Palestine Journal of Mathematics

Vol. 9(1)(2020) , 464–469

MULTIPLICATION AND COMPOSITION OPERATORS ON THE GENERALIZED SPACE OF ENTIRE FUNCTIONS

Deepali Saxena

Communicated by Ayman Badawi

MSC 2010 Classifications: 30D20, 32A15, 47B38.

Keywords and phrases: Multiplcation operator; composition operator; entire functions; vector spaces; linear transformations.

Abstract. Some interesting results, in a very lucid presentation, on multiplication and composition operators on the generalized space of entire function have been obtained. This article establishes relationship between operator theory and complex analysis.

1 Introduction

Serveral functions, in several ways, may be obtained for any two given functions f and g, viz. composing them as gof or fog or else by multiplying them as f.g under suitable conditions. Composition operator (also called substitution operator) is the concept of composition of function, while the multiplication of functions give rise to multiplication linear transformation. In what follows, are some formal definitions and concepts of the subject.

Let X and Y be two non-empty sets and let $f\ (x)$ and $f\ (y)$ are two topological vector spaces of complex valued functions, defined on them. Suppose $T : Y \to X$ be a mapping such that for $f\ o\ t \in f\ (y)$, whenever $f \in f\ (X)$. We can, hence, define a composition (substitution) transformation C_T as

$$C_T : f\ (X) \to f\ (Y)$$

by

$$C_T(f) = foT,\ f \in f(X), \tag{1.1}$$

if C_T is continuous, we call it a composition operator, induced by T. For multiplication transformation if w : x – C such that $f \in f\ (X)$, then it implies that w.$f \in f\ (X)$.
If $\mu_w : f(x) - f(Y)$
is defined by

$$\mu_w(f) = w.f, \quad f \in f(X) \tag{1.2}$$

them a continuous linear multiplication transformation is called a multiplication operator.

While we compile literature on composition operators, it may be observed that it is connected to and concentrates on L^p – space , H^p – spaces or locally convex function spaces. The literature available and the references cited in our present work on operator theory, spell a very intimate relationship between multiplication and composition operators, Singh and Kumar [6]. Infact, one may visualize the applications of multiplication operators in the study of Hilbert space operators in the works of Shields and Wallen [5], Abrahmse and Kriete [1] and Singh and Manhas [7]. In the present paper we have established some results for the multiplication operator on generalized space of entire functions. The definitions and terms employed, are those due to Halmos [3], Dugundgi [2] and Rudin [4].

In order to give some preliminaries, we may include following definitions and notations.

In operator $A \in B(H)$ is said to be of finite rank operator if the dimension of the range of A is finite.

A sequence (X_n) in a metric space X = (X,d) is said to be Cauchy, if for every $\varepsilon > 0$ there is an N=N (ε) such that

$$d(X_m, X_n) < \in, \forall m, n > N,$$

being the set of natural numbers.

Let *X* be a metric space and μ be a subset of it, then a point $x_0 \in X$ (which may or may not be a point μ) is called an accumulation point of μ, if every *nbd* or x_0 contains atleast on epoint *y* $\in \mu$, distinct form x_0.

A linear transformation $T : H \to H$, form Hilbert space *H* into itself is said to be bounded away from zero if there exists $E > 0$, such that $||T_x|| \geq E||X||$, for every $x \in H$.

An operator *A*, on a Banach space *E*, is called a Fredholm operator if the range of *A* is closed and the dimensions of kernel of *A* and co-kernel of *A* are finite.

An operator *A* on a Banach space *E* into itself is called an isometry if $\|AX - AY\| = \|X - Y\|,\ \forall\, X,\ Y \in E.$

2 Main Results

This section of the paper is devoted to three theorems, in the context of the notations and definition described in the preceding section.

Theorem 2.1. *Let α and β be two cardinal numbers, where the operation of multiplication in a set of cardinal number is commutative, additive and distributive over addition.*

Let μ_∞denotes the multiplication operator, where $\mu_w \in C\left(L,\ (x,\ \alpha)\right)$. Then μ_w is a Fredholm operator, if

(i) α /w (α) is a finite set

(ii) $E = \{\alpha : \overline{\overline{w}}(w(\alpha)) \geq 2\}$is finite set, where $\overline{\overline{w}}$ is the cardinal number of w.

(iii) There exists b > 0 such that

$$\frac{\beta(w(\alpha))}{\beta(\alpha)} \geq b,$$

for all except finitely many $x \in X$.

Proof. We suppose that$\mu_w$is a Fredholm operator. If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite set contained in a α / $w(\alpha)$, then

$F_w e_{\alpha_n} = e_{\alpha_n} \to 0$ as $n \to \infty$,

Which asserts the Ker μ_w cannot be of finite dimensional, which readily contradicts our assumption, Thus, α / $w(\alpha)$ is a finite set. Here $\{e_n : n \in N\}$ is orthonormal basis for Hilbert space *H*.

Now we prove that *E* is a finite set. Because, if the set

$$E = \{\alpha \in \alpha : \overline{\overline{w}}(w(\alpha)) \geq 2\} \tag{2.1}$$

is a finite set, then for each pair $x,y \in E$, define

$f_{xy} : L\,(x, \alpha) \to C$

as

$$f_{xy} = E_x \cdot E_y, \tag{2.2}$$

where E_X and E_Y are evaluation functions on $L(x,\alpha)$, defined by $E_x(f) = xf, f \in L\,(x,\alpha)$. Consider

$$(\mu_w^* f_{xy})f = f_{xy}(\mu_w f)$$

$$= E_x(\mu_w f) - E_y(\mu_w f)$$

$$= (\mu_w f)(x) - (\mu_w f)(y)$$

s.t. $(\mu_w^* f_{xy})f = f(w(x)) - f(w(y))$

= 0.

Since there are many infinitely distinct pairs (x_n, y_n), with $w(x_n) = w(y_n)$, so that $f_{x_n y_n} \in$ker μ_w^*, thereby we assert that ker μ_w^* cannot be of finite dimensional, which incidentally, is again a contradiction to the definition and thus, *E* must be a finite set.

Further, if the condition (iii) of the theorem is taken to be false, then we can find a sequence $\{\alpha_n\}$ in α, such that

$$w(w(\alpha_n)) = \alpha_n \text{ and } \frac{\beta(w(\alpha_n))}{\beta(\alpha_n)} < \frac{1}{n}, \forall n \in\ N.$$

Now for any $R > 1$,

$$\left\| \frac{e_w(\alpha_n)}{R^{1/\beta}(w(a_n))} \right\| = \frac{1}{R}.$$

But $\left\| \frac{\mu_w e_w(\alpha_n)}{R^{1/\beta}(w(a_n))} \right\| = \frac{1}{\beta(a_n)/R^{\beta(w(x_n))}} \to 0, \quad \text{as } \; n \to \infty,$

Which readily approves a contradiction to our assumption, that the range of μ_w is closed, hence condition (iii) must be true and hold good.

Converse of the theorem is asserted, if we consider that the conditions (i)-(iii) are satisfied.

We show that μ_w is a Fredholm operator. From conditions (i) and (ii), it is obvious to conclude that dimension of ker $\mu_w$and co-dimension of range μ_w are finite. Next we prove that range of μ_w is a closed set.

Let *f* be a limit point of R and μ_w. Also let $\mu_w f_n$ is a sequence in *L*(x,α) which converges to *f*. Thus, for given ϵ, $0 < \epsilon < 1$, choose a positive integer n_0, such that

$$|f_n(w(x)) - f_m(w(x))|^{\beta(x)} <\in, \tag{2.3}$$

For all $x \in X$ and for all n, $m \geq n_0$, and this, in turn, implies

$$|f_n(w(x)) - f_m(w(x))|^{\beta(w(x))} < \frac{\beta(w(x))}{{}_\epsilon\beta(x)} \in \epsilon^b, \tag{2.4}$$

for all $x \in X$ and for all n, $m \geq n_0$.

Now set

$$\hat{f}_n(x) = \begin{cases} f_n(x); X \in & \text{ran } w \\ 0 \qquad ; X \in & \text{ran } w. \end{cases} , \qquad \text{ran} = \text{range}.$$

It is clear from (2.4) that $\left\{ \hat{f}_n \right\}$ is a Cauchy sequence in *L*(x,α) and thus, we can find $g \in L\,(x,\alpha)$, such that

$$\lim_{n\to\infty} \hat{f}_n = g$$

or

$$\lim_{n\to\infty} \mu_w f_n = \lim_{n\to\infty} \mu_w \hat{f}_n = \mu_w g, \tag{2.5}$$

so that

$f = \mu_w g,$

which proves that μ_w is a closed set, and hence μ_w is a Fredholm operator.

The theorem is completely proved.

□

Let us quote following examples :

(i) Let $X = Z_+$ and $\alpha : Z_+ \to Z_+$ be defined by $\alpha\,(n) = n$. Define $w : Z_+ \to Z_+$ by

$w(n) = \frac{n-k}{n+k}$; for some $k \geq 0$,

then

Ker (μ_w) = span $\{e_1, e_2, \ldots e_{n-1}, \ldots\}$

Ker (μ_w) =$\{0\}$

and range of $\mu_w$is a closed set.

(II) Let $X = C$ and $\alpha : C \to C_+$ is defined by $\alpha\,(n) = n$. Define $w : C \to C$ by

$w(n) = \frac{n-k}{n+k}$; for some $k \geq 0$,

then

Ker (μ_w) = span $\{e_1, e_2, \ldots e_{n-1}, \ldots\}$

Ker (μ_w) =$\{0\}$

and range of $\mu_w$is a closed set.

Theorem 2.2. *Let $\mu_w \in C\,(L,\, C)$,where μ_w is the multiplication operator, and $w \in X \to C$, such that $w \in w\,(X)$, then μ_w is an isometry iff f $w : X \to X$ is surjective, and $\beta = \beta\ o\ w$.*

Proof. Following the definition of isometry, given in section 1 of this article, we assume that μ_w is and isometry. Then for $C > 1$,

$$d(\mu_w C_{e_x}, 0) = d(C_{e_x}, 0), \tag{2.6}$$

$$\sup(|C|^{\beta(Y)} : Y \in w(X)) = |C|^{\beta(X)}, \tag{2.7}$$

where $w(X)$ is a finite set for asserting μ_w to be a bounded operator, Thus, we can find $x_0 \in w(X)$, such that

$$|C|^{\beta(X_0)} = \sup\left\{|C|^{\beta(Y)} : Y \in w(X)\right\}. \tag{2.8}$$

Form (2.7) and (2.8), we obtain

$$\beta(X) = \beta w(X). \tag{2.9}$$

But as X is arbitrary, $\beta = \beta$ o w; while if w is not surjective, then $\mu_w\ e_x = 0$ for $X \epsilon X/w\ (X)$. This justifies that μ_w has a non-trivial kernel. This, incidentally, is a contradiction to our initial assumption, that, μ_w is an isometry. Which straightway proves the first part of theorem, the necessary condition.

Conversily, if the given condition of the theorem are satisfied, then

$$\|\mu_w f\| = \sup\left\{\ |f(w(x))|^{\beta(x)}; x\ \in\ X\right\}$$

i.e.

$$= \sup\left\{\ |f(w(x))|^{\beta(x)}; x\ \in\ X\right\}$$

$$= ||f||, \tag{2.10}$$

which proves that μ_w is an isometry, and thereby the theorem is completely asserted.

□

Following example may be of interest.
Let $X = [0,1]$ and let us define
$C : X \to Z_+$
by

$$C(x) = \begin{cases} n; X = m/n \in (0,1), (m,n) = 0 \\ 0; \quad \text{otherwise} \end{cases}$$

and $w : X \to X$ by $w(X) = (1\text{-}X)$.
If X is irrational, then clearly β ($T(X)$=1=β (X).
Again , if X = 0 or 1, then β (w(0)= β(1) = β (w(1))= β (0) =1.
Further, if $X = m/n$ and (m,n) =1, then

$$\beta\left(w\left(\frac{m}{n}\right)\right) = \beta\left(\frac{m}{n}\right) = \frac{1}{n},$$

thus, $\beta = \beta$ o w also w is surjective. Thus μ_w is an isometry i.e. isometric multiplication operator.

Theorem 2.3. *If X is a metric space and μ_w be a subspace of X, if $\mu_w \in$ C (L (x, C)), μ_w being the multiplication operator, then μ_w is compact subspace of X if μ_w has closed range.*

In order to prove this theorem, we require to prove following lemma.

Lemma :

Let μ_w be a non-empty subset of a metric space (X,d) and $\overline{\mu}_w$be its closure, then
(i) $X \in \overline{\mu}_w$, iff there is a sequence $\{X_n\}$ in μ such that $X_n \to X$.
(ii) μ_w is closed, iff $X_n \in \mu_w$, $X_n \to X$ implies that $X \in \mu_w$.

Proof. (i) Let $X \in \overline{\mu}_w$.

If $X \in \mu_w$, we have sequence $\{X, X, \ldots\}$ while if $X \in \mu_w$, it is a point of accumulation of μ . Hence for each n = 1,2,..., the ball β (x ;1/n) contains an $X_n \in \mu_w$ and $X_n \to X$, because $Y_n \to 0$ as $n \to \infty$.

Conversely, if (x_n) is in μ and $X_n \to X$ then $x \in \mu$ or every nbd. of x contains point $x_n \neq x$ is a point of accumulation of μ . Hence $X \in \overline{\mu}_w$, by definition of closure.
(ii) μ is closed , iff $\mu_w = \overline{\mu}_w$, so this follows from (i).

Proof of Main Theorem:
Let μ_w is a complete subspace of *X*, then we prove that μ_w has closed range. Let also that, *E* be a measureable subset of *S*, where

$$S = x \in X; w(x) \neq 0. \tag{2.11}$$

Then, for $\in > 0$, such that
$\| w(x)g \| \geq \in \| g \|$
for μ - almost all $x \in E$ and for all $g \in C$, for a measurable set E_n and *X* with $0 < (\mu(E_n)) < 1$ and a vector $e_n \in C$, such that

$$||w(x)e_n|| < \frac{1}{Z^n}||e_n||,$$

We let

$$g = \sum_{n=1}^{\infty} \frac{w.^x E_n e^n}{||e_n||^p \sqrt{\mu(E_n)}}. \tag{2.12}$$

Then

$$f||g(x)||^p d\mu = \sum_{n=1}^{\infty} \int_{E_n} ||w(x)e_n||^p d\mu$$

$$< \sum_{n=1}^{\infty} \frac{1}{\mu(E_n)} \int_{E_n} \left| \frac{1}{Z^n} \right|^p d\mu.$$

$$< \sum_{n=1}^{\infty} \frac{1}{Z^{np}} < \infty. \tag{2.13}$$

For

$$f = \sum_{k=1}^{n} \frac{w.^x E_k e_k}{||e_k||^p \sqrt{\mu(E_k)}}, k = 0, 1, 2, ...$$

$$||\mu_w f||^p \geq \int_S ||w(x)f(x)||^p d\mu(x)$$

$$\geq \int_S ||f(x)||^p d\mu(x). \tag{2.14}$$

Now suppose, $\mu_w f^{(n)} \to g$ for some sequence

$$\{f^{(n)}\} < L(X, C).$$

Clearly,

$$||\mu_w f^{(n)} - \mu_w f^{(m)}|| \geq \int_s ||f^{(n)} - f^{(m)}||^p d\mu.$$

Now, sequence $\{f^{(n)}\}$ is a Cauchy sequence in *L* (*X,C*), where

$$f^{(n)}(x) = \begin{cases} f^n(x); x \in S \\ 0 \quad ; \quad x \in S \end{cases}.$$

Hence, there exists $f \in L$ (*X,C*), such that
$\| f^{(n)} - f \| \to 0$, as $n \to \infty$,
so that,

$$\mu_w f^{(n)} \to \mu_w f,$$

and as such,
g = $\mu_w f$.

This shows that μ_w has closed range. Then we prove that μ_w is a complete subspace. If x_n is a Cauchy sequence in μ_w, then x_n to $x \in X$, which implies that $x \in \overline{\mu}_w$ (by (lemma)) and $x \in \mu_w$, because $\mu_w = \overline{\mu}_w$.

Hence x_n converges in μ_w, and so it proves that μ_w is a complete subspace of X.

□

References

[1] M. B. Abrahamse and T. L. Kriets, The spectral multiplicity of multiplication operator, *Indiana Univ. Math.* **22**, 845-851 (1973).

[2] J. Mugundgi, *Topology*, Allyn and Bacon, Inc. Boston (USA), (1966).

[3] P. R. Halmos, *Neasure Thoery*, Springer-Verlag, New York. (1974).

[4] W. Rudin, *Functionel Analysis*, Tata McGraw-Hill (1978).

[5] A. Shields and L.I. Wallen, The commutants of certain Hilbert space operators on H2, *Indian Univ. Math. J.* **23**, 471-796 (1973/74).

[6] R. K. Singh and A. Kumar, Compact composition operators, *J. aust. Math. Soc. Ser. A* **28**, 309-314 (1979).

[7] R. K. Singh and J. S. Manhas, Multiplication operators onweighted spaces of vector continuous functions *J. Aust. Math. Soc. Ser. A* **50**, 98-107 (1991).

Author information

Deepali Saxena, Department of Mathematics, University of Jazan, KSA.
E-mail: deepali.saxena@rediffmail.com

Received: March 2, 2018.
Accepted: March 15, 2018.