Palestine Journal of Mathematics

Vol. 9(1)(2020) , 274–288

Strict Coincidence and Common Strict Fixed Point of a Faintly Compatible Hybrid Pair of Maps via C-class function and Applications

Anita Tomar, Ritu Sharma and Arslan Hojat Ansari

Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: Primary 47H10; Secondary 54H25.

Keywords and phrases: Boundary value problem, common strict fixed point, conditional reciprocal continuity, faint compatibility and hybrid pair.

Abstract Some strict coincidence and common strict fixed point results are obtained acknowledging the notion of faint compatibility, recently introduced by Tomar et al. [Coincidence and common strict fixed point of a faintly compatible hybrid pair of mappings, Electron. J. Math. Anal. Appl., 5(2), **(2017)**, 298-305], via C-class functions that covers a large class of contractive conditions. Our results do not rely on the completeness of space / subspace, the continuity or the containment of range space of involved hybrid pair of single valued and multi-valued maps. Some interesting examples are furnished and the solutions of an initial value problem and a boundary value problem are given to demonstrate the usability of results obtained.

1 Introduction

The notion of compatibility is useful for generalizing the results in the metric fixed point theory of a continuous single valued, multi-valued or hybrid pair of maps. Recently Tomar et al. [12] introduced the notions of conditional compatibility, faint compatibility and conditional reciprocal continuity for a hybrid pair of discontinuous maps in a metric space and utilized these relatively weaker notions to establish strict coincidence and common strict fixed point of a hybrid pair using δ-distance. Aim of this paper is to demonstrate the applicability of faint compatibility and conditional reciprocal continuity for the significant C-class functions introduced by Ansari [1] that covers a large class of contractive conditions. Significance of the paper lies in establishing strict coincidence and common strict fixed point of a discontinuous faintly compatible hybrid pair of maps under non-expansive, contraction, strict contractive as well as non-contractive conditions in a non-complete metric space. This appears to be extremely important in view of the fact that even a continuous and commuting pair of single valued maps on a complete metric space need not have a coincidence or common fixed point. For instance if $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are single valued self maps such that $fx = x$ and $gx = x + 1$ for all $x \in \mathbb{R}$, then neither f and g have a coincidence point nor common fixed point. However both the maps are continuous and the pair (f, g) is commuting. On the other hand motivated by the fact that the study of two-point boundary value problem and initial value problem associated with the second order differential equation plays an important role in the real world physical problems and scientific research, we solve a two-point boundary value problem and an initial value problem of the second order differential equations arising in steady state heat flow in rod and spring mass system respectively.

2 Preliminaries

Throughout this paper, let (X, d) be a metric space and $CB(X)$ be the family of all non-empty closed and bounded subsets of X. For $A, B \in CB(X)$, functions $\delta(A, B)$ and $D(A, B)$ are defined as: $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ and $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Let H be the Hausdorff metric with respect to d, i.e.,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\},$$

where $d(x,A) = \inf\{d(x,y) : y \in A\}$, for all $A, B \in CB(X)$.
If $A = \{a\}$, then $\delta(A,B) = \delta(a,B) = D(a,B) = H(a,B)$.
If $A = \{a\}$ and $B = \{b\}$, then $\delta(A,B) = d(a,b) = D(A,B) = H(A,B)$.
It follows immediately from the definition of δ that

- $\delta(A,B) = \delta(B,A) > 0$;
- $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$;
- $\delta(A,B) = 0 \iff A = B = \{a\}$;
- $\delta(A,A) = \text{diam } A$.

If $f : X \to X$ is a single valued and $T : X \to CB(X)$ is a multivalued map of a metric space (X,d) then a pair (f,T) is known as a hybrid pair. For a hybrid pair (f,T), a point $u \in X$ is a

- coincidence point if $fu \in Tu$;
- strict coincidence point if $Tu = \{fu\}$;
- common fixed point if $u = fu \in Tu$;
- common strict fixed point if $Tu = \{fu\} = \{u\}$.

Definition 2.1. [4] A hybrid pair of maps (f,T) of a metric space (X,d) is compatible if $fTx \in CB(X)$ for all $x \in X$ and $\lim_{n\to\infty} H(fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$.

Definition 2.2. [7] A pair of single valued maps (f,g) of a metric space (X,d) is conditionally compatible if whenever the sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non-empty, there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t \in X$ and $\lim_{n\to\infty} d(fgy_n, gfy_n) = 0$.

Definition 2.3. [2] A pair of single valued maps (f,g) of a metric space (X,d) is faintly compatible iff f and g are conditionally compatible and f and g commute on a non-empty subset of coincidence points whenever the set of coincidences is nonempty.

Definition 2.4. [12] A hybrid pair of maps (f,T) is conditionally compatible iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$ is non-empty, there exists a sequence $\{y_n\}$ such that $\lim_{n\to\infty} fy_n = u \in B = \lim_{n\to\infty} Ty_n$ and $\lim_{n\to\infty} \delta(fTy_n, Tfy_n) = 0$ for some $u \in X$ and $B \in CB(X)$.

Definition 2.5. [12] A hybrid pair of maps (f,T) is faintly compatible iff f and T are conditionally compatible and f and T commute on a non-empty subset of coincidence points whenever the set of coincidences is non-empty, i.e., if $C(f,T) \neq \emptyset$ then there exists $x \in M \subseteq C(f,T)$ such that $fx \in Tx$ and $fTx \subseteq Tfx$.

Faint compatibility is an improvement of conditional compatibility that allows the existence of a common fixed point or multiple common fixed point or coincidence points or multiple coincidence points under both contractive and non-contractive conditions for single valued maps (Example 3.2 of Tomar et al. [11]). Also it does not reduce to the class of commutativity at point of coincidence like other weaker forms of commutativity (Singh and Tomar [9]).

Definition 2.6. [3] A pair of single valued maps (f,g) of a metric space (X,d) is conditionally reciprocally continuous iff whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non empty, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = t$ for some $t \in X$ such that $\lim_{n\to\infty} fgy_n = ft$ and $\lim_{n\to\infty} gfy_n = gt$.

Conditional reciprocal continuity is weaker than most of the variants of continuity existing in literature. For a brief development of variants of continuity for single valued maps and the relation between them one may refer to Tomar and Karapinar [10].

Definition 2.7. [12] A hybrid pair of maps (f, T) is called conditionally reciprocally continuous iff the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ where $t \in X$ and $A \in CB(X)$ is non-empty, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = u \in B = \lim_{n\to\infty} Ty_n$, for some $u \in X$ and $B \in CB(X)$ such that $\lim_{n\to\infty} fTy_n = fB$ and $\lim_{n\to\infty} Tfy_n = Tu$.

Definition 2.8. [1] A map $F : [0,\infty)^2 \to \mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \le s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0,\infty)$.

Note for some F we have $F(0,0) = 0$.
We denote C-class functions as $\mathcal{C}$.

Example 2.9. [1] The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0,\infty)$:

(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;
(2) $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;
(3) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $F(s,t) = \log(t + a^s)/(1 + t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(5) $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;
(6) $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;
(7) $F(s,t) = s \log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;
(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;
(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to (0,1)$ and is continuous, $F(s,t) = s \Rightarrow s = 0$;
(10) $F(s,t) = s - \frac{t}{k+t}$, $F(s,t) = s \Rightarrow t = 0$;
(11) $F(s,t) = s - \varphi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(12) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, here $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;
(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$;
(14) $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$, $F(s,t) = s \Rightarrow s = 0$;
(15) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$;
(16) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$.

Definition 2.10. [5] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.11. [1] An ultra altering distance function is a continuous, non-decreasing map $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \ge 0$.

We denote this set with Φ_u.

3 Main results

In all that follows $f : X \to X$ is a single valued, $T : X \to CB(X)$ is a multivalued and (f, T) is a hybrid pair of maps of a metric space (X, d) unless otherwise specifically mentioned. Further, ψ altering distance function, φ is ultra altering distance function and F is a C-class function .
Now as an application of faint compatibility we state and prove our first main result for non expansive type condition.

Theorem 3.1. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) satisfies*

$$\psi\left(\delta\left(Tx,Ty\right)\right)\leq F\Big(\psi\left(d\left(fx,fy\right)\right),\varphi\left(d\left(fx,fy\right)\right)\Big), \tag{3.1}$$

where $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$. If f is continuous then f and T have a unique common strict fixed point.

Proof. Faint compatibility of a hybrid pair (f,T) implies that it is conditionally compatible, i.e., there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = t\in A = \lim_{n\to\infty} Tx_n.$$

for some $t\in X$ and $A\in CB(X)$.
Also there exists a sequence $\{y_n\}$ in X satisfying $\lim\limits_{n\to\infty} fy_n = u\in B = \lim\limits_{n\to\infty} Ty_n$ such that $\lim\limits_{n\to\infty}\delta(fTy_n,Tfy_n)=0$ for some $u\in X$ and $B\in CB(X)$.
Further, since f is continuous $\lim\limits_{n\to\infty} ffy_n = fu$ and $\lim\limits_{n\to\infty} fTy_n = fB$. Thus $\lim\limits_{n\to\infty} Tfy_n = fB$.
By putting $x=u$ and $y=fy_n$ in condition (3.1), we get

$$\psi\left(\delta(Tu,Tfy_n)\right)\leq F\Big(\psi\left(d\left(fu,ffy_n\right)\right),\varphi\left(d\left(fu,ffy_n\right)\right)\Big).$$

Taking $\lim n\to\infty$ we get,

$$\psi\left(\delta(Tu,fB)\right)\leq F\Big(\psi\left(d\left(fu,fu\right)\right),\varphi\left(d\left(fu,fu\right)\right)\Big)\leq\psi\left(d\left(fu,fu\right)\right)=\psi\left(0\right)=0,$$

i.e., $\delta(Tu,fB)=0$ or $Tu=fB$.
Since, $u\in B$, $fu\in fB=Tu=\{fu\}$, i.e., fu is a strict coincidence point of f and T.
Further, faint compatibility implies $fTu\subseteq Tfu$.
For $x=fu$ and $y=u$, condition (3.1) gives,

$$\psi\left(\delta(Tfu,Tu)\right)\leq F\Big(\psi\left(d\left(ffu,fu\right)\right),\varphi\left(d\left(ffu,fu\right)\right)\Big).$$

Since, $fu\in Tu$, $ffu\in fTu\subseteq Tfu$.

$$\begin{aligned}\psi\left(d(ffu,fu)\right) &\leq \psi\left(\delta(Tfu,Tu)\right)\\ &\leq F\Big(\psi\left(d\left(ffu,fu\right)\right),\varphi\left(d\left(ffu,fu\right)\right)\Big).\end{aligned}$$

But $F\Big(\psi(d(ffu,fu)),\varphi(d(ffu,fu))\Big)<\psi(d(ffu,fu))$ gives $\psi(d(ffu,fu))<\psi(d(ffu,fu))$, a contradiction.
So $F\Big(\psi(d(ffu,fu)),\varphi(d(ffu,fu))\Big)=\psi(d(ffu,fu))$,
i.e., $\psi\left(d(ffu,fu)\right)=0$ or $\varphi\left(d(ffu,fu)\right)=0$. Thus $d(ffu,fu)=0$.
Hence, $\{fu\}=\{ffu\}=Tfu$, i.e., fu is a common strict fixed point of f and T.
For uniqueness, suppose that w is also a common strict fixed point other than $fu=z$.
Then by using condition (3.1), we have

$$\psi\left(\delta(Tz,Tw)\right)\leq F\Big(\psi\left(d\left(fz,fw\right)\right),\varphi\left(d\left(fz,fw\right)\right)\Big).$$

Since, $z=fz\in Tz$ and $w=fw\in Tw$. Therefore,

$$\begin{aligned}\psi\left(d(fz,fw)\right) &\leq \psi\left(\delta(Tz,Tw)\right)\\ &\leq F\Big(\psi\left(d\left(fz,fw\right)\right),\varphi\left(d\left(fz,fw\right)\right)\Big).\end{aligned}$$

But $F\Big(\psi(d(fz,fw)),\varphi(d(fz,fw))\Big) < \psi(d(fz,fw))$ gives $\psi(d(fz,fw)) < \psi(d(fz,fw))$, a contradiction.
So $F\Big(\psi(d(fz,fw)),\varphi(d(fz,fw))\Big) = \psi(d(fz,fw))$, i.e., $\psi\left(d(fz,fw)\right) = 0$ or $\varphi\left(d(fz,fw)\right) = 0$. Thus $d(fz,fw) = 0$.
Hence, $z = w$, i.e., fu is a unique common strict fixed point of f and T.

Now we furnish a example to demonstrate the validity of Theorem 3.1.

Example 3.2. Let $X = [0,12]$, d be the usual metric on X. Let a hybrid pair of maps (f,T) on X be defined as follows:

$$fx = \begin{cases} 2x-2 & 0 \le x < 2 \\ \frac{11x+2}{12} & 2 \le x \le 12, \end{cases} \qquad Tx = \begin{cases} [\frac{8-x}{3}, \frac{12-x}{4}] & 0 \le x < 2 \\ \{2\} & 2 \le x \le 12. \end{cases}$$

Then one may verify that f and T satisfy condition (3.1) of Theorem 3.1 on taking $F(s,t) = ks$ and $\psi\left(t\right) = t$.
Let $\{x_n\}$ be a sequence in X where $x_n = 2 - \frac{1}{n}$ and $\lim_{n\to\infty} fx_n = 2 \in [2, \dfrac{5}{2}] = \lim_{n\to\infty} Tx_n$ such that $\lim_{n\to\infty} \delta(fTx_n, Tfx_n) = \lim_{n\to\infty} \delta([2, \dfrac{59}{24}], \{2\}) \neq 0$, i.e., pair of maps (f,T) is noncompatible.

Let $\{y_n\}$ be a sequence in X where $y_n = 2$ and $\lim_{n\to\infty} fy_n = 2 \in \{2\} = \lim_{n\to\infty} Ty_n$ such that $\lim_{n\to\infty} \delta(fTy_n, Tfy_n) = \lim_{n\to\infty} \delta(2, \{2\}) = 0$. Also $fT2 \subseteq Tf2$ and strict coincidence point $2 \in X$. Thus a pair of maps (f,T) is faintly compatible. Here one may verify that f is continuous and T is discontinuous at $x = 2$.
Hence, f and T satisfy all the conditions of Theorem 3.1 and have a unique common strict fixed point at $x = 2$. Moreover, $fX \not\subseteq TX$.

On taking $F(s,t) = s - t$, in Theorem 3.1 we obtain the following Corollary:

Corollary 3.3. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) satisfies*

$$\psi\left(\delta\left(Tx,Ty\right)\right) \le \psi(d\left(fx,fy\right)) - \varphi\left(d\left(fx,fy\right)\right),$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance functions, $\varphi \in \phi_u$ and $F \in \mathcal{C}$. If f is continuous then f and T have a unique common strict fixed point.

On taking $F(s,t) = \frac{s}{(1+t)^r}$, in Theorem 3.1 we obtain the following Corollary:

Corollary 3.4. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) satisfies*

$$\psi\left(\delta\left(Tx,Ty\right)\right) \le \frac{\psi(d\left(fx,fy\right))}{\left(1+\varphi\left(d\left(fx,fy\right)\right)\right)^r},$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. If f is continuous then f and T have a unique common strict fixed point.

Corollary 3.5. *Let a faintly compatible single valued pair (f,g) of a metric space (X,d) satisfies*

$$\psi\left(d\left(gx,gy\right)\right) \le F\Big(\psi(d(fx,fy)),\varphi(d(fx,fy))\Big),$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. If f is continuous then f and g have a unique common fixed point.

Following result for contraction type condition is slightly more fascinating and follows on the similar lines as in Theorem 3.1.

Theorem 3.6. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) satisfies*

$$\psi(\delta(Tx,Ty)) \leq kF\Big(\psi(d(fx,fy)),\varphi(d(fx,fy))\Big), 0 \leq k < 1, \tag{3.2}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$.
If f is continuous then f and T have a unique common strict fixed point.

It is interesting to see that main result of Tomar et al. [12] can be obtained as the Corollary of Theorem 3.6 via C-class function.

Corollary 3.7. *[12] Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) satisfies*

$$\delta(Tx,Ty) \leq kd(fx,fy), 0 \leq k < 1.$$

If f is continuous then f and T have a unique common strict fixed point.

Next result is proved for strict contractive condition. It is worth mentioning here that a pair of continuous maps satisfying strict contractive condition may fail to have a common fixed point even on a complete metric space .

Theorem 3.8. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) satisfies*

$$\psi(\delta(Tx,Ty)) < F\Big(\psi(d(fx,fy)),\varphi(d(fx,fy))\Big), \tag{3.3}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. If f is continuous then f and T have a unique common strict fixed point.

Now we authenticate the applicability of faint compatibility and conditional reciprocal continuity to establish strict coincidence and unique common strict fixed point of a discontinuous hybrid pair of self maps satisfying non expansive type condition via C-class functions.

Theorem 3.9. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a unique common strict fixed point provided that the pair satisfies*

$$\psi\left(\delta\left(Tx,Ty\right)\right) \leq F\Big(\psi\left(m(x,y)\right),\varphi\left(m(x,y)\right)\Big), \tag{3.4}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$, $F \in \mathcal{C}$ and

$$m(x,y) = \max\Big\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{1}{2}[D(fx,Ty) + D(fy,Tx)]\Big\}.$$

Proof. Since, the hybrid pair (f,T) is conditionally reciprocally continuous, there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$ where $t \in X$ and $A \in CB(X)$.
Also there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = u \in B = \lim_{n\to\infty} Ty_n$, such that $\lim_{n\to\infty} fTy_n = fB$ and $\lim_{n\to\infty} Tfy_n = Tu$.
Also, since the pair (f,T) is faintly compatible. It is also conditionally compatible, i.e., $\lim_{n\to\infty} \delta(fTy_n, Tfy_n) = 0$. Hence, $\delta(fB,Tu) = 0$, i.e., $fB = Tu$.
Now $u \in B$ implies $fu \in fB = Tu = \{fu\}$, i.e., f and T have a strict coincidence point. So $C(T,f) \neq 0$.
Hence, there exists $u \in M \subseteq C(f,T)$ such that $ffu \in fTu \subseteq Tfu$.

Next, we prove that fu is a common strict fixed point of f and T. Suppose that $ffu \neq fu$. Then by using the condition (3.4), we have

$$\begin{aligned}
& \psi\left(\delta(Tfu, Tu)\right) \\
\leq\ & F(\psi\left(\max\{d(ffu, fu), D(ffu, Tfu), D(fu, Tu), \frac{1}{2}[D(ffu, Tu) + D(fu, Tfu)]\}\right), \\
& \varphi\left(\max\{d(ffu, fu), D(ffu, Tfu), D(fu, Tu), \frac{1}{2}[D(ffu, Tu) + D(fu, Tfu)]\}\right)) \\
=\ & F(\psi\left(\max d(ffu, fu), 0, 0, \frac{1}{2}[D(ffu, Tu) + D(fu, Tfu)]\}\right), \\
& \varphi\left(\max d(ffu, fu), 0, 0, \frac{1}{2}[D(ffu, Tu) + D(fu, Tfu)]\}\right)).
\end{aligned}$$

Since, $fu \in Tu$, $D(ffu, Tu) \leq d(ffu, fu)$ and $ffu \in fTu$, $D(fu, Tfu) \leq d(ffu, fu)$. Therefore,

$$\begin{aligned}
\psi\left(d(ffu, fu)\right) &\leq \psi\left(\delta(Tfu, Tu)\right) \\
&\leq F(\psi\left(d\left(ffu, fu\right)\right), \varphi\left(d\left(ffu, fu\right)\right)).
\end{aligned}$$

But $F\bigg(\psi(d(ffu, fu)), \varphi(d(ffu, fu))\bigg) < \psi(d(ffu, fu))$ gives $\psi(d(ffu, fu)) < \psi(d(ffu, fu))$, a contradiction.
So $F\bigg(\psi(d(ffu, fu)), \varphi(d(ffu, fu))\bigg) = \psi(d(ffu, fu))$,
i.e., $\psi\left(d(ffu, fu)\right) = 0$ or $\varphi\left(d(ffu, fu)\right) = 0$. Thus $d(ffu, fu) = 0$.
Hence, $\{fu\} = \{ffu\} = Tfu$, i.e., fu is a common strict fixed point of f and T.
For uniqueness, suppose that w is also a common strict fixed point other than $fu = z$. Then by using condition (3.4), we have

$$\begin{aligned}
& \psi\left(\delta(Tz, Tw)\right) \\
\leq\ & F(\psi\left(\max\{d(fz, fw), D(fz, Tu), D(fw, Tw), \frac{1}{2}[D(fu, Tw) + D(fw, Tu)]\}\right), \\
& \varphi\left(\max\{d(fz, fw), D(fz, Tu), D(fw, Tw), \frac{1}{2}[D(fu, Tw) + D(fw, Tu)]\}\right)) \\
=\ & F(\psi\left(\max\{d(fz, fw), 0, 0, \frac{1}{2}[D(fz, Tw) + D(fw, Tz)]\}\right), \\
& \varphi\left(\max\{d(fz, fw), 0, 0, \frac{1}{2}[D(fz, Tw) + D(fw, Tz)]\}\right)).
\end{aligned}$$

Since, $fz \in Tz$, $D(fw, Tz) \leq d(fw, fz)$ and $w \in fw$, $D(fz, Tw) \leq d(fz, fw)$. Therefore,

$$\begin{aligned}
\psi\left(d(fz, fw)\right) &\leq \psi\left(\delta(Tz, Tw)\right) \\
&\leq F(\psi\left(d\left(fz, fw\right)\right), \varphi\left(d\left(fz, fw\right)\right)).
\end{aligned}$$

But $F(\psi(d(fz, fw)), \psi(d(fz, fw))) < \psi(d(fz, fw))$ gives $\psi(d(fz, fw)) < \psi(d(fz, fw))$, a contradiction.
So $F(\psi(d(fz, fw)), \psi(d(fz, fw))) = \psi(d(fz, fw))$, i.e., $\psi\left(d(fz, fw)\right) = 0$ or $\varphi\left(d(fz, fw)\right) = 0$. Thus $d(fz, fw) = 0$.
Hence, $z = w$, i.e., fu is a unique common strict fixed point of f and T.

Now we furnish example to demonstrate the validity of Theorem 3.9.

Example 3.10. Let $X = [0, 12]$, d be the usual metric on X. Let a hybrid pair of map (f, T) on X be defined as follows:

$$fx = \begin{cases} 4 - x & 0 < x \leq 2 \\ \frac{x+10}{2} & 2 < x \leq 12, \end{cases} \qquad Tx = \begin{cases} \{2\} & 0 < x \leq 2 \\ [\frac{5}{4}, \frac{3}{2}] & 2 < x \leq 12. \end{cases}$$

Then one may verify that f and T satisfy condition (3.4) of Theorem 3.9 on taking $F(s, t) = ks$, $\psi(t) = t$.
Consider a sequence $\{x_n\}$ in X satisfying $x_n = 2 - \frac{1}{n}$ and $\lim_{n\to\infty} fx_n = 2 \in \{2\} = \lim_{n\to\infty} Tx_n$ such that $\lim_{n\to\infty} Tfx_n = [\frac{5}{4}, \frac{3}{2}] \neq T\{2\}$ and $\lim_{n\to\infty} fTx_n = 2 = f2$, i.e., pair of maps (f, T) is not reciprocally continuous. Also $\lim_{n\to\infty} \delta(fTx_n, Tfx_n) \neq 0$, i.e., pair of maps (f, T) is non-compatible.
Let $\{y_n\}$ be a sequence in X where $y_n = 2$ and $\lim_{n\to\infty} fy_n = 2 \in \{2\} = \lim_{n\to\infty} Ty_n$ such that $\lim_{n\to\infty} Tfy_n = \{2\} = T2$ and $\lim_{n\to\infty} fTy_n = 2 = f2$, i.e., $\lim_{n\to\infty} \delta(fTy_n, Tfy_n) = 0$. Also $fT2 \subseteq Tf2$ and strict coincidence point $2 \in X$. Thus pair of maps (f, T) is conditionally reciprocally continuous and faintly compatible.
Hence, f and T satisfy all the conditions of Theorem 3.9 and have a unique common strict fixed point at $x = 2$. Here one may verify that f and T are discontinuous at $x = 2$. Moreover, $fX \nsubseteq TX$.

On taking $F(s, t) = s - t$, in Theorem 3.9 we obtain the following Corollary:

Corollary 3.11. *Let a faintly compatible hybrid pair (f, T) of a metric space (X, d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a unique common strict fixed point provided that the pair satisfies*

$$\psi\left(\delta\left(Tx, Ty\right)\right) \leq \psi\left(m(x, y)\right) - \varphi\left(m(x, y)\right),$$

where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function, $\varphi \in \Phi_u$, $F \in \mathcal{C}$ and

$$m(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}.$$

On taking $F(s, t) = \frac{s}{(1+t)^r}$, in Theorem 3.9 we obtain the following Corollary:

Corollary 3.12. *Let a faintly compatible hybrid pair (f, T) of a metric space (X, d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a unique common fixed point provided that the pair satisfies*

$$\psi\left(\delta\left(Tx, Ty\right)\right) \leq \frac{\psi\left(m(x, y)\right)}{\left(1 + \varphi\left(m(x, y)\right)\right)^r},$$

where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function, $\varphi \in \Phi_u$, $F \in \mathcal{C}$ and

$$m(x, y) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{1}{2}[D(fx, Ty) + D(fy, Tx)]\}.$$

Corollary 3.13. *Let a faintly compatible pair of single valued maps (f, T) of a metric space (X, d) be conditionally reciprocally continuous. Then f and T have a coincidence point. Moreover f and T have a unique common fixed point provided that the pair satisfies*

$$\psi(d(Tx, Ty)) \leq F(\psi\left(m(x, y)\right), \varphi\left(m(x, y)\right)),$$

where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function, $\varphi \in \Phi_u$, $F \in \mathcal{C}$ and

$$m(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\}.$$

Following the similar pattern as in Theorem 3.9 we obtain the following result for contraction type condition.

Theorem 3.14. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a unique common strict fixed point provided that the pair satisfies*

$$\psi(d(Tx,Ty)) \leq kF(\psi(m(x,y)), \phi(m(x,y))), 0 \leq k < 1, \tag{3.5}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u, F \in \mathcal{C}$ and $m(x,y) = \max d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty) + d(fy,Tx)]$.

It is interesting to see that Theorem 2 of Tomar et al. [12] can be obtained as the Corollary of Theorem 3.14 via C-class function.

Corollary 3.15. *[12] Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a unique common strict fixed point provided that the pair satisfies*

$$\delta(Tx,Ty) \leq k \max\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{1}{2}[D(fx,Ty) + D(fy,Tx)]\},$$

where, $0 \leq k < 1$.

Although it is believed that the strict contractive condition does not guarantee the existence of common strict fixed points without assuming some strong conditions, we give our next result for strict contractive type condition which follows on the similar lines as in Theorem 3.9.

Theorem 3.16. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a unique common strict fixed point provided that the pair satisfies*

$$\psi\left(\delta\left(Tx,Ty\right)\right) < F(\psi\left(m(x,y)\right), \varphi\left(m(x,y)\right)). \tag{3.6}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$, $F \in \mathcal{C}$ and

$$m(x,y) = \max\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{1}{2}[D(fx,Ty) + D(fy,Tx)]\}.$$

Following result is different than the results for a hybrid pair of maps satisfying contractive, contraction as well as non-expansive type conditions as it admits the possibility of more than one strict coincidence and common strict fixed point.

Theorem 3.17. *Let a faintly compatible hybrid pair (f,T) of a metric space (X,d) be conditionally reciprocally continuous. Then f and T have a strict coincidence point. Moreover, f and T have a common strict fixed point provided that the pair satisfies*

$$\psi\left(\delta\left(Tx,Ty\right)\right) \neq F(\psi\left(m(x,y)\right), \varphi\left(m(x,y)\right)), \tag{3.7}$$

whenever the right hand side is non zero, $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi \in \Phi_u$, $F \in \mathcal{C}$ and

$$m(x,y) = \max\{d(fx,fy), D(fx,Tx), D(fy,Ty), \frac{1}{2}[D(fx,Ty) + D(fy,Tx)]\}.$$

Proof. On the similar lines as in Theorem 3.9 we may prove that f and T have a strict coincidence. So $C(T,f) \neq 0$. Next, we have to prove that fu is a common strict fixed point of f and T. Suppose that $ffu \neq fu$. Now using equation (3.7)

$$\begin{aligned}
& \psi\left(\delta(Tfu, Tu)\right) \\
\neq\; & F(\psi\left(\max\{d(ffu,fu), D(ffu,Tfu), D(fu,Tu), \frac{1}{2}[D(ffu,Tu)+D(fu,Tfu)]\}\right), \\
& \varphi\left(\max\{d(ffu,fu), D(ffu,Tfu), D(fu,Tu), \frac{1}{2}[D(ffu,Tu)+D(fu,Tfu)]\}\right)) \\
=\; & F(\psi\left(\max d(ffu,fu), 0, 0, \frac{1}{2}[D(ffu,Tu)+D(fu,Tfu)]\}\right), \\
& \varphi\left(\max d(ffu,fu), 0, 0, \frac{1}{2}[D(ffu,Tu)+D(fu,Tfu)]\}\right)).
\end{aligned}$$

Since, $fu \in Tu$, $D(ffu,Tu) \leq d(ffu,fu)$ and $ffu \in fTu$, $D(fu,Tfu) \leq d(ffu,fu)$. Therefore,

$$\begin{aligned}
\psi\left(d(ffu,fu)\right) &\leq \psi\left(\delta(Tfu,Tu)\right) \\
&\neq F(\psi\left(d\left(ffu,fu\right)\right), \varphi\left(d\left(ffu,fu\right)\right)) \\
&\leq \psi\left(d\left(ffu,fu\right)\right),
\end{aligned}$$

a contradiction. Thus $d(ffu,fu) = 0$.
Hence, $\{fu\} = \{ffu\} = Tfu$, i.e., fu is a common strict fixed point of f and T. □

Now we furnish an interesting example to demonstrate that the notion of faint compatibility allows the existence of multiple common strict fixed points and multiple strict coincidence points for a hybrid pair of maps in a metric space which is not even complete.

Example 3.18. Let $X = [0,5]$, d be the usual metric on X. Let a hybrid pair of map (f,T) on X be defined as follows:

$$fx = \begin{cases} 4-x & 0 < x \leq 2 \\ x, & 2 < x \leq 5, \end{cases} \qquad Tx = \begin{cases} [2,4] & 0 < x \leq 2 \\ \{5\} & 2 < x \leq 5. \end{cases}$$

Then one may verify that f and T satisfy condition (3.7) of Theorem 3.17 taking $F(s,t) = ks$, $\psi(t) = t$.
Consider a sequence $\{x_n\}$ in X satisfying $x_n = 2 - \frac{1}{n}$ and $\lim_{n\to\infty} fx_n = 2 \in [2,4] = \lim_{n\to\infty} Tx_n$ such that $\lim_{n\to\infty} Tfx_n = \{5\} \neq T2$ and $\lim_{n\to\infty} fTx_n = [2,4] \neq f2$, i.e., pair of maps (f,T) is not reciprocally continuous. Also $\lim_{n\to\infty} \delta(fTx_n, Tfx_n) \neq 0$, i.e., pair of maps (f,T) is non-compatible.
Let $\{y_n\}$ be a sequence in X where $y_n = 5 - \frac{1}{n}$ and $\lim_{n\to\infty} fy_n = 5 \in \{5\} = \lim_{n\to\infty} Ty_n$ such that $\lim_{n\to\infty} Tfy_n = \{5\} = T5$ and $\lim_{n\to\infty} fTy_n = 5 = f5$, i.e., $\lim_{n\to\infty} \delta(fTy_n, Tfy_n) = 0$. Also $fT2 \subseteq Tf2$, $fT5 \subseteq Tf5$ and both the strict coincidence points 2 and $5 \in X$. Thus pair of maps (f,T) is conditionally reciprocally continuous and faintly compatible.
Hence, f and T satisfy all the conditions of Theorem 3.17 and have two common strict fixed points at $x = 2$ and 5.

Remark 3.19. Faint compatibility used to establish common strict fixed point is more general than all the variants of compatibility and allows the existence of a common fixed point/common strict fixed point or multiple common fixed points/multiple common strict fixed points or coincidence points/strict coincidence points under contractive, strict contractive, non-contractive and contraction type conditions ([9], [2]) for single valued as well as hybrid pair of maps. Also conditional reciprocal continuity is more general than well known variants of continuity [10]. Our results generalize, extend and improve the results of Bisht and Shahzad [2], Pant and Bisht [7], Manro and Tomar [6], Tomar and Upadhyay [11], Tomar et al. [12] and references therein, which is demonstrated well by illustrating examples. Further, on varying the elements of $\mathcal{C}$ suitably, a variety of known contractions in the literature can be deduced.

4 Applications

4.1 Application to steady state heat flow in rod.

As an application to our Corollary 3.5, we solve a two-point boundary value problem of the second order differential equation arising in steady state heat flow in rod

$$-\frac{d^2x}{dt^2} = K(t, x(t)), t \in [0, 1]$$

$$x(0) = x(1) = 0, \tag{4.1}$$

where $K : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The Green function associated to equation (4.1) is

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Assume that $C([0, 1])$ is the set of all continuous functions defined on $[0, 1]$. Let $d : C[0, 1] \times C[0, 1] \to \mathbb{R}$ be defined as $d(x, y) = \|x - y\|_\infty = sup_{t\in[0,1]}|x(t) - y(t)|$ for all x, $y \in C[0, 1]$ such that $(C[0, 1], d)$ is a metric space.
We now state and prove the result for the existence of a solution of a two-point boundary value problem of the second order differential equation arising in steady state heat flow in rod.

Theorem 4.1. *Let $f, T : C([0, 1]) \to C([0, 1])$ be self maps of a metric space $(C([0, 1]), d)$ such that the following conditions hold:*

(i) $\psi(d(Tx, Ty)) \leq F\Big(\psi(d(fx, fy)), \varphi(d(fx, fy))\Big)$, *where $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function, $\varphi : [0, \infty) \to [0, \infty)$ is an ultra altering distance function such that $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ is the C-class function;*

(ii) $\psi(x) = ln(x + 1)$ *and* $F(x, y) = \psi(x)$*;*

(iii) $|K(t, x) - K(t, y)| \leq 8|fx - fy|$, *for $t \in [0, 1]$ and $x, y \in \mathbb{R}$;*

(iv) if $\{x_n\}$ is a sequence in $C([0, 1])$ such that $\lim_{n\to\infty} x_n = x \in C([0, 1])$ and $\lim_{n\to\infty} fx_n = fx \in C([0, 1])$;

(v) $\lim_{n\to\infty} fx_n = z = \lim_{n\to\infty} Tx_n$ *for some $z \in C([0, 1])$, there exists a sequence $\{y_n\}$ such that $\lim_{n\to\infty} fy_n = u = \lim_{n\to\infty} Ty_n$ and $\lim_{n\to\infty} fTy_n = \lim_{n\to\infty} Tfy_n$ for some $u \in C([0, 1])$. Also $x \in M = \{fx = Tx\}$ and $fTx = Tfx$.*

Then equation (4.1) has at least one solution $x^* \in C([0, 1])$.

Proof. We know that $x^* \in C([0, 1])$ is a solution of equation (4.1) if and only if it is a solution of the integral equation

$$x(t) = \int_0^1 G(t, s)K(s, x(s))ds,$$

for all $t \in [0,1]$. Let T and f be defined by

$$Tx(t) = \int_0^1 G(t,s)K(s,x(s))ds$$

and

$$fx(t) = \int_0^1 G(t,s)K'(s,x(s))ds,$$

for all $t \in [0,1]$. Therefore, the problem (4.1) is equivalent to finding $x^* \in C([0,1])$ which is a common fixed point of f and T. Let $x, y \in C([0,1])$ such that for all $t \in [0,1]$, using condition (iii) we obtain that

$$|Tx(t) - Ty(t)| = |\int_0^1 G(t,s)[K(s,x(s)) - K(s,y(s))]ds|$$

$$\leq \int_0^1 G(t,s)|K(s,x(s)) - K(s,y(s))|ds$$

$$\leq 8\int_0^1 G(t,s)|fx(s) - fy(s)|ds$$

$$\leq 8\int_0^1 G(t,s)d(fx,fy)ds$$

$$\leq 8d(fx,fy)\sup_{t\in[0,1]}\int_0^1 G(t,s)ds.$$

Since, $\int_0^1 G(t,s)ds = \frac{t}{2} - \frac{t^2}{2}$ for all $t \in [0,1]$, we have $sup_{t\in[0,1]}\int_0^1 G(t,s)ds = \frac{1}{8}$.
Therefore, $d(Tx,Ty) \leq d(fx,fy)$.
Taking logarithm on both sides,

$$ln(d(Tx,Ty)+1) \leq ln(d(fx,fy)+1).$$

Using conditions (i) and (ii)

$$\psi(d(Tx,Ty)) \leq F\bigg(\psi(d(fx,fy)), \varphi(d(fx,fy))\bigg),$$

for all $x, y \in C([0,1])$.
Hence, using conditions (iv) and (v) all the hypotheses of Corollary 3.5 are satisfied. Hence, f and T have a unique common fixed point in $C([0,1])$, i.e., there exists $x^* \in C([0,1])$ such that $x^* = Tx^* = fx^*$ is a solution of equation (4.1). □

4.2 Application to spring mass system.

As an application of our Corollary 3.5 now we solve an initial value problem of the second order differential equation arising in spring mass system. If we consider the motion of a horizontal spring that is subjected to a frictional force or a vertical spring subjected to a damping force (like a shock absorber in a car or a bicycle) then an external force also affects the motion of spring. The equation arising in such a critical damped motion is given by:

$$\begin{cases} m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = 0; \\ \\ x(0) = 0, x'(0) = a, \end{cases} \tag{4.2}$$

where $K : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}$.

The Green function associated to (4.2) is given by

$$G(t,s) = \begin{cases} -se^{\mu(s-t)}, & 0 \le s \le t \le 1 \\ \\ -te^{\mu(s-t)}, & 0 \le t \le s \le 1, \end{cases} \tag{4.3}$$

where $\mu > 0$ is a constant, calculated in terms of c and m mentioned in (4.2). Let $X = (C([0, 1]), \mathbb{R}^+)$ be the set of all non negative real valued functions defined on $[0, 1]$. For an arbitrary $x \in X$, we define

$$\|x\|_\mu = \sup_{t\in[0,1]} \{|x(t)|e^{-2\mu t}\}. \tag{4.4}$$

Define $d : X \times X \to \mathbb{R}^+$ by

$$d(fx, fy) = \|fx - fy\|_\mu = \sup_{t\in[0,1]} \{|fx(t) - fy(t)|e^{-2\mu t}\}. \tag{4.5}$$

Then clearly, (X, d) is a metric space. We now state and the prove the result for the existence of a solution of the initial value problem of the second order differential equation arising in spring mass system:

Theorem 4.2. *Let $f, T : C([0, 1]) \to C([0, 1])$ be self maps of a metric space $(C([0, 1]), d)$ such that the following conditions hold:*

- *(i)* $\psi(d(Tx, Ty)) \le F\Big(\psi(d(fx, fy)), \varphi(d(fx, fy))\Big)$*, where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi : [0,\infty) \to [0,\infty)$ is an ultra altering distance function such that $\varphi \in \Phi_u$ and $F \in \mathcal{C}$, the C-class functions;*
- *(ii)* $\psi(x) = ln(x + 1)$ *and* $F(x, y) = \psi(x)$*;*
- *(iii) K is increasing function, such that there exists $\mu > 0$ such that $|K(s, x) - K(s, y)| \le \mu^2 e^{-\mu}|fx - fy|$ for all $s \in [0, 1]$ and $x, y \in \mathbb{R}^+$;*
- *(iv) if $\{x_n\}$ is a sequence in $C([0, 1])$ such that $\lim_{n\to\infty} x_n = x \in C([0, 1])$ and $\lim_{n\to\infty} fx_n = fx \in C([0, 1])$;*
- *(v) $\lim_{n\to\infty} fx_n = z = \lim_{n\to\infty} Tx_n$ for some $z \in C([0, 1])$, there exists a sequence $\{y_n\}$ such that $\lim_{n\to\infty} fy_n = w = \lim_{n\to\infty} Ty_n$ and $\lim_{n\to\infty} fTy_n = \lim_{n\to\infty} Tfy_n$ for some $w \in C([0, 1])$. Also $x \in M = \{fx = Tx\}$ and $fTx = Tfx$.*

Then equation (4.2) has a solution.

Proof. Above problem is equivalent to the integral equation

$$x(t) = \int_0^t G(t, s)K(s, x(s))ds, \; t \in [0, 1]. \tag{4.6}$$

Consider the self maps T and S defined by

$$Tx(t) = \int_0^t G(t,s)K(s,x(s))ds$$

and

$$fx(t) = \int_0^t G(t,s)K'(s,x(s))ds,$$

$t \in [0,1]$. Then clearly, x^* is a solution of equation (4.6), if and only if x^* is a common fixed point of f and T.

From (1), for all $x, y \in X$, we have

$$|Tx(t) - Ty(t)| \leq \int_0^t G(t,s)|K(s,x(s)) - K(s,y(s))|ds$$

$$\leq \int_0^t G(t,s)\mu^2 e^{-\mu}|fx(s) - fy(s)|ds$$

$$\leq \int_0^t \mu^2 e^{-\mu} e^{2\mu s} e^{-2\mu s}|fx(s) - fy(s)|G(t,s)ds$$

$$\leq \mu^2 e^{-\mu}\|fx - fy\|_\mu \times \int_0^t e^{2\mu s}G(t,s)ds$$

$$\leq \mu^2 e^{-\mu}\|fx - fy\|_\mu \times [-\frac{e^{2\mu t}}{\mu^2}(2\mu t - \mu t e^{-\mu t} + e^{-\mu t} - 1)],$$

i.e.,

$$|Tx(t) - Ty(t)|e^{-2\mu t} \leq e^{-\mu}\|fx - fy\|_\mu \times [(1 - 2\mu t + \mu t e^{-\mu t} - e^{-\mu t})],$$

i.e.,

$$\|Tx(t) - Ty(t)\|_\mu \leq e^{-\mu}\|fx - fy\|_\mu \times [(1 - 2\mu t + \mu t e^{-\mu t} - e^{-\mu t})].$$

Clearly, $(1 - 2\mu t + \mu t e^{-\mu t} - e^{-\mu t}) \leq 1$. Hence,

$$\|Tx(t) - Ty(t)\|_\mu \leq e^{-\mu}\|fx - fy\|_\mu$$

or

$$d(Tx, Ty) \leq e^{-\mu}d(fx, fy) \leq d(fx, fy)$$

or

$$d(Tx, Ty) + 1 \leq d(fx, fy) + 1.$$

Taking logarithm,

$$ln(d(Tx, Ty) + 1) \leq ln(d(fx, fy) + 1)$$

.

Taking $\psi(x) = ln(x + 1)$ and $F(x, y) = \psi(x)$ we get

$$\psi(d(Tx, Ty)) \leq F\Big(\psi(d(fx, fy)), \varphi(d(fx, fy))\Big).$$

Clearly, all conditions of Corollary 3.5 are satisfied by operators f and T. Hence, f and T have a common fixed point which is the solution of differential equation arising in spring mass system. $\square$

5 Conclusion

Since C-class functions cover a large class of contractive conditions, the strict coincidence and common strict fixed point results established in this paper, generalize, extend and cover all corresponding results existing in the literature to a hybrid pair of maps. Furthermore, maps are not forced to be continuous at the common strict fixed point using these contractive conditions.In the sequel we identified novel elucidations to the problem of Rhoades[8] that there exists a contractive condition, which is sufficient to establish a fixed point but does not force the map to be continuous at the fixed point. We have also utilised the results obtained to find the solutions of a two-point boundary value problem of the second order differential equations arising in steady state heat flow in rod and an initial value problem of spring mass system. On the same lines we can also find the solutions of other physical problems, for instance electrical circuit equation, equation arising in the motion of pendulum, simple harmonic motion and so on.

References

[1] A. H. Ansari, *Note on "φ –ψ-contractive type mappings and related fixed point"* The 2nd Regional Conference on Mathematics and Applications, PNU, Sept. 377-380 (2014).

[2] R. K. Bisht and N. Shahzad, *Faintly compatible mappings and common fixed points,* Fixed Point Theory Appl., 2013:**156**, (2013).

[3] R. K. Bisht and R. P. Pant, *Common fixed point theorems under a new continuity condition,* Ann. Univ., Ferrara, **58**, 127-141 (2012).

[4] H. Kaneko and S. Sessa, *Fixed point theorems for compatible multi-valued and single-valued mappings,* Int. J. Math. Sci., **12**, 257-262 (1989).

[5] M. S. Khan, M. Swaleh, and S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Aust. Math. Soc., **30(1)**, 1-9 (1984).

[6] Saurabh Manro and Anita Tomar, *Faintly compatible maps and existence of common fixed points in fuzzy metric space,* Ann. Fuzzy Math. Inform, **8(2)**, 223-230 (2014).

[7] R. P. Pant and R. K. Bisht, *Occasionally weakly compatible mappings and fixed points,* Bull. Belg. Math. Soc. Simon Stevin, **19**, 655-661 (2012).

[8] B. E. Rhoades, Contractive definitions and continuity, Contemp. Math. **72**, 233-245 (1988).

[9] S. L. Singh and Anita Tomar, *Weaker forms of commuting maps and existence of fixed points,* J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., **10(03)**, 145-161 (2003).

[10] Anita Tomar and Erdal Karapinar, *On variants of continuity and existence of fixed point via meir-keeler contractions in MC-spaces,* J. Adv. Math. Stud., Vol. **9(2)**, 348-359 (2016).

[11] Anita Tomar and Shivangi Upadhyay, *Coincidence and common fixed point theorems for faintly compatible maps,* TWMS J. App. & Eng. Math., **7(1)**, 25-32 (2017).

[12] Anita Tomar, Shivangi Upadhyay and Ritu Sharma, *On existence of Strict Coincidence and Common Strict Fixed Point of a Faintly Compatible Hybrid Pair of Maps,* Electron. J. Math. Anal. Appl., **5(2)**, 298-305 (2017).

Author information

Anita Tomar, Department of Mathematics, V. S. K. C. Government P. G. College Dakpathar
Dehradun (Uttrakhand), India.
E-mail: `anitatmr@yahoo.com`

Ritu Sharma, Department of Mathematics, V. S. K. C. Government P. G. College Dakpathar
Dehradun (Uttrakhand), India.
E-mail: `ritus4184@gmail.com`

Arslan Hojat Ansari, Department of Mathematics, Karaj Branch,
Islamic Azad University, Karaj, Iran.
E-mail: `analsisamirmath2@gmail.com`

Received: August 8, 2017.
Accepted: January 28, 2018.