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MATRIX OPERATORS AND THE KLEIN FOUR GROUP

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Abstract. In this note we show that the set of operators, $S = \{I, T, P, T \circ P\}$ that consists of the identity I, the usual transpose T, the per-transpose P and their product $T \circ P$, forms a Klein Four-Group with the composition. With the introduced framework, we study in detail the properties of bisymmetric, centrosymmetric matrices and other algebraic structures, and we provide new definitions and results concerning these structures. In particular, we show that the per-tansposition allows to define a degenerate inner product of vectors, a cross product and a dyadic product of vectors with some interesting properties. In the last part of the work, we provide another realization of the Klein Group involving the tensorial product of some 2×2 matrices.

1 Introduction and background

Definition 1.1. Let $A \in \mathbb{R}^{n\times m}$. If $A = [a_{ij}]$ for all $1 \le i \le n$, $1 \le j \le m$, then $P(A)$ is the per-transpose of A, and operation defined by

$$P\Big([a_{ij}]\Big) = [a_{m-j+1,n-i+1}] \tag{1.1}$$

Consequently, $P(A) \in \mathbb{R}^{m\times n}$. Here, we list some properties:

(i) $P \circ P(A) = A$

(ii) $P(A \pm B) = P(A) \pm P(B)$ if $A, B \in \mathbb{R}^{n\times m}$

(iii) $P(\alpha A) = \alpha P(A)$ if $\alpha \in \mathbb{R}$

(iv) $P(AB) = P(B)P(A)$ if $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$

(v) $P \circ T(A) = T \circ P(A)$ where $T(A)$ is the transpose of A

(vi) $\det(P(A)) = \det(A)$

(vii) $P(A)^{-1} = P(A^{-1})$ if $\det(A) \ne 0$

The proofs of these properties follow directly from the definition. For instance, let us prove property 5. Indeed, if $A \in \mathbb{R}^{n\times m}$, then $T(A), P(A) \in \mathbb{R}^{m\times n}$ and the operators T, P act over the element $[a_{ij}]$ as $T([a_{ij}]) = [a_{ji}], P([a_{ij}]) = [a_{m-i+1,n-j+1}]$, then we have

$$\begin{aligned} P \circ T([a_{ij}]) &= P([a_{ji}]) = [a_{n-i+1,m-j+1}] \\ &= T([a_{m-j+1,n-i+1}]) = T \circ P([a_{ij}]) \end{aligned} \tag{1.2}$$

Proposition 1.2. *The set of operators $S = \{I, T, P, E\}$ that consists of the identity I, the transpose T, the per-transpose P, and $E \equiv P \circ T$, forms a Klein Four-Group, $\{S, \circ\}$ with the composition*

The group structure of $\{S, \circ\}$ can be easily shown taking into account that each operator is their own inverse, i.e. $T \circ T = I$, $P \circ P = I$, and $(P \circ T)^{-1} = T^{-1} \circ P^{-1} = T \circ P$, which implies $E \circ E = I$, and that the binary operation defined in such a way satisfies the associative property. On the other hand, by virtue of property 5, $T \circ P = P \circ T = E$. The closure property of the group under the composition can be shown by means of the Cayley table [1, 2]:

	I	T	P	E
I	I	T	P	E
T	T	I	E	P
P	P	E	I	T
E	E	P	T	I

1.1 Examples

To illustrate the differences with respect to the standard transpose, we can say that the per-transpose is an operation that converts columns into files with a rotation in the clockwise sense

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}^P = \begin{bmatrix} x_n & x_{n-1} & \dots & x_3 & x_2 & x_1 \end{bmatrix} \tag{1.4}$$

In general, let X be a $n \times 1$ matrix (column vector) their transpose and per-transpose are different file vectors related by the equation,

$$X^P = X^T J_n \tag{1.5}$$

Where $J_n \in \mathbb{R}^{n\times n}$ is known in the literature[4] as exchange matrix or permutation matrix. Their elements J_{ij} are given by

$$J_{ij} = \begin{cases} 1, & j = n - i + 1 \\ 0, & j \neq n - i + 1 \end{cases} \tag{1.6}$$

Making explicit the matrix representation

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{1.7}$$

Note that J_n satisfies an involutive property, i.e. J_n^2=1. It is easy to see that multiplying Eq.(1.5) on the right by J_n we obtain the relation,

$$X^T = X^P J_n \tag{1.8}$$

On the other hand, for square matrices the per-transpose is obtained reflecting terms with respect to the northeast-to-southwest diagonal

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^P = \begin{pmatrix} a_{33} & a_{23} & a_{13} \\ a_{32} & a_{22} & a_{12} \\ a_{31} & a_{21} & a_{11} \end{pmatrix} \tag{1.9}$$

1.2 Persymmetric matrices

There are some important classes of matrices that can be defined concisely in terms of the per-transpose operation. We begin giving a pair of definitions

Definition 1.3. (Persymmetric matrix) A real matrix A is said to be persymmetric if $P(A) = A$. In other words, if $A \in \mathbb{R}^{n\times n}$, then $[a_{ij}] = [a_{n-j+1,n-i+1}]$ $\forall 1 \leq i \leq n$, $1 \leq j \leq n$

Definition 1.4. (Skew-persymmetric matrix) A real matrix A is said to be skew-persymmetric if $P(A) = -A$. In other words, if $A \in \mathbb{R}^{n\times n}$, then $[a_{ij}] = -[a_{n-j+1,n-i+1}]$ $\forall 1 \leq i \leq n$, $1 \leq j \leq n$

For instance, the general form of a 2×2 persymmetric matrix A and skew-persymmetric matrix B will be, respectively

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} \qquad B = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \tag{1.10}$$

Proposition 1.5. *Any square matrix $A \in \mathbb{R}^{n\times n}$ can be decomposed as $A = B + C$, where B is persymmetric and C is skew-persymmetric.*

Proof. Simply define $B \equiv \frac{1}{2}\Big(A + P(A)\Big)$, and $C \equiv \frac{1}{2}\Big(A - P(A)\Big)$. From the definition it automatically follows that, $P(B) = B$, $P(C) = -C$ □

For example, the form of this decomposition for a general 2×2 square real matrix will be

$$A \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha+\delta & 2\beta \\ 2\gamma & \delta+\alpha \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha-\delta & 0 \\ 0 & \delta-\alpha \end{pmatrix} \tag{1.11}$$

Proposition 1.6. *Let $A \in \mathbb{R}^{n\times n}$. Then, the composition $T \circ P(A)$, turns out to be equal to, $T \circ P(A) = J_n A J_n$*

Proof.

$$\begin{aligned} \Big[\Big(J_n A J_n\Big)_{ij}\Big] &= \sum_{k=1}^{n}\sum_{l=1}^{n} J_{ik}[a_{kl}]J_{lj} = \sum_{k=1}^{n} J_{ik}[a_{k,n-j+1}] \\ &= [a_{n-i+1,n-j+1}] = T \circ P([a_{ij}]) \end{aligned} \tag{1.12}$$

□

Prop.1.6 allows one to study the properties of persymmetric matrices. Indeed, let us suppose that a square matrix turns out to be equal to their per-transpose, then $P(A) = A$. In that case the general relation $T \circ P(A) = J_n A J_n$ acquires the form

$$T(A) = J_n A J_n \tag{1.13}$$

Where $T(A) = A^T$ is the transpose. Now, if we multiply Eq.(1.13) on the left by the exchange matrix J we get the result

$$J_n A^T = A J_n \tag{1.14}$$

Which is a well known property that satisfy any persymmetric matrix [5, 6]. Therefore, we have shown that a persymmetric matrix is only a square matrix that turns out to be equal to their per-transpose.

On the other hand, Prop.1.6 allows as well to give as a corollary a compact expression for the per-transpose operation when square matrices are involved. Indeed, since $J_n^T = J_n$, with a bit of algebra we find

$$P(A) = J_n T(A) J_n \tag{1.15}$$

Or in a more familiar notation, $A^P = J_n A^T J_n$. From this relation, Eq.(1.14) can be more directly obtained when $A^P = A$. For skew-persymmetric matrices ($A^P = -A$), the substitution in Eq.(1.15) provides, $J_n A^T = -AJ_n$. It is easy to generalize Eq.(1.15) for non-square matrices. Indeed, if $A \in \mathbb{R}^{n\times m}$, then

$$P(A) = J_m T(A) J_n \tag{1.16}$$

1.3 Bisymmetric and centrosymmetric matrices

The per-transpose operation will also allow us to give an alternative definition of bisymmetric and centrosymmetric matrices and to prove a well known definition property in the literature of these matrices, namely, if A is bisymmetric or centrosymmetric, then satisfies the equation, $J_n A = AJ_n$, a property that can be derived as a particular case from Prop.1.6, or even more directly from Eq.(1.15)

Definition 1.7. (Bisymmetric matrix) A real matrix A is said to be bisymmetric if $P(A) = T(A) = A$. In other words, if $A \in \mathbb{R}^{n\times n}$, then $[a_{ij}] = [a_{n-j+1,n-i+1}] = [a_{ji}]$ for all $1 \le i \le n$, $1 \le j \le n$

Definition 1.8. (Skew-bisymmetric matrix) A real matrix A is said to be skew-bisymmetric if $P(A) = T(A) = -A$. In other words, if $A \in \mathbb{R}^{n\times n}$, then $[a_{ij}] = -[a_{n-j+1,n-i+1}] = -[a_{ji}]$ for all $1 \le i \le n$, $1 \le j \le n$

Note that if a square matrix A is bisymmetric, namely, it turns out to be equal to their transpose and per-transpose simultaneously, then by virtue of Eq.(1.15) we have

$$P(A) = A = J_n A J_n \tag{1.17}$$

Which automatically implies that, $AJ_n = J_n A$. Such relation is also satisfied by any centrosymmetric matrix, as we will show. In fact, these properties, usually accepted in the literature as part of the definitions of these matrices, are derived in this work as a consequence of the novel and more fundamental definitions introduced.

Proposition 1.9. *Let $A \in \mathbb{R}^{2\times 2}$. The matrix $B \in \mathbb{R}^{2\times 2}$ defined as, $B \equiv T(A) + P(A)$ is bisymmetric*

Proof. If $B \equiv T(A) + P(A)$, then their per-transpose $P(B)$ will be equal to

$$P(B) = P \circ T(A) + A \tag{1.18}$$

where by virtue of the Caley table of Prop. 1.2 for the composition we have that, $P \circ P = I$, where I is the identity, and therefore $P \circ P(A) = A$. On the other hand, the transpose $T(B)$ will be

$$T(B) = A + T \circ P(A) \tag{1.19}$$

where have used the involutive property of the transpose, namely, $T \circ T = I$, and therefore, $T \circ T(A) = A$. Since, $T \circ P = P \circ T$, as we proved in Eq.(1.2) (the Klein Group is commutative), then we can conclude that $T(B) = P(B)$. On the other hand, let us consider a 2×2 general square matrix A given by, $A \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Therefore, the matrix B defined above will be

$$B \equiv T(A) + P(A) = \begin{pmatrix} \alpha+\delta & \beta+\gamma \\ \gamma+\beta & \alpha+\delta \end{pmatrix} \tag{1.20}$$

Which is bisymmetric, i.e. it turns out to be equal to their transpose and per-transpose simultaneously. □

It is worth noting that given an arbitrary matrix $A \in \mathbb{R}^{n\times n}$, the construction $B \equiv T(A) + P(A)$ will not be bisymmetric in the general case. For example, let us consider the case of a general matrix $A \in \mathbb{R}^{3\times 3}$ given by,

$$A \equiv \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \eta & \theta & \iota \end{pmatrix} \tag{1.21}$$

The construction $B = T(A) + P(A)$ for this example gives the result

$$B = \begin{pmatrix} \alpha+\iota & \delta+\zeta & \gamma+\eta \\ \beta+\theta & 2\epsilon & \beta+\theta \\ \gamma+\eta & \delta+\zeta & \alpha+\iota \end{pmatrix} \tag{1.22}$$

This matrix is centrosymmetric (symmetric about its centre, in this case the element 2ϵ) but it is not a bisymmetric matrix. Then, given an arbitrary square matrix A, the construction $B \equiv T(A)+P(A)$, will only provide a bisymmetric matrix in the 2×2 case. Nevertheless, there exist a method to build a bisymmetric matrix from a general square matrix of arbitrary dimension. This is achieved by adding to the transpose and per-transpose two additional pieces, where the two other operators of the Klein 4-Group $\{I, T, P, E\}$ play a role.

Proposition 1.10. *Let $A \in \mathbb{R}^{n\times n}$. Then, the matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$ defined as, $\mathcal{B} \equiv P\Big(A + T(A)\Big) + A + T(A)$ is bisymmetric*

Proof. First, let us develop the parenthesis,

$$\mathcal{B} \equiv P\Big(A+T(A)\Big) + A + T(A) = P(A) + P\circ T(A) + A + T(A) \tag{1.23}$$

The per-transpose $P(\mathcal{B})$ and transpose $T(\mathcal{B})$ are identical and also agree with the own $\mathcal{B}$ matrix, since

$$\begin{aligned} P(\mathcal{B}) &= P\circ P(A) + P\circ P\circ T(A) + P(A) + P\circ T(A) \\ &= A + I\circ T(A) + P(A) + P\circ T(A) = \mathcal{B} \\ &= T\circ T(A) + T(A) + T\circ T\circ P(A) + T\circ P(A) \\ &= T(\mathcal{B}) \end{aligned} \tag{1.24}$$

□

Remark 1.11. The construction, $\mathcal{C} \equiv T\Big(A + P(A)\Big) + A + P(A)$ is also a bisymmetric matrix. Furthermore, since $T\circ P = P\circ T$, it is easy to see that, $\mathcal{B} = \mathcal{C}$.

Note that in the bisymmetric matrix $\mathcal{B}$ above, appear all the four operators of the Klein 4-Group defined by Prop. 1.2. Indeed, $\mathcal{B}$ can be rewritten in the form $\mathcal{B} = (T + P + I + E)(A)$, where $E = T\circ P$.

Definition 1.12. (Centrosymmetric matrix) A real matrix A is said to be centrosymmetric if $P(A) = T(A) \neq A$. In other words, if $A \in \mathbb{R}^{n\times n}$ with elements, $[a_{ij}]$, then $[a_{n-j+1,n-i+1}] = [a_{ji}]$ for all $1 \leq i \leq n$, $1 \leq j \leq n$

Proposition 1.13. *If a real square matrix $A \in \mathbb{R}^{n\times n}$ is centrosymmetric, then it satisfies the property, $J_n A = A J_n$*

Proof. By virtue of Prop .1.6, if $P(A) = T(A)$ we have

$$T\circ P(A) = T\circ T(A) = A = J_n A J_n \tag{1.25}$$

□

Proposition 1.14. *Let $A \in \mathbb{R}^{n\times n}$. The pair of matrices defined as, $\mathcal{R} \equiv P(A) + T(A)$, and $\mathcal{S} \equiv P(A)T(A) + T(A)P(A)$ are both centrosymmetric*

Proof. Since $P \circ T = T \circ P$, it is direct to verify that $P(\mathcal{R}) = T(\mathcal{R})$. On the other hand, by virtue of property 4 of the per-transpose, $P(\mathcal{S})$ will be equal to

$$\begin{aligned}
P(\mathcal{S}) &= P \circ \Big(P(A)T(A)\Big) + P \circ \Big(T(A)P(A)\Big) \\
&= P \circ T(A) P \circ P(A) + P \circ P(A) P \circ T(A) \\
&= \Big(T \circ P(A)\Big)A + A\Big(T \circ P(A)\Big) \\
&= T \circ P(A) T \circ T(A) + T \circ T(A) T \circ P(A) \\
&= T \circ \Big(T(A)P(A)\Big) + T \circ \Big(P(A)T(A)\Big) \\
&= T(\mathcal{S})
\end{aligned} \tag{1.26}$$

□

As an example of the later constructions, let us consider a random 3×3 real matrix, for instance

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \\ 0 & 6 & -1 \end{pmatrix} \tag{1.27}$$

Their transpose and per-transpose will be

$$T(A) = \begin{pmatrix} 1 & 4 & 0 \\ -2 & 5 & 6 \\ 3 & -2 & -1 \end{pmatrix} \qquad P(A) = \begin{pmatrix} -1 & -2 & 3 \\ 6 & 5 & -2 \\ 0 & 4 & 1 \end{pmatrix} \tag{1.28}$$

Therefore, the two centrosymmetric matrices $\mathcal{R} \equiv P(A)+T(A)$, $\mathcal{S} \equiv T(A)P(A)+P(A)T(A)$, and the bisymmetric $\mathcal{B} \equiv P\Big(A + T(A)\Big) + A + T(A)$, for this example are given by

$$\mathcal{R} = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 10 & 4 \\ 3 & 2 & 0 \end{pmatrix} \qquad \mathcal{S} = \begin{pmatrix} 55 & -2 & -20 \\ 22 & 106 & 22 \\ -20 & -2 & 55 \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 0 & 6 & 6 \\ 6 & 20 & 6 \\ 6 & 6 & 0 \end{pmatrix} \tag{1.29}$$

1.4 Degenerate scalar product, dyadic product and cross product of vectors.

The per-transpose operation will allow us to define an alternative (degenerate) scalar product of vectors which share some properties with the ordinary one. In addition, by means of the per-transpose we can also define an alternative dyadic product of vectors, which takes a pair of vectors and returns a square matrix, and finally another operation that converts two vectors into another vector, an operation similar to the ordinary vector product.

Definition 1.15. Let $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, we define the following inner product $< \mathbf{x}, \mathbf{y} >^* \equiv \mathbf{x}^{\mathbf{P}}\mathbf{y}$, where $\mathbf{x}^P$ is the per-transpose of $\mathbf{x}$

By virtue of Eq.(1.4) if $\mathbf{x}$ is a column vector, then $\mathbf{x}^P = \begin{bmatrix} x_n & x_{n-1} & \ldots & x_3 & x_2 & x_1 \end{bmatrix}$, therefore, the scalar product $\mathbf{x^P y}$ will be

$$< \mathbf{x}, \mathbf{y} >^{*} \equiv \mathbf{x^P y} = \begin{bmatrix} x_n & x_{n-1} \dots & x_3 & x_2 & x_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$= x_n y_1 + x_{n-1} y_2 + \dots x_2 y_{n-1} + x_1 y_n = \sum_{i=1}^{n} x_i \cdot y_{n-i+1} \tag{1.30}$$

Making use of Eq.(1.5) we can see that the exchange matrix J_n appears in this scalar product in a natural way:

$$\mathbf{x^P y} = \mathbf{x^T} J_n \mathbf{y} = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_{n-1} & x_n \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \tag{1.31}$$

Then, if the elements δ_{ij} of the identity matrix can be interpreted as the standard scalar products $\mathbf{e}_i \cdot \mathbf{e}_j$ of the elements of the canonical basis, the elements of the exchange matrix J_{ij} can also be interpreted in the same way, namely

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \qquad \mathbf{e}_i^{\mathbf{P}} \cdot \mathbf{e}_j = J_{ij} \tag{1.32}$$

where $\mathbf{e}_i^{\mathbf{P}}$ stands for the per-transpose of $\mathbf{e}_i$. It is therefore clear that such scalar product should be degenerate, since $\mathbf{e}_i^{\mathbf{P}} \cdot \mathbf{e}_i = J_{ii} = 0$. In other words, $< \mathbf{x}, \mathbf{x} >^{*} = 0$ does not necessarily imply that $\mathbf{x} = 0$. We list now some algebraic properties that follow from the definition.

(i) $\mathbf{x^P y} = \mathbf{y^P x} \quad \forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

(ii) $\mathbf{x^P}(\alpha \mathbf{y} + \beta \mathbf{y}) = \alpha \mathbf{x^P y} + \beta \mathbf{x^P y} \quad \forall\, \alpha, \beta \in \mathbb{R}$

(iii) $(\alpha \mathbf{x})^{\mathbf{P}} (\beta \mathbf{y}) = \alpha \beta \mathbf{x^P y}$

These properties can be easily proved from the definition. Furthermore, there exists certain interesting property that deserves to be treated separately:

$$(\mathbf{x^T})^{\mathbf{P}} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{y}(\mathbf{x^P z}) - \mathbf{z}(\mathbf{x^P y}) \tag{1.33}$$

Where, $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$. This property is the analogue to the Lagrange triple product property that involves a cross product of three-dimensional vectors defined by

$$(\mathbf{a^T})^{\mathbf{P}} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_3 & a_2 & a_1 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{1.34}$$

And attending to their components

$$\left((\mathbf{a^T})^{\mathbf{P}} \times \mathbf{b} \right)_i = \sum_j \sum_k \epsilon_{ijk} a_{4-j} b_k \tag{1.35}$$

It is obvious from the definition that, such as the standard cross product, this product is not commutative i.e. $(\mathbf{a^T})^{\mathbf{P}} \times \mathbf{b} \neq (\mathbf{b^T})^{\mathbf{P}} \times \mathbf{a}$. Notice that if $\mathbf{a}$ is certain file vector given by $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$, then the operation $(\mathbf{a^T})^{\mathbf{P}} \equiv P \circ T(\mathbf{a})$ provides another file vector formed by the permutation among the first and third elements, i.e. $(\mathbf{a^T})^\mathbf{P} = \begin{bmatrix} a_3 & a_2 & a_1 \end{bmatrix}$. The exact algebraic relation among the vector product $(\mathbf{a^T})^\mathbf{P} \times \mathbf{b}$ and their symmetric is given by

$$(\mathbf{a^T})^\mathbf{P} \times \mathbf{b} = \left[\left((\mathbf{b^T})^\mathbf{P} \times \mathbf{a}\right)^\mathbf{T}\right]^\mathbf{P} \tag{1.36}$$

On the other hand, given a pair of column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we can define a dyadic product different from the standard $\mathbf{xy^T}$, which returns another $n \times n$ square matrix. Indeed, notice that the product $\mathbf{xy^P}$ provides a different matrix from $\mathbf{xy^T}$, where $\mathbf{y^P} \equiv P(\mathbf{y})$ is as usual the per-transpose of $\mathbf{y}$.
Making explicit the matrix representation we have

$$\mathbf{xy^P} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \begin{bmatrix} y_n & y_{n-1} \dots & y_3 & y_2 & y_1 \end{bmatrix} = \begin{bmatrix} x_1 y_n & x_1 y_{n-1} & \cdots & x_1 y_1 \\ x_2 y_n & x_2 y_{n-1} & \cdots & x_2 y_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_n & x_n y_{n-1} & \cdots & x_n y_1 \end{bmatrix} \tag{1.37}$$

The trace of this matrix allows to find a relation among the dyadic product and the scalar product defined before:

$$\text{tr}(\mathbf{xy^P}) = \mathbf{x^P y} \tag{1.38}$$

1.5 The exchange matrix and the Klein-Group

In the literature, there exists several realizations of the Klein Group[7] involving square matrices. In this last part of the note we give another realization of the Klein Group which involves the exchange matrix and the Kronecker product. Our building blocks are the two matrices:

$$J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.39}$$

Now, let us define the 4×4 matrices: $J_4 \equiv J_2 \otimes J_2$, $I_4 \equiv I_2 \otimes I_2$, $K_4 \equiv J_2 \otimes I_2$, $L_4 \equiv I_2 \otimes J_2$, where $\otimes$ stands for the Kronecker product. Making explicit the matrix representation, these matrices are constructed by the blocks

$$J_4 \equiv J_2 \otimes J_2 = \begin{bmatrix} 0 & J_2 \\ J_2 & 0 \end{bmatrix} \quad I_4 \equiv I_2 \otimes I_2 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} \tag{1.40}$$

$$K_4 \equiv J_2 \otimes I_2 = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \quad L_4 \equiv I_2 \otimes J_2 = \begin{bmatrix} J_2 & 0 \\ 0 & J_2 \end{bmatrix} \tag{1.41}$$

It is not difficult to see that the set $S = \{I_4, J_4, K_4, L_4\}$ forms a group with the standard matrix multiplication. Indeed, note that each element is their own inverse i.e. $J_4 J_4 = I_4 = K_4 K_4 = L_4 L_4$, and also $J_4 K_4 = L_4$, $K_4 L_4 = J_4$, $L_4 J_4 = K_4$. Note that the group is commutative. The Cayley table of the group is given by

	J_4	I_4	L_4	K_4
J_4	I_4	J_4	K_4	L_4
I_4	J_4	I_4	L_4	K_4
L_4	K_4	L_4	I_4	J_4
K_4	L_4	K_4	J_4	I_4

References

[1] Cayley, A. "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$", Philosophical Magazine, Vol. 7 (1854), pp. 40-47

[2] Cayley, A. "On the Theory of Groups", American Journal of Mathematics, Vol. 11, No. 2 (Jan 1889), pp. 139-157.

[3] Cayley, A. "A Memoir on the Theory of Matrices" Philosophical Transactions of the Royal Society of London, Vol. 148 (1858), pp. 17-37

[4] Horn, Roger A.; Johnson, Charles R. (2012), Matrix Analysis (2nd ed.), Cambridge University Press, p. 33, ISBN 9781139788885.

[5] Golub, Gene H.; Van Loan, Charles F. (1996), Matrix Computations (3rd ed.), Baltimore: Johns Hopkins, ISBN 978-0-8018-5414-9. See page 193.

[6] Muir, Thomas (1960), Treatise on the Theory of Determinants, Dover Press.

[7] Klein, F. Vorlesungen ueber das Ikosaeder und die Aufloesung der Gleichungen vom fuenften Grade. 1884. Reprinted as Klein, F. Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree, 2nd rev. ed. New York: Dover, 1956.

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