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A DOUBLE-SEQUENCE HYBRID S-ITERATION SCHEME FOR FIXED POINT OF LIPSCHITZ PSEUDOCONTRACTIONS IN BANACH SPACE

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Abstract. For any closed convex non-empty subset C of a real Banach space E, we proved that a double sequence Hybrid S-Iteration scheme converges to a fixed point of Lipschitz pseudocontractive map T which maps C into C.

1 Introduction

In this article, we only consider a real Banach space. For a Banach space E, the normalized duality map from E to 2^{E^*} is denoted by J and is defined by

$$J(x) = \{f^* \in E^* \colon \langle x, f^* \rangle = \|\, x \,\|^2,\ \|\, f^* \,\| = \|\, x \,\|\}\ ,\quad for\ all\ \ x \in E,$$

where E^* denotes the dual space of E and $\langle .\,,. \rangle$ denotes the generalized duality pairing. We will denote single-valued duality map by j.

The following definitions have been studied widely and deeply by many authors; see, e.g., [1-12] for more details.

Definition 1.1. Let C be non-empty closed convex subset of a Banach space E and let $T \colon C \to C$ be a mapping. Then

(i) The mapping T is said to be nonexpansive if

$$\|\, Tx - Ty \,\| \leq \|\, x - y \,\|\ ,\quad for\ all\ \ x, y \in C$$

(ii) The mapping T is said to be Lipschitzian if there exists a constant L >1 such that

$$\|\, Tx - Ty \,\| \leq\ L \,\|\, x - y \,\|\ ,\quad for\ all\ \ x, y \in C$$

Now let us recall pseudocontractive and strongly pseudocontractive mapping.

Definition 1.2. The mapping $T \colon C \to C$ is said to be pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty\ ,\ j(x-y) \rangle\ \leq \|\, x - y \,\|^2 \quad for\ all\ x, y \in C.$$

Definition 1.3. The mapping T is said to be strongly pseudocontractive if there exists $0 < k < 1$ such that

$$\langle Tx - Ty\ ,\ j(x-y) \rangle\ \leq\ k \,\|\, x - y \,\|^2 \quad for\ all\ x, y \in C.$$

In [11], C. Moore introduced the concept of double sequence iteration process in fixed point theory. Let $\mathcal{N}$ denote the set of all the non-negative integers and let E be a normed linear space. By a double sequence in E is meant a function $f \colon \mathcal{N} \times \mathcal{N} \to E$ defined by $f(n,m) = x_{n,m}$ which is in E.

Definition 1.4. The double sequence $\{x_{n,m}\}$ is said to converge strongly to x^* if given any $\epsilon > 0$ there exist integer $N, M > 0$ such that $\forall n \geq N, m \geq M$, we have that $\| x_{n,m} - x^* \| < \epsilon$. If $\forall n, r \geq N$ and if $m, t \geq M$, we have that $\| x_{n,r} - x_{m,t} \| < \epsilon$, then the double sequence is said to be Cauchy. Furthermore, if for each fixed n, $x_{n,m} \to x_n^*$ as $m \to \infty$ and then $x_n^* \to x^*$ as $n \to \infty$ then $x_{n,m} \to x^*$ as $n, m \to \infty$.

Many types of iteration process have been established for the constructive approximation of the solution to a family of nonlinear operator equations and several convergence results built using these iterative processes in the last few years (see, e.g [1-12] and the reference cited therein). Especially the iteration process of Mann, Ishikawa and Hybrid S- types have been used to find most of the convergence result as the iterative solution for the approximation of fixed point of nonlinear maps.

The concept of Mann-type double sequence iteration process introduced by C. Moore and he proved that it converges strongly to a fixed point of a continuous pseudocontraction map which maps a bounded closed convex non-empty subset of a real Hilbert space into itself.

2 Main Results

In this section, we mainly prove that the strong convergence of double sequence of Hybrid S-iterative scheme to a fixed point of a Lipschitz pseudocontraction map which maps a bounded closed convex non-empty subset of a real Banach space into itself. To prove our main result, we need the following two lemmas.

Lemma 2.1. (see [2]). Let $J \colon E \to 2^{E^*}$ be the normalized duality mapping. Then for any $x, y \in E$, one has

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2.2. (see [12]). Let $\{\rho_n\}$ and $\{\theta_n\}$ be non-negative sequences satisfying

$$\rho_{n+1} \leq (1-\theta_n)\rho_n + \omega_n,$$

where $\theta_n \in [0,1]$, $\sum_{n\geq 1} \theta_n = \infty$, and $\omega_n = o(\theta_n)$. Then $\lim_{n\to\infty} \rho_n = 0$.

Now our main result states that

Theorem 2.3. *For any non-empty bounded closed convex subset C of a real Banach space E, let $S \colon C \to C$ be nonexpansive mapping satisfying $\| x - Sy \| \leq \| Sx - Sy \| \; \forall x, y \in C$ and let $T \colon C \to C$ be a Lipschitz pseudocontractive map. If $\{\beta_n\}_{n\geq 0}, \{a_k\}_{k\geq 0} \subseteq (0,1)$ are real sequences satisfying the following conditions*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$, $\quad (ii)$ $\lim_{n\to\infty} \beta_n = 0$, $\quad (iii)$ $\lim_{k\to\infty} a_k = 0$.

For any arbitrary but fixed $\mu \in C$ and for each $k \geq 0$, define $T_k \colon C \to C$ by $T_k x = (1-a_k)\mu + a_k Tx$, and satisfying condition that $\| x - T_k y \| \leq \| T_k x - T_k y \| \quad \forall x, y \in C$. Then, the double sequence $\{x_{k,n}\}_{k\geq 0, n\geq 0}$ generated from an arbitrary $x_{0,0} \in C$ by

$$\begin{aligned} x_{k,n+1} &= Sy_{k,n} \\ y_{k,n} &= (1-\beta_n)x_{k,n} + \beta_n T_k x_{k,n}, \quad k, n \geq 0. \end{aligned} \tag{2.1}$$

converges strongly to fixed point x_∞^* of T in C.

Proof. Since T is Lipschitz pseudocontractive, so

$$\| T_k x - T_k y \| = a_k \| Tx - Ty \| \leq L a_k \| x - y \| \tag{2.2}$$

where L is Lipschitz constant of T.

$$\langle T_k x - T_k y, j(x-y)\rangle = a_k \langle Tx - Ty, j(x-y)\rangle \leq a_k \| x - y \|^2 \tag{2.3}$$

So that for all $k \geq 0$, T_k is continuous and strongly pseudocontractive. Also C is invariant under T_k, for all k by convexity. Hence, T_k has unique fixed point $x_k^* \in C$, for all $k \geq 0$. It thus suffices to prove the following

(i) for each fixed $k \geq 0$, $x_{k,n} \to x_k^* \in C$ as $n \to \infty$;

(ii) $x_k^* \to x_\infty^* \in C$ as $k \to \infty$;

(iii) $x_\infty^* \in F(T)$.

Since $\lim\limits_{n\to\infty} \beta_n = 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$,

$$\beta_n \leq min\left\{\frac{(1-a_k)}{(1+3La_k)(1+a_kL)}, \frac{1}{4a_k}\right\} \tag{2.4}$$

Consider

$$\begin{aligned}
|| x_{k,n+1} - x_k^* ||^2 &= \langle x_{k,n+1} - x_k^*, j(x_{k,n+1} - x_k^*)\rangle \\
&= \langle Sy_{k,n} - x_k^*, j(x_{k,n+1} - x_k^*)\rangle \\
&= \langle T_k x_{k,n+1} - x_k^*, j(x_{k,n+1} - x_k^*)\rangle + \langle Sy_{k,n} - T_k x_{k,n+1}, j(x_{k,n+1} - x_k^*)\rangle \\
&\leq a_k || x_{k,n+1} - x_k^* ||^2 + || Sy_{k,n} - T_k x_{k,n+1} |||| x_{k,n+1} - x_k^* || \qquad (2.5)
\end{aligned}$$

Now, consider

$$\begin{aligned}
|| Sy_{k,n} - T_k x_{k,n+1} || &\leq || x_{k,n} - Sy_{k,n} || + || x_{k,n} - T_k y_{k,n} || + || T_k y_{k,n} - T_k x_{k,n+1} || \\
&\leq || Sx_{k,n} - Sy_{k,n} || + || T_k x_{k,n} - T_k y_{k,n} || + || T_k y_{k,n} - T_k x_{k,n+1} ||
\end{aligned}$$

Since S is nonexpansive and using equation (2.2) in above inequality, we get

$$|| Sy_{k,n} - T_k x_{k,n+1} || \leq || x_{k,n} - y_{k,n} || + a_k L\{|| x_{k,n} - y_{k,n} || + || y_{k,n} - x_{k,n+1} ||\} \tag{2.6}$$

Also,

$$\begin{aligned}
|| y_{k,n} - x_{k,n+1} || &\leq || x_{k,n} - y_{k,n} || + || x_{k,n} - x_{k,n+1} || \\
&= || x_{k,n} - y_{k,n} || + || x_{k,n} - Sy_{k,n} || \\
&\leq || x_{k,n} - y_{k,n} || + || Sx_{k,n} - Sy_{k,n} || \\
&\leq 2 || x_{k,n} - y_{k,n} ||
\end{aligned}$$

Using this inequality in equation (2.6), we get

$$\begin{aligned}
|| Sy_{k,n} - T_k x_{k,n+1} || &\leq (1+3a_kL) || x_{k,n} - y_{k,n} || \\
&= (1+3a_kL) || x_{k,n} - (1-\beta_n)x_{k,n} - \beta_n T_k x_{k,n} || \\
&= \beta_n(1+3a_kL) || x_{k,n} - T_k x_{k,n} || \\
&\leq \beta_n(1+3a_kL)\{|| x_{k,n} - x_k^* || + || x_k^* - T_k x_{k,n} ||\} \\
&\leq \beta_n(1+3a_kL)(1+a_kL)) || x_{k,n} - x_k^* ||
\end{aligned}$$

Substitute this in equation (2.5), we get

$$|| x_{k,n+1} - x_k^* ||^2 \leq a_k || x_{k,n+1} - x_k^* ||^2 + \beta_n(1+3a_kL)(1+a_kL)) || x_{k,n} - x_k^* |||| x_{k,n+1} - x_k^* ||$$

which implies that

$$|| x_{k,n+1} - x_k^* || \leq \frac{\beta_n(1+3a_kL)(1+a_kL)}{1-a_k} || x_{k,n} - x_k^* ||$$

using equation (2.4) in the above inequality, we get

$$|| x_{k,n+1} - x_k^* || \leq || x_{k,n} - x_k^* ||$$

So, from the above discussion, we can conclude that the sequence $\{x_{k,n} - x_k^*\}$ is bounded. Since T_k is Lipschitzian, so $\{T_k x_{k,n} - x_k^*\}$ is also bounded.
Let $M_k' = \sup_{n\geq 1} || x_{k,n} - x_k^* || + \sup_{n\geq 1} || T_k x_{k,n} - x_k^* ||$. Now

$$\begin{aligned} || x_{k,n} - y_{k,n} || &= || x_{k,n} - (1-\beta_n)x_{k,n} - \beta_n T_k x_{k,n} || \\ &= \beta_n || x_{k,n} - T_k x_{k,n} || \\ &\leq \beta_n (|| x_{k,n} - x_k^* || + || T_k x_{k,n} - x_k^* ||) \\ &\leq \beta_n M_k' \\ &\to 0 \end{aligned}$$

as $n \to \infty$, implying that $\{x_{k,n} - y_{k,n}\}$ is bounded, so let $M_k'' = \sup_{n\geq 1} || x_{k,n} - y_{k,n} || + M_k'$. Further,

$$\begin{aligned} || y_{k,n} - x_k^* || &\leq || y_{k,n} - x_{k,n} || + || x_{k,n} - x_k^* || \\ &\leq M_k'' \end{aligned}$$

which implies that $\{y_{k,n} - x_k^*\}$ is bounded. Therefore, $\{T_k y_{k,n} - x_k^*\}$ is also bounded. Let

$$M_k''' = \sup_{n\geq 1} || y_{k,n} - x_k^* || + \sup_{n\geq 1} || T_k y_{k,n} - x_k^* ||$$

Denote $M_k = M_k' + M_k'' + M_k'''$. Obviously $M_k < \infty$. Now from (2.1) for all $n \geq 1$, we obtain

$$|| x_{k,n+1} - x_k^* ||^2 = || S y_{k,n} - x_k^* ||^2 \leq || y_{k,n} - x_k^* ||^2, \tag{2.7}$$

and by Lemma (2.1) we get

$$\begin{aligned} || y_{k,n} - x_k^* ||^2 &= || (1-\beta_n)x_{k,n} + \beta_n T_k x_{k,n} - x_k^* ||^2 \\ &= || (1-\beta_n)(x_{k,n} - x_k^*) + \beta_n (T_k x_{k,n} - x_k^*) ||^2 \\ &\leq (1-\beta_n)^2 || x_{k,n} - x_k^* ||^2 + 2\beta_n \langle T_k x_{k,n} - x_k^*, j(y_{k,n} - x_k^*)\rangle \\ &= (1-\beta_n)^2 || x_{k,n} - x_k^* ||^2 + 2\beta_n \langle T_k y_{k,n} - x_k^*, j(y_{k,n} - x_k^*)\rangle \\ &\quad + 2\beta_n \langle T_k x_{k,n} - T_k y_{k,n}, j(y_{k,n} - x_k^*)\rangle \\ &\leq (1-\beta_n)^2 || x_{k,n} - x_k^* ||^2 + 2\beta_n a_k || y_{k,n} - x_k^* ||^2 \\ &\quad + 2\beta_n || T_k x_{k,n} - T_k y_{k,n} |||| y_{k,n} - x_k^* || \\ &\leq (1-\beta_n)^2 || x_{k,n} - x_k^* ||^2 + 2\beta_n a_k || y_{k,n} - x_k^* ||^2 + 2\beta_n a_k L M_k || x_{k,n} - y_{k,n} || \end{aligned}$$

which implies that

$$\begin{aligned} || y_{k,n} - x_k^* ||^2 &\leq \frac{(1-\beta_n)^2}{1-2\beta_n a_k} || x_{k,n} - x_k^* ||^2 + \frac{2\beta_n a_k L M_k}{1-2\beta_n a_k} || x_{k,n} - y_{k,n} || \\ &\leq (1-\beta_n) || x_{k,n} - x_k^* ||^2 + 4\beta_n a_k L M_k || x_{k,n} - y_{k,n} || \end{aligned}$$

because by (2.4), we have $(1-\beta_n)/(1-2\beta_n a_k) \leq 1$ and $(1/(1-2\beta_n a_k)) \leq 2$.
Hence (2.7) gives us

$$|| x_{k,n+1} - x_k^* ||^2 \leq (1-\beta_n) || x_{k,n} - x_k^* ||^2 + 4\beta_n a_k L M_k || x_{k,n} - y_{k,n} || \tag{2.8}$$

for all $n \geq 1$, put

$$\begin{aligned} \rho_n &= || x_{k,n} - x_k^* ||^2, \\ \theta_n &= \beta_n, \\ \omega_n &= 4\beta_n a_k L M_k || x_{k,n} - y_{k,n} || \end{aligned}$$

then accordng to Lemma (2.2), we obtain from (2.8) that

$$\lim_{n\to\infty} \| x_{k,n} - x_k^* \| = 0$$

So the first part is proved. Now,

$$\begin{aligned}
\| x_k^* - Tx_k^* \| &\leq \| x_k^* - x_r^* \| + \| x_r^* - Tx_k^* \| \qquad where\, 0 < k < r \\
&= \| T_k x_k^* - T_r x_r^* \| + \| T_r x_r^* - Tx_k^* \| \\
&\leq \| (1-a_k)\mu + a_k Tx_k^* - (1-a_r)\mu - a_r Tx_r^* \| + \| T_r x_r^* - T_r T x_k^* \| \\
&\leq \| \mu(a_r - a_k) + (a_k - a_r)Tx_k^* \| + a_r \| Tx_k^* - Tx_r^* \| + a_r \| x_r^* - Tx_k^* \| \\
&\leq | a_k - a_r | \| Tx_k^* - \mu \| + a_r 2d + a_r 2d \qquad where\; d = diamC \\
&\leq 2d \, | a_k - a_r | + 4a_r d
\end{aligned}$$

So that

$$\lim_{k\to\infty} \| x_k^* - Tx_k^* \| = 0$$

and hence $\{x_k^*\}$ is an approximate fixed point sequence for T. Also, supposing that x_∞^* is a fixed point of T, then

$$\lim_{k\to\infty} \| x_\infty^* - T_k x_\infty^* \| \leq 0$$

Now, for all $0 < m \leq k$

$$\begin{aligned}
\| x_k^* - x_m^* \|^2 &= \langle x_k^* - x_m^*, j(x_k^* - x_m^*) \rangle \\
&= \langle T_k x_k^* - T_m x_m^*, j(x_k^* - x_m^*) \rangle \\
&= \langle (a_m - a_k)\mu + (a_k - a_m)Tx_m^* + a_k(Tx_k^* - Tx_m^*), j(x_k^* - x_m^*) \rangle \\
&= | a_k - a_m | \langle \mu, j(x_k^* - x_m^*) \rangle + | a_k - a_m | \langle Tx_m^*, j(x_k^* - x_m^*) \rangle \\
&\quad + a_k \langle Tx_k^* - Tx_m^*, j(x_k^* - x_m^*) \rangle \\
&\leq | a_k - a_m | \| \mu \| \| x_k^* - x_m^* \| + | a_k - a_m | \| Tx_m^* \| \| x_k^* - x_m^* \| \\
&\quad + a_k \| x_k^* - x_m^* \|^2
\end{aligned}$$

which implies that, we get

$$\| x_k^* - x_m^* \| \leq \frac{| a_k - a_m |}{1 - a_k} \{ \| \mu \| + \| Tx_m^* \| \}$$

and hence,

$$\lim_{k,r\to\infty} \| x_k^* - x_m^* \| \leq 2d \lim_{k,r\to\infty} \frac{| a_k - a_m |}{1 - a_k} = 0$$

Thus, $\{x_k^*\}$ is a Cauchy sequence and hence, there exists $x_\infty^* \in C$ such that $x_k^* \to x_\infty^*$ as $k \to \infty$. So, the second part is proved.
By continuity, $Tx_k^* \to Tx_\infty^*$ as $k \to \infty$. But $x_k^* - Tx_k^* \to 0$ as $k \to \infty$. Hence, $x_\infty^* \in F(T)$. This completes the proof.

References

[1] Arif Rafiq and Byung Soo Lee, On Asymptotically demicontractive mappings in an arbitrary Banach spaces, *East Asian Mathematical Journal,* **28(5)**, 569–578(2012).

[2] C.E. Chidume and C. Moore, The solution by iteration of nonlinear equations in uniformly smooth Banach spaces, *J. Math. Anal. Appl.* **215 (1)**, 132–146(1997).

[3] C.E. Chidume and C. Moore, Fixed point iteration for pseudocontractive maps, *Proc. Amer. Math. Soc.* **127 (4)**, 1163–1170(1999).

[4] C.E. Chidume and C. Moore, Steepest descent method for equilibrium points of nonlinear systems with accretive operators, *J. Math. Anal. Appl.* **245 (1)**, 142–160(2000).

[5] C.E. Chidume and M.O. Osilike, Nonlinear accretive and pseudocontractive operator equations in Banach spaces, *Nonlinear Anal.* **31 (7)**, 779–789(1998).

[6] S.Gopinath, J. Gnanaraj and S.Lalithambigai, Strong convergence for Hybrid S-Iteration scheme of non-expansive and asymptotically demicontractive mappings, *Int. J. Pure Appl. Math.* **110(1)**, 153–163(2016).

[7] S. Ishikawa, Fixed points by a new iteration method, *Proceedings of the American Mathematical Society*, **44**, 147 – 150(1974).

[8] S. M. Kang, A. Rafiq and Y. C. Kwun, Strong convergence for Hybrid S-Iterative scheme, *Journal of Applied Mathematics*, **Article ID 705814** , 1–4(2013).

[9] W. R. Mann, Mean value methods in iteration, *Proceedings of the American Mathematical Society*, **4**, 506 – 510(1953).

[10] C. Moore, The solution by iteration of nonlinear equations involving psi-strongly accretive operators in Banach spaces, *Nonlinear Analysis,* **37 (1)**, 125–138(1999).

[11] C. Moore, A Double-Sequence Iteration process for fixed points of continuous pseudocontraction, *An Int. J. Computers and Mathematics with Applications,* **43**, 1585–1589(2002).

[12] X. Weng, Fixed point iteration for local stricly pseudocontractive mapping, *Proceedings of the American Mathematical Society*, **113(3)**, 727 – 731(1991).

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