+}^{\alpha,\beta} x(t) = f(t, x(t), Hx(t)), \; 0 < \alpha < 1, \; 0 \leq \beta \leq 1, \\ {}^\rho I_{a_+}^{1-\gamma} x(a) = x_0, \end{cases} $$

if and only if x satisfies the following Volterra integral equation

$$ x(t) = \frac{x_0}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds. \tag{2.6} $$

3 Existence results

We make the following hypotheses to prove our main results.

(H1) For all $x_1, x_2, y_1, y_2 \in R$, there exists a positive constant $L > 0$ such that

$$ |f(t, x_1, x_2) - f(t, y_1, y_2)| \leq L(|x_1 - y_2| + |x_2 - y_2|). $$

(H2) : Let $h : \Delta \times R \to R$ be continuous and there exists a constant $H > 0$, such that

$$ \int_0^t |h(t,s,x) - h(t,s,y)| \leq H |x - y|. $$

(H3) Let $f : J \times R \times R \to R$ a function and there exists a function $\mu \in C[a,b]$ such that

$$ |f(t,x,y)| \leq \mu(t), \; \forall \, t \in J, \; x, y \in R. $$

Theorem 3.1. *Assume that [H1] - [H3] are satisfied. Then, (1.1) has at least one solution.*

Proof. Consider the operator $N : C_{1-\gamma,\rho}[a,b] \to C_{1-\gamma,\rho}[a,b]$. The equivalent integral equation (2.6) which can be written in the operator form

$$ x(t) = Nx(t) $$

where

$$(Nx)(t) = \frac{x_0}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s))ds. \quad (3.1)$$

Consider the ball

$$B_r = \{x \in C_{1-\gamma,\rho}[a,b] : \|x\| \le r\}$$

It is obvious that the operator N is well defined. Clearly, the fixed points of the operator N are solutions of the problem. For any $x \in C_{1-\gamma,\rho}[a,b]$ and each $t \in J$ we have,

$$\begin{aligned}
\left|(Nx)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| &= \left|\frac{x_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s))ds\right| \\
&\le \frac{x_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|f(s, x(s), Hx(s))\right| ds \\
&\le \frac{x_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|\mu(s)\right| ds \\
&\le \frac{x_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1} B(\gamma,\alpha)\left\|\mu\right\|_{C_{1-\gamma,\rho}} \\
&\le \frac{x_0}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha} B(\gamma,\alpha)\left\|\mu\right\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

This proves that N transforms the ball $B_r = \left\{x \in C_{1-\gamma,\rho}[a,b] : \|x\|_{C_{1-\gamma,\rho}} \le r\right\}$ into itself. The proof is divided into several steps.

Step 1: The operator N is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma,\rho}[a,b]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|((Nx_n)(t) - (Nx)(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\le \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, x_n(s), Hx_n(s))ds\right. \\
&\quad \left. -\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s))ds\right| \\
&\le \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|f(s, x_n(s), Hx_n(s)) - f(s, x(s), Hx(s))\right| ds\right| \\
&\le \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1} B(\gamma,\alpha)\left\|f(\cdot, x_n(\cdot), Hx_n(\cdot)) - f(\cdot, x(\cdot), Hx(\cdot))\right\|_{C_{1-\gamma,\rho}},
\end{aligned}$$

which implies

$$\|Nx_n - Nx\|_{C_{1-\gamma,\rho}} \le B(\gamma,\alpha)\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\frac{1}{\Gamma(\alpha)}\left\|f(\cdot, x_n(\cdot), Hx_n(\cdot)) - f(\cdot, x(\cdot), Hx(\cdot))\right\|_{C_{1-\gamma,\rho}}.$$

It implies that N is continuous.

Step 2: $N(B_r)$ is uniformly bounded.

It is clear that $N(B_r) \subset B_r$ is bounded.

Step 3: $N(B_r)$ is relatively compact.

It follows from $N(B_r) \subset B_r$ that $N(B_r)$ is uniformly boundeed. Moreover, to show that N is an equicontinuous operator. Let $t_1, t_2 \in J, t_1 < t_2, B_r$ be a bounded set of $C_{1-\gamma,\rho}[a,b]$. Then,

$$\begin{aligned}
&|((Nx)(t_1) - (Nx)(t_2))| \\
&\le \frac{x_0}{\Gamma(\gamma)}\left|\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\gamma-1}\right| + \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1}\right)\|f\|_{C_{1-\gamma,\rho}}
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of claim 1 to 3, together with Arzela-Ascoli theorem, we can conclude that $N : C_{1-\gamma,\rho}[a,b] \to C_{1-\gamma,\rho}[a,b]$ is continuous and completely continuous.

□

Theorem 3.2. *Assume that hypothesis (H1) and (H2) are fulfilled. If*

$$\frac{L(1+H)}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^\alpha B(\gamma.\alpha) < 1$$

then, Eq. (1.1) *has unique solution.*

4 Stability Analysis

Next, we shall give the definitions and the criteria of Ulam-Hyers stability and Ulam-Hyers-Rassias stability for fractional integro-differential Eq.(1.1). Let $\epsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities

$$\left|{}^\rho D^{\alpha,\beta}_{a_+} z(t) - f(t, z(t), Hz(t))\right| \le \epsilon, \quad t \in J, \tag{4.1}$$

$$\left|{}^\rho D^{\alpha,\beta}_{a_+} z(t) - f(t, z(t), Hz(t))\right| \le \epsilon\varphi(t), \quad t \in J, \tag{4.2}$$

$$\left|{}^\rho D^{\alpha,\beta}_{a_+} z(t) - f(t, z(t), Hz(t))\right| \le \varphi(t), \quad t \in J. \tag{4.3}$$

Definition 4.1. Eq. (1.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.1) there exists a solution $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le C_f\, \epsilon, \quad t \in J.$$

Definition 4.2. Eq. (1.1) is generalized Ulam-Hyers stable if there exist $\varphi \in C_{1-\gamma,\rho}[a,b]$, $\varphi_f(0) = 0$ such that for each solution $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.1) there exists a solution $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le \varphi_f\, \epsilon, \quad t \in J.$$

Definition 4.3. Eq. (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\rho}[a,b]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.2) there exists a solution $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le C_{f,\varphi}\, \epsilon\varphi(t), \quad t \in J.$$

Definition 4.4. Eq. (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C_{1-\gamma,\rho}[a,b]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.3) there exists a solution $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) with

$$|z(t) - x(t)| \le C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5. Clearly,
1. Definition 4.1 $\Rightarrow$ Definition 4.2.
2. Definition 4.3 $\Rightarrow$ Definition 4.4.
3. Definition 4.3 for $\varphi(t) = 1 \Rightarrow$ Definition 4.1

Remark 4.6. A function $z \in C_{1-\gamma,\rho}[a,b]$ is a solution of the inequality (4.1) if and only if there exists a function $g \in C_{1-\gamma,\rho}[a,b]$ such that

$$\left|{}^\rho D^{\alpha,\beta}_{a_+} z(t) - f(t, z(t), Hz(t))\right| \le \epsilon, \quad t \in J,$$

if and only if there exist a function $g \in C_{1-\gamma,\rho}[a,b]$ such that

(i) $|g(t)| \leq \epsilon, t \in J$.

(ii) ${}^{\rho}D^{\alpha,\beta}_{a_+} z(t) = f(t, z(t), Hz(t)) + g(t),\ t \in J$.

One can have similar remarks for the inequalities (4.2) and (4.3).

Remark 4.7. Let $0 < \alpha < 1$, if z is solution of the inequality (4.1) then z is a solution of the following integral inequality

$$\left| z(t) - \frac{z_0}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} f(s, z(s), Hz(s)) ds \right| \leq \epsilon \frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)}.$$

Indeed, by Remark 4.6 we have that

$${}^{\rho}D^{\alpha,\beta}_{a_+} z(t) = f(t, z(t), Hz(t)) + g(t),\ t \in J.$$

Then

$$z(t) = \frac{z_0}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \left(f(s, z(s), Hz(s)) + g(s) \right) ds.$$

From this it follows that

$$\begin{aligned} &\left| z(t) - \frac{z_0}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} f(s, z(s), Hz(s)) ds \right| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} g(s) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} |g(s)| \, ds \\ &\leq \epsilon \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha} \frac{1}{\Gamma(\alpha+1)}. \end{aligned}$$

We have similar remarks for the inequality (4.2) and (4.3).
Now, we give the main results, generalised Ulam-Hyers-Rassias stable results, in this section.

[H3]: There exists an increasing finctions $\varphi \in C_{1-\gamma,\rho}[a, b]$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$${}^{\rho}I^{\alpha}_{a_+} \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Theorem 4.8. *The hypothesis [H1] and [H2] holds. Then Eq.*(1.1) *is generalised Ulam-Hyers-Rassias stable.*

Proof. Let z be solution of 4.3 and by Theorem 3.2 there x is unique solution of the problem

$$\begin{aligned} &{}^{\rho}D^{\alpha,\beta}_{a_+} x(t) = f(t, x(t), Hx(t)), \quad t \in J, \\ &{}^{\rho}I^{1-\gamma}_{a_+} x(a) = {}^{\rho}I^{1-\gamma}_{a_+} z(a). \end{aligned}$$

Then we have

$$x(t) = \frac{z_0}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds.$$

By differentiating inequality (4.3), we have

$$\left| z(t) - \frac{z_0}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} f(s, z(s), Hz(s)) ds \right| \leq \lambda_\varphi \varphi(t).$$

Hence it follows

$$|z(t)-x(t)| \le \left| z(t) - \frac{z_0}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s,x(s),Hx(s))ds \right|$$

$$\le \left| z(t) - \frac{z_0}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s,z(s),Hz(s))ds \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s,z(s),Hz(s))ds \right.$$

$$\left. - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s,x(s),Hx(s))ds \right|$$

$$\le \lambda_\varphi \varphi(t) + \frac{L(1+H)}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |z(s)-x(s)|\, ds.$$

By Lemma 2.10, there exists a constant $M^* > 0$ independent of $\lambda_\varphi \varphi(t)$ such that

$$|z(t)-x(t)| \le M^* \lambda_\varphi \varphi(t) := C_{f,\varphi}\varphi(t).$$

Thus, Eq.(1.1) is generalized Ulam-Hyers-Rassias stable. □

Remark 4.9. (i) Under the assumption of Theorem 4.8, we consider (1.1) and the inequality (4.2). One can repeat the same process to verify that Eq.(1.1) is Ulam-Hyers-Rassias stable.

(ii) Under the assumption of Theorem 4.8, we consider (1.1) and the inequality (4.1). One can repeat the same process to verify that Eq.(1.1) is Ulam-Hyers stable.

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