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Some new Ostrowski type inequalities via Caputo k-fractional derivatives concerning $(n+1)$-differentiable generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvex mappings

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Abstract. In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvex mappings. And then, a new identity concerning $(n+1)$-differentiable mappings defined on m-invex set via Caputo k-fractional derivatives is derived. By using the notion of generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Ostrowski type inequalities via Caputo k-fractional derivatives are established. It is pointed out that some new special cases can be deduced from main results of the article.

1 Introduction

The subsequent inequality is known as Ostrowski inequality which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x\in[a,b]$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $a,b\in I^\circ$ with $a<b$. If $|f'(x)|\le M$ for all $x\in[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le M(b-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right],\quad \forall x\in[a,b]. \tag{1.1}$$

Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n-times differentiable mappings etc. appeared in a number of papers, see [2]-[4],[11]-[13],[15],[16],[18],[19],[22],[23],[27],[28],[30],[33],[36],[38],[39],[45],[47],[55],[57],[59],[61]. In recent years, one more dimension has been added to this studies, by introducing a number of integral inequalities involving various fractional operators like Riemann-Liouville, Erdelyi-Kober, Katugampola, conformable fractional integral operators etc. by many authors, see [1],[43],[48]-[53]. Riemann-Liouville fractional integral operators are the most central between these fractional operators.

In numerical analysis many quadrature rules have been established to approximate the definite integrals, see [14],[21],[32],[34],[35],[40],[44],[56],[58]. Ostrowski inequality provides the bounds for many numerical quadrature rules. In recent decades Ostrowski inequality is studied in fractional calculus point of view by many mathematicians, see [6]-[10],[24]-[26],[31],[41],[46].

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1.2. The Euler beta function is defined for $a,b>0$ as

$$\beta(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Definition 1.3. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{1.2}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.3}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

For $k = 1$, (1.3) gives integral representation of gamma function.

Definition 1.4. For $k \in \mathbb{R}^+$ and $x, y \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt. \tag{1.4}$$

For $k = 1$, (1.4) gives integral representation of beta function.

Theorem 1.5. *Let $x, y > 0$, then for k-gamma and k-beta function the following equality holds:*

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{1.5}$$

Definition 1.6. [26] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and are continuous on $[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$$^cD^\alpha_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \; x > a \tag{1.6}$$

and

$$^cD^\alpha_{b-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \; x < b. \tag{1.7}$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $\left(^cD^\alpha_{a+} f\right)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$\left(^cD^0_{a+} f\right)(x) = \left(^cD^0_{b-} f\right)(x) = f(x) \tag{1.8}$$

where $n = 1$ and $\alpha = 0$.

Definition 1.7. [20] Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$$^cD^{\alpha,k}_{a+} f(x) = \frac{1}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \; x > a \tag{1.9}$$

and

$$^cD^{\alpha,k}_{b-} f(x) = \frac{(-1)^n}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \; x < b. \tag{1.10}$$

Definition 1.8. [60] A set $M_\varphi \subseteq \mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \; \forall\, x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]. \tag{1.11}$$

Definition 1.9. [60] A function f is named as a relative convex (φ-convex) on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{1.12}$$

$\forall\, x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]$.

Definition 1.10. [14] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 1.11. [5] A set $K \subseteq \mathbb{R}^n$ is said to be invex respecting the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Definition 1.12. [32] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.13}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 1.12, f becomes a preinvex function, see [42]. If the mapping $\eta(y,x) = y - x$ in Definition(1.12), then the non-negative function f reduces to h-convex mappings, see [58].

Definition 1.13. [56] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function. A function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex on K, if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.14}$$

grips for all $x, y \in K$ and $t \in (0,1)$.

Definition 1.14. [30] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.15}$$

Definition 1.15. [35] A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, among $m \in (0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.16}$$

Definition 1.16. [41] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y, mx)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.17}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

Let us recall the Gauss-Jacobi type quadrature formula as follows.

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{1.18}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$, see [54].

In [29], Liu obtained integral inequalities for P-function related to the left-hand side of (1.18), and in [37], Özdemir et al. also presented several integral inequalities concerning the left-hand side of (1.18) via some kinds of convexity.

Motivated by the above literatures, the main objective of this article is to establish integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula and some new estimates on Ostrowski type inequalities via Caputo k-fractional derivatives associated with generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article.

2 Main results involving Gauss-Jacobi type quadrature formula

The following definitions will be used in this section.

Definition 2.1. [17] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ grips for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\varphi : I \longrightarrow K$ are continuous functions. A mapping $f : K \longrightarrow (0,+\infty)$, is said to be generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\big) \leq M_r(h_1(t), h_2(t); mf(x), f(y), p, q) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$, for $p, q > -1$ with some fixed $m \in (0,1]$, where

$$M_r(h_1(t), h_2(t); mf(x), f(y), p, q) := \begin{cases} \Big[mh_1^p(t)f^r(x) + h_2^q(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \big[mf(x)\big]^{h_1^p(t)}\big[f(y)\big]^{h_2^q(t)}, & \text{if } r = 0 \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2.4. In Definition 2.3, if we choose $r = p = q = 1$ and $\varphi(x) = x$, then we get Definition 1.16.

Remark 2.5. For $p = q = 1$, let us discuss some special cases in Definition 2.3 as follows.

(I) Taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m, s)-$ Breckner-preinvex mappings.
(II) Taking $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m, P)-preinvex mappings.
(III) Taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m, s)-Godunova-Levin-Dragomir-preinvex mappings.
(IV) Taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m, h)-preinvex mappings.
(V) Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m, tgs)-preinvex mappings.
(VI) Taking $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Let see the following example of a generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings which is not convex.

Example 2.6. Let taking $m = r = \frac{1}{2}$, $h_1(t) = t^l$, $h_2(t) = t^s$, for all $l, s \in [0,1]$, for any fixed $p, q \geq 1$ and $\varphi(x) = x$. Consider the mapping $f : [0,+\infty) \longrightarrow [0,+\infty)$ as follows

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

Define a bifunction $\eta : [0,+\infty) \times [0,+\infty) \times \{\frac{1}{2}\} \longrightarrow \mathbb{R}$ by

$$\eta(y, x, m) = \begin{cases} -y, & 0 \leq y \leq 1; \\ x + y, & y > 1. \end{cases}$$

Then f is generalized relative semi-$(\frac{1}{2}; \frac{1}{2}, p, q, t^l, t^s)$-preinvex mapping for any fixed $p, q \geq 1$ and for all $l, s \in [0, 1]$. But f is not preinvex with respect to η and also it is not convex (consider $x = 0, y = 2$ and $t \in (0, 1]$).

We claim the following integral identity.

Lemma 2.7. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous mapping on K° (the interior of K) with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and $p, q > 0$, we have*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
&= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1 - t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.
\end{aligned} \tag{2.2}$$

Proof. It is easy to observe that

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
&= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\
&\quad \times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q \\
&\quad \times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt \\
&= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1 - t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.
\end{aligned}$$

This completes the proof of the lemma. □

With the help of Lemma 2.7, we have the following results.

Theorem 2.8. *Let $k > 1$ and $0 < r \leq 1$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for $\eta(\varphi(b), \varphi(a), m) > 0$. If $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r; m, \overline{p}, \overline{q}, h_1, h_2)$-preinvex mappings on an open m-invex set K for some fixed $m \in (0, 1]$, where $\overline{p}, \overline{q} > -1$, then for any fixed $p, q > 0$, we have*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
&\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1) \\
&\quad \times \left[m f^{\frac{rk}{k-1}}(a)\Psi^r(h_1(t); r, \overline{p}) + f^{\frac{rk}{k-1}}(b)\Psi^r(h_2(t); r, \overline{q}) \right]^{\frac{k-1}{rk}},
\end{aligned} \tag{2.3}$$

where

$$\Psi(h_1(t); r, \overline{p}) := \int_0^1 h_1^{\frac{\overline{p}}{r}}(t)dt, \quad \Psi(h_2(t); r, \overline{q}) := \int_0^1 h_2^{\frac{\overline{q}}{r}}(t)dt. \tag{2.4}$$

Proof. Since $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r; m, \overline{p}, \overline{q}, h_1, h_2)$-preinvex mappings on K, combining with Lemma 2.7, Hölder inequality and Minkowski inequality, we get

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
&\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\int_0^1 t^{kp}(1 - t)^{kq} dt \right]^{\frac{1}{k}}
\end{aligned}$$

$$\times\left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{k-1}{k}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$
$$\times\left[\int_0^1\left[mh_1^{\overline{p}}(t)f^{\frac{rk}{k-1}}(a)+h_2^{\overline{q}}(t)f^{\frac{rk}{k-1}}(b)\right]^{\frac{1}{r}}dt\right]^{\frac{k-1}{k}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$
$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h_1^{\frac{\overline{p}}{r}}(t)f^{\frac{k}{k-1}}(a)dt\right)^r+\left(\int_0^1 h_2^{\frac{\overline{q}}{r}}(t)f^{\frac{k}{k-1}}(b)dt\right)^r\right\}^{\frac{k-1}{rk}}$$
$$=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$
$$\times\left[mf^{\frac{rk}{k-1}}(a)\Psi^r(h_1(t);r,\overline{p})+f^{\frac{rk}{k-1}}(b)\Psi^r(h_2(t);r,\overline{q})\right]^{\frac{k-1}{rk}}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 2.8.

Corollary 2.9. *In Theorem 2.8 for $r=\overline{p}=\overline{q}=1$ and $h_1(t)=h(1-t)$, $h_2(t)=h(t)$, we have the following inequality for generalized relative semi-(m,h)-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\Psi^{\frac{k-1}{k}}(h(t);1,1)\left[mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}. \tag{2.5}$$

Corollary 2.10. *In Theorem 2.8 for $r=\overline{p}=\overline{q}=1$ and $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{s+1}\right]^{\frac{k-1}{k}}. \tag{2.6}$$

Corollary 2.11. *In Theorem 2.8 for $r=\overline{p}=\overline{q}=1$ and $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{1-s}\right]^{\frac{k-1}{k}}. \tag{2.7}$$

Corollary 2.12. *In Theorem 2.8 for $r=\overline{p}=\overline{q}=1$ and $h_1(t)=h_2(t)=t(1-t)$, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)}{6}\right]^{\frac{k-1}{k}}. \tag{2.8}$$

Corollary 2.13. *In Theorem 2.8 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\ &\le \left(\frac{\pi}{4}\right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\Big[mf^{\frac{k}{k-1}}(a)+f^{\frac{k}{k-1}}(b)\Big]^{\frac{k-1}{k}}.\end{aligned} \tag{2.9}$$

Theorem 2.14. *Let $l \ge 1$ and $0 < r \le 1$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,+\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for $\eta(\varphi(b),\varphi(a),m) > 0$. If f^l is generalized relative semi-$(r; m, \overline{p}, \overline{q}, h_1, h_2)$-preinvex mappings on an open m-invex set K for some fixed $m \in (0,1]$, where $\overline{p}, \overline{q} > -1$, then for any fixed $p, q > 0$, we have*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\ &\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\ &\times\Big[mf^{rl}(a)I^r(h_1(t);r,p,q,\overline{p})+f^{rl}(b)I^r(h_2(t);r,p,q,\overline{q})\Big]^{\frac{1}{rl}},\end{aligned} \tag{2.10}$$

where

$$I(h_1(t);r,p,q,\overline{p}) := \int_0^1 t^p(1-t)^q h_1^{\frac{\overline{p}}{r}}(t)dt,\quad I(h_2(t);r,p,q,\overline{q}) := \int_0^1 t^p(1-t)^q h_2^{\frac{\overline{q}}{r}}(t)dt. \tag{2.11}$$

Proof. Since f^l is generalized relative semi-$(r; m, \overline{p}, \overline{q}, h_1, h_2)$-preinvex mappings on K, combining with Lemma 2.7, the well-known power mean inequality and Minkowski inequality, we get

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\ &= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}\Big[t^p(1-t)^q\Big]^{\frac{1}{l}}\\ &\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\\ &\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\\ &\times\left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{1}{l}}\\ &\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\ &\times\left[\int_0^1 t^p(1-t)^q\Big[mh_1^{\overline{p}}(t)f^{rl}(a)+h_2^{\overline{q}}(t)f^{rl}(b)\Big]^{\frac{1}{r}}dt\right]^{\frac{1}{l}}\\ &\le \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\ &\times\left\{\left(\int_0^1 m^{\frac{1}{r}}t^p(1-t)^q h_1^{\frac{\overline{p}}{r}}(t)f^l(a)dt\right)^r+\left(\int_0^1 t^p(1-t)^q h_2^{\frac{\overline{q}}{r}}(t)f^l(b)dt\right)^r\right\}^{\frac{1}{rl}}\\ &= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\ &\times\Big[mf^{rl}(a)I^r(h_1(t);r,p,q,\overline{p})+f^{rl}(b)I^r(h_2(t);r,p,q,\overline{q})\Big]^{\frac{1}{rl}}.\end{aligned}$$

So, the proof of this theorem is complete. □

Let us discuss some special cases of Theorem 2.14.

Corollary 2.15. *In Theorem 2.14 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, we have the following inequality for generalized relative semi-(m,h)-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.12}$$
$$\times\Big[mf^l(a)I(h(t);1,p,q,1) + f^l(b)I(h(t);1,q,p,1)\Big]^{\frac{1}{l}}.$$

Corollary 2.16. *In Theorem 2.14 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.13}$$
$$\times\Big[mf^l(a)\beta(p+1,q+s+1) + f^l(b)\beta(q+1,p+s+1)\Big]^{\frac{1}{l}}.$$

Corollary 2.17. *In Theorem 2.14 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.14}$$
$$\times\Big[mf^l(a)\beta(p+1,q-s+1) + f^l(b)\beta(q+1,p-s+1)\Big]^{\frac{1}{l}}.$$

Corollary 2.18. *In Theorem 2.14 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = h_2(t) = t(1-t)$, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\beta^{\frac{1}{l}}(p+2,q+2)\Big[mf^l(a) + f^l(b)\Big]^{\frac{1}{l}}. \tag{2.15}$$

Corollary 2.19. *In Theorem 2.14 for $r = \overline{p} = \overline{q} = 1$ and $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.16}$$
$$\times\left[mf^l(a)\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right) + f^l(b)\beta\left(q+\frac{1}{2},p+\frac{3}{2}\right)\right]^{\frac{1}{l}}.$$

3 Other results involving Caputo k-fractional derivatives

For establishing our main results regarding some new Ostrowski type integral inequalities associated with generalized relative semi-$(r;m,p,q,h_1,h_2)$-preinvexity via Caputo k-fractional derivatives, we need the following lemma.

Lemma 3.1. *Let $\alpha > 0$, $k \geq 1$, $r \geq 0$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Also, let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$. Assume that $f : K \longrightarrow \mathbb{R}$ is a mapping on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. Then we have the following equality for Caputo k-fractional derivatives:*

$$\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(a),m)f^{(n)}(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{(r+1)^{n-\frac{\alpha}{k}}\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(b),m)f^{(n)}(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{(r+1)^{n-\frac{\alpha}{k}}\eta(\varphi(b),\varphi(a),m)}$$
$$+(-1)^{n+1}\frac{(nk-\alpha)\Gamma_k\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Bigg[{}^cD^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))^-}f\left(m\varphi(a)+\frac{r}{r+1}\eta(\varphi(x),\varphi(a),m)\right)$$
$$-{}^cD^{\alpha,k}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))^-}f\left(m\varphi(b)+\frac{r}{r+1}\eta(\varphi(x),\varphi(b),m)\right)\Bigg]$$
$$=\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{(r+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)} \tag{3.1}$$
$$\times\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(a),m)\right)dt$$
$$-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{(r+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}$$
$$\times\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}\left(m\varphi(b)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(b),m)\right)dt.$$

We denote

$$I_{f,\eta,\varphi}(x;\alpha,k,r,n,m,a,b) := \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{(r+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)} \tag{3.2}$$
$$\times\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(a),m)\right)dt$$
$$-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{(r+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}$$
$$\times\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}\left(m\varphi(b)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(b),m)\right)dt.$$

Proof. Integrating by parts, we get

$$I_{f,\eta,\varphi}(x;\alpha,k,r,n,m,a,b) = \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(a),m)}{(r+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}$$
$$\times\left[\frac{(r+1)t^{n-\frac{\alpha}{k}}f^{(n)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(a),m)\right)}{\eta(\varphi(x),\varphi(a),m)}\Bigg|_0^1\right.$$
$$\left.-\frac{(r+1)\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(x),\varphi(a),m)}\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(m\varphi(a)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(a),m)\right)dt\right]$$
$$-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),\varphi(b),m)}{(r+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[\frac{(r+1)t^{n-\frac{\alpha}{k}}f^{(n)}\left(m\varphi(b)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(b),m)\right)}{\eta(\varphi(x),\varphi(b),m)}\Bigg|_0^1\right.$$

$$\left.-\frac{(r+1)\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(x),\varphi(b),m)}\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}\left(m\varphi(b)+\left(\frac{r+t}{r+1}\right)\eta(\varphi(x),\varphi(b),m)\right)dt\right]$$

$$=\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(a),m)f^{(n)}(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{(r+1)^{n-\frac{\alpha}{k}}\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),\varphi(b),m)f^{(n)}(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{(r+1)^{n-\frac{\alpha}{k}}\eta(\varphi(b),\varphi(a),m)}$$

$$+(-1)^{n+1}\frac{(nk-\alpha)\Gamma_k\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[{}^cD^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))^-}f\left(m\varphi(a)+\frac{r}{r+1}\eta(\varphi(x),\varphi(a),m)\right)\right.$$

$$\left.-{}^cD^{\alpha,k}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))^-}f\left(m\varphi(b)+\frac{r}{r+1}\eta(\varphi(x),\varphi(b),m)\right)\right].$$

This completes the proof of the lemma. □

Using Lemma 3.1, we now state the following theorems for the corresponding version for power of $(n+1)$-derivative.

Theorem 3.2. *Let $\alpha>0$, $k\geq 1$, $r_1\geq 0$, $0<r\leq 1$, $p_1,p_2>-1$ and $\alpha\notin\{1,2,3,\dots\}$, $n=[\alpha]+1$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Suppose $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$. Assume that $f:K\longrightarrow(0,+\infty)$ is a mapping on K° such that $f\in C^{n+1}[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m)>0$. If $\left(f^{(n+1)}\right)^q$ is generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex mappings, $q>1$, $p^{-1}+q^{-1}=1$, then the following inequality for Caputo k-fractional derivatives holds:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,r_1,n,m,a,b)\right|\leq\frac{1}{(r_1+1)^{n-\frac{\alpha}{k}+1}}\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1,p_1)\right.\right.$$

$$\left.+\left(f^{(n+1)}(x)\right)^{rq}I^r(h_2(t);r,r_1,p_2)\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(b)\right)^{rq}I^r(h_1(t);r,r_1,p_1)\right.$$

$$\left.\left.+\left(f^{(n+1)}(x)\right)^{rq}I^r(h_2(t);r,r_1,p_2)\right]^{\frac{1}{rq}}\right\},\tag{3.3}$$

where

$$I(h_i(t);r,r_1,p_i):=\int_0^1 h_i^{\frac{p_i}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt,\ \forall\, i=1,2.$$

Proof. From Lemma 3.1, generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvexity of $\left(f^{(n+1)}\right)^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,r_1,n,m,a,b)\right|$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(m\varphi(b)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right|dt$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(n-\frac{\alpha}{k}\right)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(n-\frac{\alpha}{k}\right)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left(f^{(n+1)}\left(m\varphi(b)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(n-\frac{\alpha}{k}\right)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[mh_1^{p_1}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}+h_2^{p_2}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(x)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(n-\frac{\alpha}{k}\right)}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[mh_1^{p_1}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(b)\right)^{rq}+h_2^{p_2}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(x)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(n-\frac{\alpha}{k}\right)}dt\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q h_1^{\frac{p_1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1\left(f^{(n+1)}(x)\right)^q h_2^{\frac{p_2}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{p\left(n-\frac{\alpha}{k}\right)}dt\right)^{\frac{1}{p}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(b)\right)^q h_1^{\frac{p_1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1\left(f^{(n+1)}(x)\right)^q h_2^{\frac{p_2}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{1}{(r_1+1)^{n-\frac{\alpha}{k}+1}}\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(a)\right)^{rq}I^{r}(h_1(t);r,r_1,p_1)\right.\right.$$

$$\left.+\left(f^{(n+1)}(x)\right)^{rq}I^{r}(h_2(t);r,r_1,p_2)\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(b)\right)^{rq}I^{r}(h_1(t);r,r_1,p_1)\right.$$

$$\left.\left.+\left(f^{(n+1)}(x)\right)^{rq}I^{r}(h_2(t);r,r_1,p_2)\right]^{\frac{1}{rq}}\right\}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 3.2.

Corollary 3.3. *In Theorem 3.2 for* $h_1(t)=h_2(t)=h(t)$, $p_1=p_2=m=k=r=1, r_1=0$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$ *and* $f^{(n+1)}\leq K$, *we get the following inequality for Caputo fractional derivatives:*

$$\left|\left[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\right]f^{(n)}(x)+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\left[{}^{c}D^{\alpha}_{x-}f(a)-{}^{c}D^{\alpha}_{x-}f(b)\right]\right|$$

$$\leq\frac{2^{\frac{1}{q}}K}{\left[p(n-\alpha)+1\right]^{\frac{1}{p}}}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right].\tag{3.4}$$

Corollary 3.4. *In Theorem 3.2 for* $r_1=0$, $h_1(t)=h(1-t)$ *and* $h_2(t)=h(t)$, *we have the following inequality for generalized relative semi-*$(r;m,p_1,p_2,h)$*-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right|\leq\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}\tag{3.5}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\right.$$

$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}I^{r}(h(1-t);r,0,p_1)+\left(f^{(n+1)}(x)\right)^{rq}I^{r}(h(t);r,0,p_2)\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}$$

$$\left.\times\left[m\left(f^{(n+1)}(b)\right)^{rq}I^{r}(h(1-t);r,0,p_1)+\left(f^{(n+1)}(x)\right)^{rq}I^{r}(h(t);r,0,p_2)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 3.5. *In Theorem 3.2 for* $r_1=0$, $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, *we have the following inequality for generalized relative semi-*$(r;m,p_1,p_2,s)$*-Breckner-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right|\leq\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}\tag{3.6}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(a)\right)^{rq}\left(\frac{r}{r+sp_1}\right)^{r}+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{r}{r+sp_2}\right)^{r}\right]^{\frac{1}{rq}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(b)\right)^{rq}\left(\frac{r}{r+sp_1}\right)^{r}+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{r}{r+sp_2}\right)^{r}\right]^{\frac{1}{rq}}\right\}.$$

Corollary 3.6. *In Theorem 3.2 for $r_1 = 0$, $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, s)$–Godunova-Levin-Dragomir-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \leq \left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.7}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(a)\right)^{rq}\left(\frac{r}{r-sp_1}\right)^{r}+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{r}{r-sp_2}\right)^{r}\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(b)\right)^{rq}\left(\frac{r}{r-sp_1}\right)^{r}+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{r}{r-sp_2}\right)^{r}\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3.7. *In Theorem 3.2 for $r_1 = 0$, $h_1(t) = h_2(t) = t(1-t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, tgs)$-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \leq \left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.8}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}\beta^{r}\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\beta^{r}\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(b)\right)^{rq}\beta^{r}\left(\frac{p_1}{r}+1,\frac{p_1}{r}+1\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\beta^{r}\left(\frac{p_2}{r}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3.8. *In Theorem 3.2 for $r_1 = 0$, $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2)$-MT-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \leq \left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.9}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^{r}\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^{r}\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(b)\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^{r}\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^{r}\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Theorem 3.9. *Let $\alpha > 0$, $k \geq 1$, $r_1 \geq 0$, $0 < r \leq 1$, $p_1, p_2 > -1$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$. Assume that $f : K \longrightarrow (0,+\infty)$ is a mapping on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. If $\left(f^{(n+1)}\right)^q$ is generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvex mappings, $q \geq 1$, then the following inequality for Caputo k-fractional derivatives holds:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,r_1,n,m,a,b)\right| \leq \frac{1}{(r_1+1)^{n-\frac{\alpha}{k}+1}}\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.10}$$
$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(a)\right)^{rq} I^r(h_1(t);r,r_1,\alpha,k,n,p_1)$$
$$+\left(f^{(n+1)}(x)\right)^{rq} I^r(h_2(t);r,r_1,\alpha,k,n,p_2)\Big]^{\frac{1}{rq}}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(b)\right)^{rq} I^r(h_1(t);r,r_1,\alpha,k,n,p_1)$$
$$+\left(f^{(n+1)}(x)\right)^{rq} I^r(h_2(t);r,r_1,\alpha,k,n,p_2)\Big]^{\frac{1}{rq}}\Bigg\},$$

where

$$I(h_i(t);r,r_1,\alpha,k,n,p_i) := \int_0^1 t^{n-\frac{\alpha}{k}} h_i^{\frac{p_i}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt,\ \forall i = 1,2.$$

Proof. From Lemma 3.1, generalized relative semi-$(r;m,p_1,p_2,h_1,h_2)$-preinvexity of $\left(f^{(n+1)}\right)^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,r_1,n,m,a,b)\right|$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right|dt$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}\left(m\varphi(b)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right|dt$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(a),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left(f^{(n+1)}\left(m\varphi(b)+\left(\frac{r_1+t}{r_1+1}\right)\eta(\varphi(x),\varphi(b),m)\right)\right)^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left[mh_1^{p_1}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}+h_2^{p_2}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(x)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left[mh_1^{p_1}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(b)\right)^{rq}+h_2^{p_2}\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(x)\right)^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q t^{n-\frac{\alpha}{k}}h_1^{\frac{p_1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1\left(f^{(n+1)}(x)\right)^q t^{n-\frac{\alpha}{k}}h_2^{\frac{p_2}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}}{(r_1+1)^{n-\frac{\alpha}{k}+1}\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(b)\right)^q t^{n-\frac{\alpha}{k}}h_1^{\frac{p_1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1\left(f^{(n+1)}(x)\right)^q t^{n-\frac{\alpha}{k}}h_2^{\frac{p_2}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{1}{(r_1+1)^{n-\frac{\alpha}{k}+1}}\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\frac{1}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1,\alpha,k,n,p_1)\right.\right.$$

$$\left.+\left(f^{(n+1)}(x)\right)^{rq}I^r(h_2(t);r,r_1,\alpha,k,n,p_2)\right]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\left[m\left(f^{(n+1)}(b)\right)^{rq}I^r(h_1(t);r,r_1,\alpha,k,n,p_1)\right.$$

$$\left.\left.+\left(f^{(n+1)}(x)\right)^{rq}I^r(h_2(t);r,r_1,\alpha,k,n,p_2)\right]^{\frac{1}{rq}}\right\}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 3.9.

Corollary 3.10. *In Theorem 3.9 for* $h_1(t)=h_2(t)=h(t)$, $p_1=p_2=m=k=r=1, r_1=0$, $\eta(\varphi(y),\varphi(x),m)=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $\forall x\in I$ *and* $f^{(n+1)}\leq K$, *we get the following inequality for Caputo fractional derivatives:*

$$\left|\left[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\right]f^{(n)}(x)+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\left[{}^cD^{\alpha}_{x-}f(a)-{}^cD^{\alpha}_{x-}f(b)\right]\right|$$

$$\leq 2^{\frac{1}{q}}K\left(\frac{1}{n-\alpha+1}\right)^{1-\frac{1}{q}}I^{\frac{1}{q}}(h(t);1,0,\alpha,k,n,1)\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right]. \quad (3.11)$$

Corollary 3.11. *In Theorem 3.9 for $r_1 = 0$, $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, we have the following inequality for generalized relative semi-$(r;m,p_1,p_2,h)$-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \le \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.12}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(a)\right)^{rq} I^r(h(1-t);r,0,\alpha,k,n,p_1)$$

$$+\left(f^{(n+1)}(x)\right)^{rq} I^r(h(t);r,0,\alpha,k,n,p_2)\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(b)\right)^{rq} I^r(h(1-t);r,0,\alpha,k,n,p_1)$$

$$+\left(f^{(n+1)}(x)\right)^{rq} I^r(h(t);r,0,\alpha,k,n,p_2)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3.12. *In Theorem 3.9 for $r_1 = 0$, $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we have the following inequality for generalized relative semi-$(r;m,p_1,p_2,s)$-Breckner-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \le \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.13}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(a)\right)^{rq} \beta^r\left(n-\frac{\alpha}{k}+1,\frac{sp_1}{r}+1\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{n+\frac{sp_2}{r}-\frac{\alpha}{k}+1}\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(b)\right)^{rq} \beta^r\left(n-\frac{\alpha}{k}+1,\frac{sp_1}{r}+1\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{n+\frac{sp_2}{r}-\frac{\alpha}{k}+1}\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3.13. *In Theorem 3.9 for $r_1 = 0$, $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$, we have the following inequality for generalized relative semi-$(r;m,p_1,p_2,s)$–Godunova-Levin-Dragomir-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \le \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.14}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(a)\right)^{rq} \beta^r\left(n-\frac{\alpha}{k}+1,1-\frac{sp_1}{r}\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{n-\frac{sp_2}{r}-\frac{\alpha}{k}+1}\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(b)\right)^{rq} \beta^r\left(n-\frac{\alpha}{k}+1,1-\frac{sp_1}{r}\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{n-\frac{sp_2}{r}-\frac{\alpha}{k}+1}\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3.14. *In Theorem 3.9 for $r_1 = 0$, $h_1(t) = h_2(t) = t(1-t)$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2, tgs)$-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \leq \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.15}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}\beta^r\left(n+\frac{p_1}{r}-\frac{\alpha}{k}+1,\frac{p_1}{r}+1\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\beta^r\left(n+\frac{p_2}{r}-\frac{\alpha}{k}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(b)\right)^{rq}\beta^r\left(n+\frac{p_1}{r}-\frac{\alpha}{k}+1,\frac{p_1}{r}+1\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\beta^r\left(n+\frac{p_2}{r}-\frac{\alpha}{k}+1,\frac{p_2}{r}+1\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 3.15. *In Theorem 3.9 for $r_1 = 0$, $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, we have the following inequality for generalized relative semi-$(r; m, p_1, p_2)$-MT-preinvex mappings:*

$$\left|I_{f,\eta,\varphi}(x;\alpha,k,0,n,m,a,b)\right| \leq \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b),\varphi(a),m)} \tag{3.16}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^r\left(n-\frac{\alpha}{k}-\frac{p_1}{2r}+1,1+\frac{p_1}{2r}\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^r\left(n+\frac{p_2}{2r}-\frac{\alpha}{k}+1,1-\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{n-\frac{\alpha}{k}+1}\Bigg[m\left(f^{(n+1)}(b)\right)^{rq}\left(\frac{1}{2}\right)^{p_1}\beta^r\left(n-\frac{\alpha}{k}-\frac{p_1}{2r}+1,1+\frac{p_1}{2r}\right)$$

$$+\left(f^{(n+1)}(x)\right)^{rq}\left(\frac{1}{2}\right)^{p_2}\beta^r\left(n+\frac{p_2}{2r}-\frac{\alpha}{k}+1,1-\frac{p_2}{2r}\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Remark 3.16. For $k = 1$, by our Theorems 3.2 and 3.9, we can get some new special Ostrowski type inequalities associated with generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex mappings via Caputo fractional derivatives of order α.

Remark 3.17. Also, applying our Theorems 3.2 and 3.9, we can deduce some new inequalities using special means associated with generalized relative semi-$(r; m, p_1, p_2, h_1, h_2)$-preinvex mappings.

4 Conclusion

Motivated by this new interesting class of generalized relative semi-$(r; m, p, q, h_1, h_2)$-preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Ostrowski, Hermite-Hadamard and Simpson type integral inequalities for various kinds of preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

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