Palestine Journal of Mathematics

Vol. 9(1)(2020) , 289–298

Complex Fuzzy Soft Rings

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Communicated by Najeeb Mahdou

MSC 2010 Classifications: Primary 03E72 ; Secondary 20N25.

Keywords and phrases: $\pi-$fuzzy set, $\pi-$fuzzy subring, homogeneous complex fuzzy soft set, complex fuzzy soft ideal, complex fuzzy soft ring..

Abstract In this paper, we introduced the concepts of complex fuzzy soft rings and complex fuzzy soft ideals, then we investigated some of their characteristics. Also, we defined the image and inverse image of complex fuzzy soft rings under complex fuzzy soft homomorphism and we studied their elementary properties.

1 Introduction

Zadeh [9] introduced the concept of fuzzy sets in 1965. A fuzzy set A in U is defined by a membership function $\mu_A : U \longrightarrow [0,1]$, where U is nonempty set, called universe. In 1971, Rosenfeld [8] introduced the concept of fuzzy subgroup. In 2002, Ramot et al.[7] introduced the concept of the complex fuzzy sets, they extended the closed interval $[0,1]$ to the unite circle in the complex plane. In 2010, Nadia [6] introduced the concept of complex fuzzy soft sets which is a combination of complex fuzzy set and soft sets theory. In 2016, as a previous work we [1, 3] introduced the concepts of complex fuzzy subgroups and complex fuzzy soft groups.

In this paper, we introduced the concepts of complex fuzzy soft rings and complex fuzzy soft ideals, then we investigated some of their characteristics. Also, we defined the image and inverse image of complex fuzzy soft rings under complex fuzzy soft homomorphism and we studied their elementary properties.

2 Preliminaries

Definition 2.1. [7] A complex fuzzy set, defined on a universe of discourse U, is characterized by a membership function $\mu_A(x)$ that assigns any element a complex-valued grade of membership in A. By definition, the values $\mu_A(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_A(x).e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, $r_A(x)$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0,1]$, $\omega_A(x) \in [0,2\pi]$. The complex fuzzy set may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in U\}.$$

Definition 2.2. Let $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in U\}$ and $B = \{(x, \mu_B(x)) : \mu_B(x) = r_B(x)e^{i\omega_B(x)}, x \in U\}$ be two complex fuzzy sets of the same universe U. Then

$$\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)} = min\{r_A(x), r_B(x)\}\, e^{imin\{\omega_A(x),\omega_B(x)\}}.$$

Definition 2.3. [5] Let R be a ring and $A = \{(x, \mu_A(x)) : x \in U\}$ be a fuzzy set. Then A is said to be a fuzzy subring if the following hold

(i) $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all x, y in R.

(ii) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$ for all x, y in R.

Definition 2.4. [5] Let R be a ring and $A = \{(x, \mu_A(x)) : x \in U\}$ be a fuzzy set. Then A is said to be a fuzzy ideal if the following hold:

(i) $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$ for all x, y in R.

(ii) $\mu_A(xy) \geq max\{\mu_A(x), \mu_A(y)\}$ for all x, y in R.

Definition 2.5. [1] Let $A = \{(x, \mu_A(x)) : x \in U\}$ be a fuzzy set. Then the set $A_\pi = \{(x, \gamma_{A_\pi}(x)) : \gamma_{A_\pi}(x) = 2\pi\mu_A(x), x \in U\}$ is said to be a $\pi-$fuzzy set.

Definition 2.6. [2] Let R be a ring and A_π be a $\pi-$fuzzy set. Then A_π is said to be a $\pi-$fuzzy subring if the following hold

(i) $\gamma_{A_\pi}(x - y) \geq min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\}$ for all x, y in R.

(ii) $\gamma_{A_\pi}(xy) \geq min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\}$ for all x, y in R.

Definition 2.7. [2] Let R be a ring and A_π be a $\pi-$fuzzy set. Then A_π is said to be a $\pi-$fuzzy ideal if the following hold:

(i) $\gamma_{A_\pi}(x - y) \geq min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\}$ for all x, y in R.

(ii) $\gamma_{A_\pi}(xy) \geq max\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\}$ for all x, y in R.

Proposition 2.8. *[2] A $\pi-$fuzzy set A_π is a $\pi-$fuzzy subring ($\pi-$fuzzy ideal) if and only if A is a fuzzy subring (fuzzy ideal).*

Definition 2.9. [6] Let U be the initial universe and E be the set of parameters. let S^U denote the set of all complex fuzzy sets of U, $A \subset E$ and $f : A \longmapsto S^U$. A pair (f, A) is said to be a complex fuzzy soft set over U.

Let U be a universe set and (f, A) be a complex fuzzy soft set over U. Then (f, A) yields two fuzzy soft sets over U as follows

(i) The fuzzy soft set $(\overline{f}, A)$, where $\overline{f} : A \longmapsto \overline{s}^U$ and $\overline{s}^U$ is the set of all fuzzy sets of the form $\{(x, r_{f(a)}(x)) : x \in U, a \in A\}$ such that $\mu_{f(a)}(x) = r_{f(a)}(x)e^{i\omega_{f(a)}(x)}$ is the membership function of the complex fuzzy set $f(a)$.

(ii) The $\pi-$fuzzy soft set $(\underline{f}, A)$, where $\underline{f} : A \longmapsto \underline{s}^U$ and $\underline{s}^U$ is the set of all $\pi-$fuzzy sets of the form $\{(x, \omega_{f(a)}(x)) : x \in U, a \in A\}$ such that $\mu_{f(a)}(x) = r_{f(a)}(x)e^{i\omega_{f(a)}(x)}$ is the membership function of the complex fuzzy set $f(a)$.

Definition 2.10. [4] A fuzzy soft set $(\overline{f}, A)$ is said to be a fuzzy soft ring (fuzzy soft ideal) if and only if $\overline{f}(a)$ is a fuzzy subring (fuzzy ideal) for all $a \in A$.

Definition 2.11. A $\pi-$fuzzy soft set $(\underline{f}, A)$ is said to be a $\pi-$fuzzy soft ring ($\pi-$fuzzy soft ideal) if and only if $\underline{f}(a)$ is a $\pi-$fuzzy subring ($\pi-$fuzzy ideal) for all $a \in A$.

Definition 2.12. [6] The intersection of two complex fuzzy soft sets (f, A) and (g, B) over U, denoted by $(f, A) \cap (g, B)$, is the complex fuzzy soft set (h, C), where $C = A \cap B$, and $h(c) = f(c) \cap g(c)$ for all $c \in C$.

Definition 2.13. [6] The union of two complex fuzzy soft sets (f, A) and (g, B) over U, denoted by $(f, A) \cup (g, B)$, is the complex fuzzy soft set (h, C), where $C = A \cup B$, and for all $c \in C$,

$$h(c) = \begin{cases} f(c) & .\text{if } c \in A - B \\ g(c) & ,\text{if } c \in B - A \\ f(c) \cup g(c) & ,\text{if } c \in A \cap B \end{cases}$$

Definition 2.14. [1] Let $A = \{(x, \mu_A(x)) : x \in U\}$ and $B = \{(x, \mu_B(x)) : x \in U\}$ be two complex fuzzy subsets of U, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. Then

(i) A complex fuzzy subset A is said to be a homogeneous complex fuzzy set if for all $x, y \in U$

$$r_A(x) \leq r_A(y) \text{ if and only if } \omega_A(x) \leq \omega_A(y).$$

(ii) A complex fuzzy subset A is said to be homogeneous with B, if for all $x, y \in U$

$$r_A(x) \leq r_B(y) \text{ if and only if } \omega_A(x) \leq \omega_B(y) \text{ for all } x, y \in U.$$

Theorem 2.15. *[2] Let R be a ring and $A = \{(x, \mu_A(x)) : x \in R\}$ be a homogeneous complex fuzzy set with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$. Then A is a complex fuzzy subring (ideal) of R if and only if:*

(i) The fuzzy set $\overline{A} = \{(x, r_A(x)) : x \in R, r_A(x) \in [0, 1]\}$ is a fuzzy subring (ideal).

(ii) The $\pi-$fuzzy set $\underline{A} = \{(x, \omega_A(x)) : x \in R, \omega_A(x) \in [0, 2\pi]\}$ is a $\pi-$fuzzy subring (ideal).

3 Complex Fuzzy Soft Rings

Definition 3.1. [3] Let (f, A) and (g, B) be two complex fuzzy soft sets over a universe set U. Then

(i) A complex fuzzy soft set (f, A) is said to be a homogeneous complex fuzzy soft set if and only if $f(a)$ is a homogeneous complex fuzzy set for all $a \in A$.

(ii) A complex fuzzy soft set (f, A) is said to be a completely homogeneous complex fuzzy soft set if and only if $f(a)$ is a homogeneous with $f(b)$ for all $a, b \in A$.

(iii) A complex fuzzy soft set (f, A) is said to be homogeneous with (g, B) if and only if $f(a)$ is a homogeneous with $g(a)$ for all $a \in A \cap B$.

Definition 3.2. Let R be a ring and (f, A) be a homogeneous complex fuzzy soft set over R. Then (f, A) is said to be a complex fuzzy soft ring shortly (CFSR) over R if and only if the following hold

(i) $\mu_{f(a)}(x - y) \geq min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$ for all $a \in A$ and $x, y \in R$.

(ii) $\mu_{f(a)}(xy) \geq min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$ for all $a \in A$ and $x, y \in R$.

Definition 3.3. Let R be a ring and (f, A) be a homogeneous complex fuzzy soft set over R. Then (f, A) is said to be a complex fuzzy soft ideal shortly (CFSI) over R if and only if the following hold

(i) $\mu_{f(a)}(x - y) \geq min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$ for all $a \in A$ and $x, y \in R$.

(ii) $\mu_{f(a)}(xy) \geq max\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$ for all $a \in A$ and $x, y \in R$.

Theorem 3.4. *Let R be a ring and (f, A) be a homogeneous complex fuzzy soft set over R. Then (f, A) is a complex fuzzy soft ring of R if and only if:*

(i) The fuzzy soft set $(\overline{f}, A)$ is a fuzzy soft ring.

(ii) The $\pi-$fuzzy soft set $(\underline{f}, A)$ is a $\pi-$fuzzy soft ring.

Proof. Let (f, A) be a CFSR and $x, y \in R$. Then for all $a \in A$ we have

$$\begin{aligned} r_{f(a)}(x-y)e^{i\omega_{f(a)}(x-y)} &= \mu_{f(a)}(x-y) \\ &\geq min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\} \\ &= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \\ &= min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}} \\ &\text{(since } (f, A) \text{ is homogeneous).} \end{aligned}$$

So $r_{f(a)}(x - y) \geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}$ and $\omega_{f(a)}(x - y) \geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$. On the other hand

$$\begin{aligned} r_{f(a)}(xy)e^{i\omega_{f(a)}(xy)} &= \mu_{f(a)}(xy) \\ &\geq min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\} \\ &= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \\ &= min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}} \\ &\text{(since } (f, A) \text{ is homogeneous).} \end{aligned}$$

Which implies $r_{f(a)}(xy) \geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}$ and
$\omega_{f(a)}(xy) \geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$.

So $(\overline{f}, A)$ is a fuzzy soft ring and $(\underline{f}, A)$ is a $\pi-$fuzzy soft ring.
Conversely, let $(\overline{f}, A)$ be a fuzzy soft ring and $(\underline{f}, A)$ be a $\pi-$fuzzy soft ring, then for all $a \in A$ we have

$$r_{f(a)}(x-y) \geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}, \omega_{f(a)}(x-y) \geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$$
$$r_{f(a)}(xy) \geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} \text{ and } \omega_{f(a)}(xy) \geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}.$$

Now,

$$\begin{aligned}
\mu_{f(a)}(x-y) &= r_{f(a)}(x-y)e^{i\omega_{f(a)}(x-y)} \\
&\geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}} \\
&= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \text{ (homogeneity).} \\
&= min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu_{f(a)}(xy) &= r_{f(a)}(xy)e^{i\omega_{f(a)}(xy)} \\
&\geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}} \\
&= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \text{ (homogeneity).} \\
&= min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\}.
\end{aligned}$$

So $f(a)$ is a complex fuzzy subring, thus (f, A) is a complex fuzzy soft ring. □

Theorem 3.5. *Let R be a ring and (f, A) be a homogeneous complex fuzzy soft set over R. Then (f, A) is a complex fuzzy soft ideal of R if and only if:*

(i) The fuzzy soft set $(\overline{f}, A)$ is a fuzzy soft ideal.

(ii) The $\pi-$fuzzy soft set $(\underline{f}, A)$ is a $\pi-$fuzzy soft ideal.

Proof. Let (f, A) be a CFSI and $x, y \in R$. Then for all $a \in A$ we have

$$\begin{aligned}
r_{f(a)}(x-y)e^{i\omega_{f(a)}(x-y)} &= \mu_{f(a)}(x-y) \\
&\geq min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\} \\
&= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \\
&= min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}} \\
&\text{(since } (f, A) \text{ is homogeneous).}
\end{aligned}$$

So $r_{f(a)}(x-y) \geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}$ and
$\omega_{f(a)}(x-y) \geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$. On the other hand

$$\begin{aligned}
r_{f(a)}(xy)e^{i\omega_{f(a)}(xy)} &= \mu_{f(a)}(xy) \\
&\geq max\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\} \\
&= max\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \\
&= max\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imax\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}} \\
&\text{(since } (f, A) \text{ is homogeneous).}
\end{aligned}$$

Which implies $r_{f(a)}(xy) \geq max\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}$ and $\omega_{f(a)}(xy) \geq max\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$.

So $(\overline{f}, A)$ is a fuzzy soft ideal and $(\underline{f}, A)$ is a $\pi-$fuzzy soft ideal.
Conversely, let $(\overline{f}, A)$ be a fuzzy soft ideal and $(\underline{f}, A)$ be a $\pi-$fuzzy soft ideal, then for all $a \in A$ we have

$$r_{f(a)}(x-y) \geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}, \omega_{f(a)}(x-y) \geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$$
$$r_{f(a)}(xy) \geq max\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} \text{ and } \omega_{f(a)}(xy) \geq max\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}.$$

Now,

$$\begin{aligned}\mu_{f(a)}(x-y) &= r_{f(a)}(x-y)e^{i\omega_{f(a)}(x-y)} \\ &\geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x),\omega_{f(a)}(y)\right\}} \\ &= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \text{ (homogeneity).} \\ &= min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\}.\end{aligned}$$

On the other hand

$$\begin{aligned}\mu_{f(a)}(xy) &= r_{f(a)}(xy)e^{i\omega_{f(a)}(xy)} \\ &\geq max\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imax\left\{\omega_{f(a)}(x),\omega_{f(a)}(y)\right\}} \\ &= max\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \text{ (homogeneity).} \\ &= max\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\}.\end{aligned}$$

So $f(a)$ is a complex fuzzy ideal, thus (f, A) is a complex fuzzy soft ideal. □

Theorem 3.6. *Let $\{(f_i, A_i) : i \in I\}$ be a collection of CFSRs over a ring R such that (f_j, A_j) is homogeneous with (f_k, A_k) for all $j, k \in I$. Then $\cap_{i\in I}(f_i, A_i)$ is a CFSR.*

Proof. Let $\cap_{i\in I}(f_i, A_i) = (h, C)$ where $C = \cap_{i\in I}A_i$. Then we have $f_i(c)$ is a complex fuzzy subring for all $i \in I$, so $r_{f_i(c)}(x)$ is a fuzzy subring and $\omega_{f_i(c)}(x)$ is a $\pi-$fuzzy subring (Theorem 2.15). Now, for all $x, y \in R$ we have

$$\begin{aligned}\mu_{h(c)}(x-y) =& \mu_{\cap_{i\in I}f_i(c)}(x-y) \\ &= r_{\cap_{i\in I}f_i(c)}(x-y)e^{i\omega_{\cap_{i\in I}f_i(c)}(x-y)} \\ &= min_{i\in I}\left\{r_{f_i(c)}(x-y)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(x-y)\right\}} \\ &\geq min_{i\in I}\left\{min\left\{r_{f_i(c)}(x), r_{f_i(c)}(y)\right\}\right\} e^{imin_{i\in I}\left\{min\left\{\omega_{f_i(c)}(x),\omega_{f_i(c)}(y)\right\}\right\}} \\ &= min\left\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\}, min_{i\in I}\left\{r_{f_i(c)}(y)\right\}\right\} \\ &\qquad e^{imin\left\{min_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}, min_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}\right\}} \\ &= min\left\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}},\right. \\ &\qquad \left. min_{i\in I}\left\{r_{f_i(c)}(y)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}}\right\} \\ &\quad \text{(since } (f_j, A_j) \text{ is homogeneous with } (f_k, A_k) \text{ for all } j, k \in I) \\ &= min\left\{\mu_{\cap_{i\in I}f_i(c)}(x), \mu_{\cap_{i\in I}f_i(c)}(y)\right\} \\ &= min\left\{\mu_{h(c)}(x), \mu_{h(c)}(y)\right\}.\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu_{h(c)}(xy) =&\mu_{\cap_{i\in I} f_i(c)}(xy)\\
&= r_{\cap_{i\in I} f_i(c)}(xy)e^{i\omega_{\cap_{i\in I} f_i(c)}(xy)}\\
&= min_{i\in I}\left\{r_{f_i(c)}(xy)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(xy)\right\}}\\
&\geq min_{i\in I}\left\{min\left\{r_{f_i(c)}(x), r_{f_i(c)}(y)\right\}\right\} e^{imin_{i\in I}\left\{min\left\{\omega_{f_i(c)}(x),\omega_{f_i(c)}(y)\right\}\right\}}\\
&= min\left\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\}, min_{i\in I}\left\{r_{f_i(c)}(y)\right\}\right\}\\
&\qquad e^{imin\left\{min_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}, min_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}\right\}}\\
&= min\Big\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}},\\
&\qquad min_{i\in I}\left\{r_{f_i(c)}(y)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}}\Big\}\\
&\quad \text{(since } (f_j, A_j) \text{ is homogeneous with } (f_k, A_k) \text{ for all } j,k\in I)\\
&= min\left\{\mu_{\cap_{i\in I} f_i(c)}(x), \mu_{\cap_{i\in I} f_i(c)}(y)\right\}\\
&= min\left\{\mu_{h(c)}(x), \mu_{h(c)}(y)\right\}.
\end{aligned}$$

□

Theorem 3.7. *Let $\{(f_i, A_i) : i\in I\}$ be a collection of CFSIs over a ring R such that (f_j, A_j) is homogeneous with (f_k, A_k) for all $j,k\in I$. Then $\cap_{i\in I}(f_i, A_i)$ is a CFSI.*

Proof. Let $\cap_{i\in I}(f_i, A_i) = (h, C)$ where $C = \cap_{i\in I} A_i$. Then we have $f_i(c)$ is a complex fuzzy ideal for all $i\in I$, so $r_{f_i(c)}(x)$ is a fuzzy ideal and $\omega_{f_i(c)}(x)$ is a $\pi-$fuzzy ideal (Theorem 2.15). Now, for all $x, y\in R$ we have

$$\begin{aligned}
\mu_{h(c)}(x-y) =&\mu_{\cap_{i\in I} f_i(c)}(x-y)\\
&= r_{\cap_{i\in I} f_i(c)}(x-y)e^{i\omega_{\cap_{i\in I} f_i(c)}(x-y)}\\
&= min_{i\in I}\left\{r_{f_i(c)}(x-y)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(x-y)\right\}}\\
&\geq min_{i\in I}\left\{min\left\{r_{f_i(c)}(x), r_{f_i(c)}(y)\right\}\right\} e^{imin_{i\in I}\left\{min\left\{\omega_{f_i(c)}(x),\omega_{f_i(c)}(y)\right\}\right\}}\\
&= min\left\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\}, min_{i\in I}\left\{r_{f_i(c)}(y)\right\}\right\}\\
&\qquad e^{imin\left\{min_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}, min_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}\right\}}\\
&= min\Big\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}},\\
&\qquad min_{i\in I}\left\{r_{f_i(c)}(y)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}}\Big\}\\
&\quad \text{(since } (f_j, A_j) \text{ is homogeneous with } (f_k, A_k) \text{ for all } j,k\in I)\\
&= min\left\{\mu_{\cap_{i\in I} f_i(c)}(x), \mu_{\cap_{i\in I} f_i(c)}(y)\right\}\\
&= min\left\{\mu_{h(c)}(x), \mu_{h(c)}(y)\right\}.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
\mu_{h(c)}(xy) =& \mu_{\cap_{i\in I} f_i(c)}(xy) \\
&= r_{\cap_{i\in I} f_i(c)}(xy) e^{i\omega_{\cap_{i\in I} f_i(c)}(xy)} \\
&= min_{i\in I}\left\{r_{f_i(c)}(xy)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(xy)\right\}} \\
&\geq min_{i\in I}\left\{max\left\{r_{f_i(c)}(x), r_{f_i(c)}(y)\right\}\right\} e^{imin_{i\in I}\left\{max\left\{\omega_{f_i(c)}(x),\omega_{f_i(c)}(y)\right\}\right\}} \\
&\geq max\left\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\}, min_{i\in I}\left\{r_{f_i(c)}(y)\right\}\right\} \\
&\qquad e^{imax\left\{min_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}, min_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}\right\}} \\
&= max\Big\{min_{i\in I}\left\{r_{f_i(c)}(x)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(x)\right\}}, \\
&\qquad min_{i\in I}\left\{r_{f_i(c)}(y)\right\} e^{imin_{i\in I}\left\{\omega_{f_i(c)}(y)\right\}}\Big\} \\
&\quad \text{(since } (f_j, A_j) \text{ is homogeneous with } (f_k, A_k) \text{ for all } j,k\in I) \\
&= max\left\{\mu_{\cap_{i\in I} f_i(c)}(x), \mu_{\cap_{i\in I} f_i(c)}(y)\right\} \\
&= max\left\{\mu_{h(c)}(x), \mu_{h(c)}(y)\right\}.
\end{aligned}
$$

□

Theorem 3.8. *Let (f,A) and (g,B) be disjoint CFSRs (CFSIs). Then $(f,A)\cup(g,B)$ is CFSR (CFSI).*

Proof. Straightforward. □

Definition 3.9. [3]Let (f,A) be a complex fuzzy soft set over a universe U. Then for all $\alpha\in[0,1]$ and $\beta\in[0,2\pi]$, the set $(f,A)_{(\alpha,\beta)} = \{f(a)_{(\alpha,\beta)} : a\in A\}$ is called an $(\alpha,\beta)-$level soft set of the complex fuzzy soft set (f,A), where $f(a)_{(\alpha,\beta)} = \{x\in U : r_{f(a)}(x)\geq\alpha, \omega_{f(a)}(x)\geq\beta\}$ is an $(\alpha,\beta)-$level set of the complex fuzzy set $f(a)$. Here, for each $\alpha\in[0,1]$ and $\beta\in[0,2\pi]$, $(f,A)_{(\alpha,\beta)}$ is a soft set in the classical case.

Theorem 3.9. *Let (f,A) be a complex fuzzy soft set over a ring R. Then (f,A) is a CFSR over a ring R if and only if $\forall a\in A$ and for arbitrary $\alpha\in[0,1]$ and $\beta\in[0,2\pi]$, with $f(a)_{(\alpha,\beta)}\neq\phi$ the $(\alpha,\beta)-$level soft set $(f,A)_{(\alpha,\beta)}$ is a soft ring over R in classical case.*

Proof. Let (f,A) be a CFSR over a ring R. Then for all $a\in A$, $f(a)$ is a complex fuzzy subring of R. For arbitrary $\alpha\in[0,1]$, $\beta\in[0,2\pi]$ and $a\in A$ with $f(a)_{(\alpha,\beta)}\neq\phi$, let $x,y\in f(a)_{(\alpha,\beta)}$. Then we have $r_{f(a)}(x)\geq\alpha$ and $\omega_{f(a)}(x)\geq\beta$, also, $r_{f(a)}(y)\geq\alpha$ and $\omega_{f(a)}(y)\geq\beta$.
Now,

$$
\begin{aligned}
r_{f(a)}(x-y)e^{i\omega_{f(a)}(x-y)} &= \mu_{f(a)}(x-y) \\
&\geq min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\} \\
&= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \\
&= min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x),\omega_{f(a)}(y)\right\}}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
r_{f(a)}(x-y) &\geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} \\
&\geq min\{\alpha,\alpha\} \\
&= \alpha.
\end{aligned}
$$

And

$$\begin{aligned}\omega_{f(a)}(x-y) &\geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\} \\ &\geq min\left\{\beta, \beta\right\} \\ &= \beta.\end{aligned}$$

So $x-y \in f(a)_{(\alpha,\beta)}$. On the other hand we have

$$\begin{aligned}r_{f(a)}(xy)e^{i\omega_{f(a)}(xy)} &= \mu_{f(a)}(xy) \\ &\geq min\left\{\mu_{f(a)}(x), \mu_{f(a)}(y)\right\} \\ &= min\left\{r_{f(a)}(x)e^{i\omega_{f(a)}(x)}, r_{f(a)}(y)e^{i\omega_{f(a)}(y)}\right\} \\ &= min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} e^{imin\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}}.\end{aligned}$$

This implies

$$\begin{aligned}r_{f(a)}(xy) &\geq min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\} \\ &\geq min\left\{\alpha, \alpha\right\} \\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\omega_{f(a)}(xy) &\geq min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\} \\ &\geq min\left\{\beta, \beta\right\} \\ &= \beta.\end{aligned}$$

So $xy \in f(a)_{(\alpha,\beta)}$. Therefore $f(a)_{(\alpha,\beta)}$ is a subring of R, thus $(f,A)_{(\alpha,\beta)}$ is a soft ring over R. Conversely, for all $a \in A$ and for arbitrary $\alpha \in [0,1]$ and $\beta \in [0, 2\pi]$ let $(f,A)_{(\alpha,\beta)}$ be a soft ring over R. Let $x, y \in R$, assume $r_{f(a)}(x) = \lambda$, $r_{f(a)}(y) = \delta$, $\omega_{f(a)}(x) = \theta$ and $\omega_{f(a)}(y) = \eta$. Suppose $\alpha = min\left\{\lambda, \delta\right\}$ and $\beta = min\left\{\theta, \eta\right\}$, this implies $x, y \in f(a)_{(\alpha,\beta)}$. By hypothesis, $f(a)_{(\alpha,\beta)}$ is a subring of R, so $x-y \in f(a)_{(\alpha,\beta)}$ and $xy \in f(a)_{(\alpha,\beta)}$. Thus $r_{f(a)}(x-y) \geq \alpha = min\left\{\lambda, \delta\right\} = min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}$ and $\omega_{f(a)}(x-y) \geq \beta = min\left\{\theta, \eta\right\} = min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$, therefore $\mu_{f(a)}(x-y) \geq min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$. Also, $r_{f(a)}(xy) \geq \alpha = min\left\{\lambda, \delta\right\} = min\left\{r_{f(a)}(x), r_{f(a)}(y)\right\}$ and $\omega_{f(a)}(xy) \geq \beta = min\left\{\theta, \eta\right\} = min\left\{\omega_{f(a)}(x), \omega_{f(a)}(y)\right\}$, therefore $\mu_{f(a)}(xy) \geq min\{\mu_{f(a)}(x), \mu_{f(a)}(y)\}$, so (f,A) is a CFSR over a ring R. □

4 Homomorphism of Complex Fuzzy Soft Rings

Theorem 4.1. *[4] Let (f,A) and (g,B) be two fuzzy soft rings over a ring R and S, respectively and (φ,ψ) be a fuzzy soft homomorphism from R to S. Then*

(i) The image of (f,A) under the fuzzy soft homomorphism (φ,ψ) is a fuzzy soft ring over S.

(ii) The pre-image of (g,B) under the fuzzy soft homomorphism (φ,ψ) is a fuzzy soft ring over R.

Definition 4.2. [3] Let $\varphi : U \longrightarrow V$ and $\psi : A \longrightarrow B$ be two functions, where A and B are parameter sets for the crisp sets U and V, respectively. Then the pair (φ,ψ) is called a complex fuzzy soft function from U to V.

Definition 4.3. [3] Let (f,A) and (g,B) be two complex fuzzy soft sets over U and V, respectively. Let (φ,ψ) be a complex fuzzy soft function from U to V. Then

(i) The image of (f,A) under the complex fuzzy soft function (φ,ψ), denoted by $(\varphi,\psi)(f,A)$, is the complex fuzzy soft set over V defined by $(\varphi,\psi)(f,A) = (\varphi(f), \psi(A))$, with membership function

$$\mu_{\varphi(f)(b)}(y) = \begin{cases} \bigvee\limits_{\varphi(x)=y} \ \bigvee\limits_{\psi(a)=b} \mu_{f(a)}(x) & \text{if } \varphi^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

where $b \in \psi(A)$ and $y \in V$.

(ii) The pre-image of (g, B) under the complex fuzzy soft function (φ, ψ), denoted by $(\varphi, \psi)^{-1}(g, B)$, is the complex fuzzy soft set over U defined by $(\varphi, \psi)^{-1}(g, B) = (\varphi^{-1}(g), \psi^{-1}(B))$ such that

$$\mu_{\varphi^{-1}(g)(a)}(x) = \mu_{g(\psi(a))}(\varphi(x))$$

where $a \in \psi^{-1}(B)$, $x \in U$.

Lemma 4.4. *[3] Let (f, A) and (g, B) be two completely homogeneous complex fuzzy soft sets over U and V, respectively. Let (φ, ψ) be a complex fuzzy soft function from U to V. Then*

(i) $\mu_{\varphi(f)(a)}(y) = r_{\varphi(f)(a)}(y)e^{i\omega_{\varphi(f)(a)}(y)}$.

(ii) $\mu_{\varphi^{-1}(g)(b)}(x) = r_{\varphi^{-1}(g)(b)}(x)e^{i\omega_{\varphi^{-1}(g)(b)}(x)}$.

Theorem 4.5. *Let (f, A) and (g, B) be two completely homogeneous CFSRs over a ring R and S, respectively, and (φ, ψ) be a complex fuzzy soft homomorphism from R to S. Then*

(i) The image of (f, A) under the complex fuzzy soft homomorphism (φ, ψ) is a CFSR over S.

(ii) The pre-image of (g, B) under the complex fuzzy soft homomorphism (φ, ψ) is a CFSR over R.

Proof. (1) Since (f, A) is a complex fuzzy soft ring, then by Theorem 3.4 we have the fuzzy soft set $(\overline{f}, A)$ is a fuzzy soft ring and the $\pi-$fuzzy soft set $(\underline{f}, A)$ is a $\pi-$fuzzy soft ring. Thus by Theorem 4.1 and Proposition 2.8 the image of $(\overline{f}, A)$ and $(\underline{f}, A)$ are fuzzy soft ring and $\pi-$fuzzy soft ring, respectively, therefore for all $a \in \psi(A)$ and $x, y \in S$ we have:
$r_{\varphi(f)(a)}(x - y) \geq \min\left\{r_{\varphi(f)(a)}(x), r_{\varphi(f)(a)}(y)\right\}$,
$r_{\varphi(f)(a)}(xy) \geq \min\left\{r_{\varphi(f)(a)}(x), r_{\varphi(f)(a)}(y)\right\}$,
$\omega_{\varphi(f)(a)}(x - y) \geq \min\left\{\omega_{\varphi(f)(a)}(x), \omega_{\varphi(f)(a)}(y)\right\}$ and
$\omega_{\varphi(f)(a)}(xy) \geq \min\left\{\omega_{\varphi(f)(a)}(x), \omega_{\varphi(f)(a)}(y)\right\}$

Now, by Lemma 4.4

$$\begin{aligned}\mu_{\varphi(f)(a)}(x - y) &= r_{\varphi(f)(a)}(x - y)e^{i\omega_{\varphi(f)(a)}(x-y)} \\ &\geq min\left\{r_{\varphi(f)(a)}(x), r_{\varphi(f)(a)}(y)\right\} e^{imin\left\{\omega_{\varphi(f)(a)}(x),\omega_{\varphi(f)(a)}(y)\right\}} \\ &= min\left\{r_{\varphi(f)(a)}(x)e^{i\omega_{\varphi(f)(a)}(x)}, r_{\varphi(f)(a)}(y)e^{i\omega_{\varphi(f)(a)}(y)}\right\} \\ &= min\left\{\mu_{\varphi(f)(a)}(x), \mu_{\varphi(f)(a)}(y)\right\}.\end{aligned}$$

Also,

$$\begin{aligned}\mu_{\varphi(f)(a)}(xy) &= r_{\varphi(f)(a)}(xy)e^{i\omega_{\varphi(f)(a)}(xy)} \\ &\geq min\left\{r_{\varphi(f)(a)}(x), r_{\varphi(f)(a)}(y)\right\} e^{imin\left\{\omega_{\varphi(f)(a)}(x),\omega_{\varphi(f)(a)}(y)\right\}} \\ &= min\left\{r_{\varphi(f)(a)}(x)e^{i\omega_{\varphi(f)(a)}(x)}, r_{\varphi(f)(a)}(y)e^{i\omega_{\varphi(f)(a)}(y)}\right\} \\ &= min\left\{\mu_{\varphi(f)(a)}(x), \mu_{\varphi(f)(a)}(y)\right\}.\end{aligned}$$

(2) Since (g, B) is a complex fuzzy soft group, then by Theorem 3.4 we have the fuzzy soft set $(\overline{g}, B)$ is a fuzzy soft ring and the $\pi-$fuzzy soft set $(\underline{g}, B)$ is a $\pi-$fuzzy soft ring. Thus by Theorem 4.1 and Proposition 2.8 the pre-image of $(\overline{g}, B)$ and $(\underline{g}, B)$ are fuzzy soft ring and $\pi-$fuzzy soft ring, respectively, therefore for all $b \in \psi^{-1}(B)$ and $x, y \in R$ we have:
$r_{\varphi^{-1}(g)(b)}(x - y) \geq \min\left\{r_{\varphi^{-1}(g)(b)}(x), r_{\varphi^{-1}(g)(b)}(y)\right\}$,
$r_{\varphi^{-1}(g)(b)}(xy) \geq \min\left\{r_{\varphi^{-1}(g)(b)}(x), r_{\varphi^{-1}(g)(b)}(y)\right\}$,
$\omega_{\varphi^{-1}(g)(b)}(x - y) \geq \min\left\{\omega_{\varphi^{-1}(g)(b)}(x), \omega_{\varphi^{-1}(g)(b)}(y)\right\}$ and
$\omega_{\varphi^{-1}(g)(b)}(xy) \geq \min\left\{\omega_{\varphi^{-1}(g)(b)}(x), \omega_{\varphi^{-1}(g)(b)}(y)\right\}$

Now, by Lemma 4.4

$$\begin{aligned}
\mu_{\varphi^{-1}(g)(b)}(x-y) &= r_{\varphi^{-1}(g)(b)}(x-y)e^{i\omega_{\varphi^{-1}(g)(b)}(x-y)} \\
&\geq min\left\{r_{\varphi^{-1}(g)(b)}(x), r_{\varphi^{-1}(g)(b)}(y)\right\} e^{imin\left\{\omega_{\varphi^{-1}(g)(b)}(x), \omega_{\varphi^{-1}(g)(b)}(y)\right\}} \\
&= min\left\{r_{\varphi^{-1}(g)(b)}(x)e^{i\omega_{\varphi^{-1}(g)(b)}(x)}, r_{\varphi^{-1}(g)(b)}(y)e^{i\omega_{\varphi^{-1}(g)(b)}(y)}\right\} \\
&= min\left\{\mu_{\varphi^{-1}(g)(b)}(x), \mu_{\varphi^{-1}(g)(b)}(y)\right\}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_{\varphi^{-1}(g)(b)}(xy) &= r_{\varphi^{-1}(g)(b)}(xy)e^{i\omega_{\varphi^{-1}(g)(b)}(xy)} \\
&\geq min\left\{r_{\varphi^{-1}(g)(b)}(x), r_{\varphi^{-1}(g)(b)}(y)\right\} e^{imin\left\{\omega_{\varphi^{-1}(g)(b)}(x), \omega_{\varphi^{-1}(g)(b)}(y)\right\}} \\
&= min\left\{r_{\varphi^{-1}(g)(b)}(x)e^{i\omega_{\varphi^{-1}(g)(b)}(x)}, r_{\varphi^{-1}(g)(b)}(y)e^{i\omega_{\varphi^{-1}(g)(b)}(y)}\right\} \\
&= min\left\{\mu_{\varphi^{-1}(g)(b)}(x), \mu_{\varphi^{-1}(g)(b)}(y)\right\}.
\end{aligned}$$

□

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Received: November 7, 2017.
Accepted: July 25, 2018