Palestine Journal of Mathematics

Vol. 9(1)(2020) , 485–492

THE GENERALIZED HYPER-STABILITY OF CUBIC FUNCTIONAL EQUATION

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Communicated by José Luis López Bonilla

MSC 2010 Classifications: Primary 39B82; Secondary 39B52.

Keywords and phrases: Stability, hyperstability, cubic functional equation.

Abstract. The aim of this paper is to prove hyperstability results for the following cubic functional equation on a restricted domain X

$$f(rx+y)+f(rx-y)=rf(x+y)+rf(x-y)+2(r^3-r)f(x)$$

for all $x, y \in X$ and r is a fixed positive integre $r \geq 2$.

1 Introduction

Let X be a nonempty subset symmetric with respect to 0 and Y be a Banach space . In the paper, we prove the hyper-stability of the cubic functional equation on a restricted domain. We say that a function $f : X \to Y$ satisfies the cubic functional equation on X if

$$f(rx+y)+f(rx-y)=rf(x+y)+rf(x-y)+2(r^3-r)f(x) \tag{1.1}$$

for all $x, y \in X$ such that $x+y, x-y \in X$. We will show that (1.1) is hyper-stable for each function $f : X \to Y$ (under some additional assumption on X) satisfying the inequality :

$$\|\frac{1}{2(r^3-r)}f(rx+y)+\frac{1}{2(r^3-r)}f(rx-y)-\frac{1}{2(r^2-1)}f(x+y)-\frac{1}{2(r^2-1)}f(x-y)-f(x)\| \leq \frac{c}{2(r^3-r)}\|x\|^p\|y\|^q \tag{1.2}$$

for all $x, y \in X$ such that $x+y, x-y \in X$ with $p+q < 0$ and $0 < p+q < 1$ must satisfy the cubic equation (1.2).

The method of the proof of the main theorem is motivated by an idea used by Brzdęk in [4] and further by Piszczek in [21] . It is based on a fixed point theorem for functional spaces obtained by Brzdęk et al. In [6] . some generalizations of their result were proved by cădariu et al. In [15] , The case of fixed point theorem for non-Archimedean metric spaces was also studied by Brzdęk and Ciepliñski in [9] . It is worth mentioning that using fixed point theorems is now one of the most popular methods of investigating the stability of functional equations in single as well as in several variables. Let us recall a few recent approaches of jung in [19], More information on the application of the fixed point method was collected by Ciepliñski in [17] . First, we take the following three hypotheses (all notations come from [16]).

$(H_1)X$ is a nonempty set, Y a Banach spaces, and $f_1, ..., f_k : X \to X$ and $L_1, ..., L_K : X \to R_+$ are given .

$(H_2)\mathcal{F} : Y^X \to Y^X$is an operator satisfying the inequality

$$\|\mathcal{F}\xi(x)-\mathcal{F}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x))-\mu(f_i(x))\| \quad \xi, \mu \in Y^X, x \in X$$

$(H_3)\Lambda : R_+^X \to R_+^X$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \quad \delta \in R_+^X, \quad x \in X$$

the mentioned fixed point theorem is stated as follows.

Theorem 1.1. *Let hypotheses* $(H_1)-(H_2)$ *be valid, functions* $\varepsilon : X \to R_+$ *and* $\varphi : X \to Y$ *fulfill the following two conditions:*

i) $\|\mathcal{F}\varphi(x)-\varphi(x)\| \leq \epsilon(x), x \in X$

ii) $\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in X$

Then there exists a unique fixed point ψ *of* $\mathcal{F}$ *with* $\|\varphi(x)-\psi(x)\| \leq \varepsilon^*(x), \quad x \in X$ *.*

Moreover $\psi(x) = \lim_{n\to\infty} \mathcal{F}^n\varphi(x), \quad x \in X$

Throughout the paper, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{N}_{m_0}$ denote the set of all positive integers, the set of all nonnegative integers and the set of all integers greater than or equal to m_0, respectively.

2 Main results

In this section, we prove the hyperstability results of the generalized cubic functional equation.

Theorem 2.1. *Assume that* X *is a nonempty symmetric with respect to* 0 *subset of a normed space such that* $0 \notin X$ *and there exist* $n_0 \in \mathbb{N}$ *with* $nx \in X$ *for* $x \in X$ *and* $n \in N_{n_0}$*. Let* Y *be a Banach space,* $c \geq 0$*, and* $p+q<0$ *. If* $f : X \to Y$ *satisfies*

$$\|\frac{1}{2(r^3-r)}f(rx+y)+\frac{1}{2(r^3-r)}f(rx-y)-\frac{1}{2(r^2-1)}f(x+y)-\frac{1}{2(r^2-1)}f(x-y)-f(x)\| \leq \frac{c}{2(r^3-r)}\|x\|^p\|y\|^q \tag{2.1}$$

for all $x, y \in X$ *such that* $x+y$*,* $x-y \in X$ *, then* f *satisfies the cubic equation on* X*.*

Proof. First observe that there exists $m_0 \in N_{m_0}$ such that

$$\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}+\frac{1}{2(r^2-1)}(m-1)^{p+q} < 1$$

for $m \geq m_0$, Assume that $q<0$ and Replacing y with mx in (2.1 we get :

$$\|\frac{1}{2(r^3-r)}f((r+m)x)+\frac{1}{2(r^3-r)}f((r-m)x)-\frac{1}{2(r^2-1)}f((1-m)x)-\frac{1}{2(r^2-1)}f((1+m)x)-f(x)\| \leq \frac{c}{2(r^3-r)}m^q\|x\|^p\|y\|^q$$

such that $x \in X$.

Further put

$$\mathcal{F}_m\xi(x) := \frac{1}{2(r^3-r)}\xi((r+m)x)+\frac{1}{2(r^3-r)}\xi((r-m)x)-\frac{1}{2(r^2-1)}\xi((1+m)x)-\frac{1}{2(r^2-1)}\xi((1-m)x)$$

$x \in X$, $\quad \xi \in Y^X$ and $\epsilon_m(x) := \frac{c}{2(r^3-r)}m^q\|x\|^p\|y\|^q$

Then the inequality (2.1) takes the form $\|\mathcal{F}_m f(x) - f(x)\| \leq \epsilon_m(x)$.

The operator

$$\Delta\delta(x) := \frac{1}{2(r^3-r}\delta((r+m)x)+\frac{1}{2(r^3-r}\delta((m-r)x)+\frac{1}{2(r^2-1)}\delta((1+m)x)+\frac{1}{2(r^2-1)}\delta((m-1)x),$$

such that $\quad \delta \in R_+^X, \quad x \in X$

has the form described in (H_3) with $k=4$ and

$$f_1(x) = (m+r)x, f_2(x) = (r-m)x, f_3(x) = (1+m)x, f_4(x) = (1-m)x,$$

$$L_1(x) = L_2(x) = \frac{1}{2(r^3-r)}, L_3(x) = L_4(x) = \frac{1}{2(r^2-1)}$$

Moreover, for every $\xi;\mu \in X^X$ and $x \in X$

$$\|\mathcal{F}_m\xi(x) - \mathcal{F}_m\mu(x)\| \leq \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|$$

So : (H_2) is valid:

Next we can find $m_0 \in N$ such that

$$\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}+\frac{1}{2(r^2-1)}(m-1)^{p+q}<1$$

Therefore we obtain that

$$\begin{aligned}\epsilon^*(x) :&= \sum_{n=0}^{\infty} \triangle^n \epsilon(x)\\ &= \frac{c}{2(r^3-r)}m^q\|x\|^{p+q}\sum_{n=0}^{\infty}(\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}\\ &+\frac{1}{2(r^2-1)}(m-1)^{p+q})^n\\ &= \frac{\frac{c}{2(r^3-r)}m^q\|x\|^{p+q}}{1-(\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}+\frac{1}{2(r^2-1)}(m-1)^{p+q})}\end{aligned}$$

Thus according to theorem 1.1 there exists a unique solution $F : X \to Y$ of the equation:

$$F(x)=\frac{1}{2(r^3-r)}F((r+m)x)+\frac{1}{2(r^3-r)}F((r-m)x)-\frac{1}{2(r^2-1)}F((1+m)x)-\frac{1}{2(r^2-1)}F((1-m)x)$$

such that

$$\|f(x)-F(x)\| \leq \frac{\frac{c}{2(r^3-r)}m^q\|x\|^{p+q}}{1-(\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}+\frac{1}{2(r^2-1)}(m-1)^{p+q})}$$

Moreover: $F(x) = \lim_{n\to\infty} \mathcal{F}^n f(x)$.

To prove that F satisfies the cubic equation on X, observe that

$$\begin{aligned}&\|\frac{1}{2(r^3-r)}\mathcal{F}^n f(rx+y)+\frac{1}{2(r^3-r)}\mathcal{F}^n f(rx-y)-\frac{1}{2(r^2-1)}\mathcal{F}^n f(x+y)\\ -\ &\frac{1}{2(r^2-1)}\mathcal{F}^n f(x-y)-\mathcal{F}^n f(x)\|\\ \leq\ &\frac{c}{2(r^3-r)}(\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}\\ +\ &\frac{1}{2(r^2-1)}(m-1)^{p+q})^n\|x\|^p\|y\|^q\end{aligned}\tag{2.2}$$

for every $x, y \in X$, $x+y \in X$, $x-y \in X$. Indeed : if $n = 0$ then, (2.2) is simply . So, fix

$n \in N_0$ and suppose that (2.2 holds for n and $x, y \in X$ such that $x+y, x-y \in X$. Then

$$\begin{aligned}
&\|\tfrac{1}{2(r^3-r)}\mathcal{F}^{n+1}f(rx+y) + \tfrac{1}{2(r^3-r)}\mathcal{F}^{n+1}f(rx-y) - \tfrac{1}{2(r^2-1)}\mathcal{F}^{n+1}f(x+y)\\
&-\tfrac{1}{2(r^2-1)}\mathcal{F}^{n+1}f(x-y) \ -\mathcal{F}^{n+1}f(x)\|\\
&= \|\tfrac{1}{2(r^3-r)}(\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r+m)(2x+y) + \tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r-m)(rx+y)\\
&-\tfrac{1}{2(r^2-1)}F^n(f((1+m)(rx+y)) - \tfrac{1}{2(r^2-1)}\mathcal{F}^n(f((1-m)(rx+y)))\\
&+\tfrac{1}{2(r^3-r)}(\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r+m)(2x-y) + \tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r-m)(rx-y)\\
&-\tfrac{1}{2(r^2-1)}\mathcal{F}^n(f((1+m)(rx-y)) - \tfrac{1}{2(r^2-1)}\mathcal{F}^n(f((1-m)(rx-y)))\\
&-\tfrac{1}{2(r^2-1)}(\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r+m)(x+y) + \tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r-m)(x+y)\\
&-\tfrac{1}{2(r^2-1)}\mathcal{F}^n(f((1+m)(x+y)) - \tfrac{1}{2(r^2-1)}\mathcal{F}^n(f((1-m)(x+y)))\\
&-\tfrac{1}{2(r^2-1)}(\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r+m)(x-y) + \tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r-m)(x-y)\\
&-\tfrac{1}{2(r^2-1)}F^n(f((1+m)(x-y)) - \tfrac{1}{2(r^2-1)}\tfrac{1}{2(r^2-1)}F^n(f((1-m)(x-y)))\\
&-\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r+m)(x)) - \tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r-m)(x))\\
&+\tfrac{1}{2(r^2-1)}F^n(f((1+m)(x)) + \tfrac{1}{2(r^2-1)}F^n(f((1-m)(x))\|\\
&\leq \|\tfrac{1}{2(r^3-r)}\Big(\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r+m)(rx+y)) + \tfrac{1}{2(r^3-r)}(\mathcal{F}^n f((r+m)(rx-y))\\
&-\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((r+m)(x+y)) - \tfrac{1}{2(r^2-1)}\mathcal{F}^n f((r+m)(x-y))\\
&-\mathcal{F}^n f((r+m)x)\Big)\| + \|\tfrac{1}{2(r^3-r)}\Big(\tfrac{1}{2(r^3-r)}\mathcal{F}^n f((r-m)(rx+y))\\
&+\tfrac{1}{2(r^3-r)}(\mathcal{F}^n f((r-m)(rx-y)) - \tfrac{1}{2(r^2-1)}(\mathcal{F}^n f((r-m)(x+y)))\\
&-\tfrac{1}{2(r^2-1)}(\mathcal{F}^n f((r-m)(x-y))) - \mathcal{F}^n f((r-m)x)\Big)\|\\
&+|\tfrac{1}{2(r^3-r)}\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1+m)(rx+y)) + \tfrac{1}{2(r^3-r)}(\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1+m)(rx-y))\\
&-\tfrac{1}{2(r^2-1)}(\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1+m)(x+y))) - \tfrac{1}{2(r^2-1)}(\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1+m)(x-y)))\\
&-\tfrac{1}{2(r^2-1)}\mathcal{F}^n f(1+m)x)\| + |\tfrac{1}{2(r^3-r)}\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1-m)(rx+y))\\
&+\tfrac{1}{2(r^3-r)}(\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1-m)(rx-y)) - \tfrac{1}{2(r^2-1)}(\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1-m)(x+y)))\\
&-\tfrac{1}{2(r^2-1)}(\tfrac{1}{2(r^2-1)}\mathcal{F}^n f((1-m)(x-y))) - \tfrac{1}{2(r^2-1)}\mathcal{F}^n f(1-m)x)\|\\
&\leq \tfrac{c}{2(r^3-r)}(\tfrac{1}{2(r^3-r)}(r+m)^{p+q} + \tfrac{1}{2(r^3-r)}(m-r)^{p+q}\\
&+\tfrac{1}{2(r^2-1)}(1+m)^{p+q} + \tfrac{1}{2(r^2-1)}(m-1)^{p+q})^n\|x\|^p\|y\|^q(\tfrac{(r+m)^{p+q}}{2(r^3-r)}\\
&+\tfrac{(m-r)^{p+q}}{2(r^3-r)} + \tfrac{(m+1)^{p+q}}{2(r^2-1)} + \tfrac{(m-1)^{p+q}}{2(r^2-1)})\\
&= \tfrac{c}{2(r^3-r)}(\tfrac{1}{2(r^3-r)}(2+m)^{p+q} + \tfrac{1}{2(r^3-r)}(m-2)^{p+q}\\
&+\tfrac{1}{2(r^2-1)}(1+m)^{p+q} + \tfrac{1}{2(r^2-1)}(m-1)^{p+q})^{(n+1)}\|x\|^p\|y\|^q
\end{aligned}$$

By induction, we have shown that (2.2) holds. Letting $n \to +\infty$ in (2.2) we obtain $F(rx+y) + F(rx-y) = rF(x+y) + rF(x-y) + 2(r^3-r)F(x)$

Thus, we have proved that for every $m \in N_{m_0}$ there exists a function $F_m : X \to Y$ such that F_m is a solution of the cubic equation on X and

$$\|f(x) - F_m(x)\| \leq \frac{\frac{c}{2(r^3-r)}m^q\|x\|^{p+q}}{1 - (\frac{1}{2(r^3-r)}(r+m)^{p+q} + \frac{1}{2(r^3-r)}(m-r)^{p+q} + \frac{1}{2(r^2-1)}(1+m)^{p+q} + \frac{1}{2(r^2-1)}(m-1)^{p+q})}$$

Since $p+q<0$ with $q<0$, the sequence

$$(\frac{\frac{c}{2(r^3-r)}m^q\|x\|^{p+q}}{1-(\frac{1}{2(r^3-r)}(r+m)^{p+q}+\frac{1}{2(r^3-r)}(m-r)^{p+q}+\frac{1}{2(r^2-1)}(1+m)^{p+q}+\frac{1}{2(r^2-1)}(m-1)^{p+q})})_{m\in N_{m_0}}$$

tends to Zero. Consequently f satisfies the cubic equation on X as the pointwise limit of $(F_m)_{m\in\mathbb{N}_{m_0}}$.

Theorem 2.2. *If $f:X\to Y$ satisfies*

$$\begin{aligned}\|\frac{1}{2(r^3-r)}f(rx+y)+\frac{1}{2(r^3-r)}f(rx-y) \quad &- \quad \frac{1}{2(r^2-1)}f(x+y)-\frac{1}{2(r^2-1)}f(x-y)-f(x)\| \\ &\le \quad \frac{c}{2(r^3-r)}\|x\|^p\|y\|^q \qquad (2.3)\end{aligned}$$

for all $x,y\in X$ such that $x+y$;$x-y\in X$; and $0<p+q<1$. Then f satisfies the cubic functional equation on X .

Proof. Assume that $q>0$ and replacing y with $\frac{x}{m}$ we get :

$$\|\frac{1}{2(r^3-r)}f((r+\frac{1}{m})x)+\frac{1}{2(r^3-r)}f((r-\frac{1}{m})x)-\frac{1}{2(r^2-1)}f((1+\frac{1}{m})x)-$$

$$\frac{1}{2(r^2-1)}f(1-\frac{1}{m})x)-f(x)\|\le\frac{c}{2(r^3-r)}\frac{1}{m^q}\|x\|^{p+q}=\epsilon_m(x)$$

Such that $x\in X$ Similarly as previously we define

$$\begin{aligned}\mathcal{F}_m\xi(x):=&\frac{1}{2(r^3-r)}\xi((r+\frac{1}{m})x)+\frac{1}{2(r^3-r)}\xi((r-\frac{1}{m})x)-\frac{1}{2(r^2-1)}\xi(1+\frac{1}{m}x)\\ &-\frac{1}{2(r^2-1)}\xi((1-\frac{1}{m})x),x\in X,\xi\in Y^X\end{aligned}$$

and $\Delta_m\delta(x):=\frac{1}{2(r^3-r)}\delta((r+\frac{1}{m})x)+\frac{1}{2(r^3-r)}\delta((\frac{1}{m}-r)x)\frac{1}{2(r^2-1)}\delta((1+\frac{1}{m})x)+\frac{1}{2(r^2-1)}\delta((\frac{1}{m}-1)x)$, $\delta\in R_+^X$, $x\in X$ and see that (2.2) is: $\|\mathcal{F}_mf(x)-f(x)\|\le\epsilon_m(x),x\in X$ Obiouvsly Δ_m has the form described in (H_3) whith $k=4$ and $f_1(x)=(r+\frac{1}{m})x$ $f_2(x)=(r-\frac{1}{m})x$, $f_3(x)=(1+\frac{1}{m})x$, $f_4(x)=(1-\frac{1}{m})x$, $L_1(x)=L_2(x)=\frac{1}{2(r^3-r)}$, $L_3(x)=L_4(x)=\frac{1}{2(r^2-1)}$

$$\|\mathcal{F}_m\xi(x)-\mathcal{F}_m\mu(x)\|\le\sum_{i=1}^{4}L_i(x)\|(\xi-\mu)(f_i(x))\|$$

So (H_2) is valid . Next we can find $m_0\in N_{n_0}$ such that

$$\frac{1}{2(r^3-r)}(r+\frac{1}{m})^{p+q}+\frac{1}{2(r^3-r)}(r-\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1+\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1-\frac{1}{m})^{p+q}<1$$

For all $m\ge m_0$,

Therefore we obtain that

$$\begin{aligned}\epsilon^*(x)&:=\sum_{n=0}^{\infty}\triangle^n\epsilon(x)\\ &=cm^q\|x\|^{p+q}\sum_{n=0}^{\infty}(\frac{1}{2(r^3-r)}(r+\frac{1}{m})^{p+q}+\frac{1}{2(r^3-r)}(r-\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1+\frac{1}{m})^{p+q}\\ &+\frac{1}{2(r^2-1)}(1-\frac{1}{m})^{p+q})^n\\ &=\frac{cm^q\|x\|^{p+q}}{1-(\frac{1}{2(r^3-r)}(r+\frac{1}{m})^{p+q}+\frac{1}{2(r^3-r)}(r-\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1+\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1-\frac{1}{m})^{p+q})}\end{aligned}$$

Thus, according to theorem 1.1 there exists a unique solution $F : X \to Y$ of the equation

$$F_m(x) = \frac{1}{2(r^3-r)}F_m((r+\frac{1}{m})x) + \frac{1}{2(r^3-r)}F_m((r-\frac{1}{m})x) - \frac{1}{2(r^2-1)}F_m((1+\frac{1}{m})x) -$$

$$\frac{1}{2(r^2-1)}F_m((1-\frac{1}{m})x)$$

such that $\|f(x) - F_m(x)\| \leq \frac{\frac{c}{2(r^3-r)}m^q\|x\|^{p+q}}{1-(\frac{1}{2(r^3-r)}(r+\frac{1}{m})^{p+q}+\frac{1}{2(r^3-r)}(r-\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1+\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1-\frac{1}{m})^{p+q})}$

and $F_m(2x+y) + F_m(2x-y) = 2F_m(x+y) + 2F_m(x-y) + 12F_m(x)$,$x \in X$, $y \in X$, $x+y \in X$, $x-y \in X$

In this way we obtain a sequence $(F_m)_{m \in N_{m_0}}$ of cubic functions on X such that $\|f(x) - F_m(x)\| \leq \frac{\frac{c}{2(r^3-r)}m^q\|x\|^{p+q}}{1-(\frac{1}{2(r^3-r)}(r+\frac{1}{m})^{p+q}+\frac{1}{2(r^3-r)}(r-\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1+\frac{1}{m})^{p+q}+\frac{1}{2(r^2-1)}(1-\frac{1}{m})^{p+q})}$ It follows; with $m \to \infty$, that f is cubic on X .

Remark 2.3. In the case $p > 1$, the considered cubic equation is not hyperstable.

Show for example: $X = R - \{[-\sqrt{2(r^3-1)}; \sqrt{2(r^3-1)}]\}$ and $f : X \to R$ be a constant $f(x) = c, x \in X$ for some $c > 0$. In this case f satisfies the inequality

$$\|\frac{1}{2(r^3-r)}f(rx+y) + \frac{1}{2(r^3-r)}f(rx-y) - \frac{1}{2(r^2-1)}f(x+y) - \frac{1}{2(r^2-1)}f(x+y) - f(x)\|$$

$$\leq \frac{c}{2(r^3-r)}\|x\|^p\|y\|^q$$

for all $x, y \in x$ such that $x+y, x-y \in X$, with $p > 1$, but is not a solution of the cubic equation on X.

Theorem 2.4. *Assume that X is a nonempty, symmetric with respect to 0 subset of a normed space such that $0 \notin X$ and there existd $n_0 \in N$ with $nx \in X$ for $x \in X$ and $n \in N_{n_0}$. Let Y be a Banach space, $c \geq 0$, and $p < 0$. If $f : X \to Y$ satisfies*

$$\|f(rx+y) + f(rx-y) - rf(x+y) - rf(x-y) - 2(r^3-r)f(x)\| \leq c(\|x\|^p + \|y\|^p) \quad (2.4)$$

for all $x, y \in X$ such that $x+y$;$x-y \in X$, then f satisfies the cubic equation on X .

Proof. Replacing (x, y) by $(mx, (rm)x)$, where $m \in N^* - \{1; 2\}$ in (2.4), we get

$$\begin{aligned} &\|f(x) + f((2rm-1)x) - rf(((r+1)m-1)x) - rf(((1-r)m+1)x) - 2(r^3-r)f(mx)\| \\ \leq\ & \frac{c}{2(r^3-r)}(m^p + (rm-1)^p)\|x\|^p \qquad (2.5) \end{aligned}$$

for all $x \in X$

Further put

$$\mathcal{F}_m\xi(x) := 2(r^3-r)\xi((m)x) + r\xi(((1-r)m+1)x) + r\xi(((r+1)m-1)x) - \xi((2rm-1)x)$$

$x \in X$, $\xi \in Y^X$ and $\epsilon_m(x) := c(m^p + (rm-1)^p)\|x\|^p$

Then the inequality (2.5) takes the form $\|\mathcal{F}_m f(x) - f(x)\| \leq \epsilon_m(x)$. $x \in X$

The operator $\Delta_m\delta(x) := 2(r^3-r)\delta(mx) + r\delta(((1-r)m+1)x) + r\delta(((r+1)m-1)x) + \delta((2rm-1)x)$

$\delta \in R_+^X, x \in X$

has the form described in (H_3) with $k = 4$ and $f_1(x) = mx$;$f_2(x) = ((1-r)m+1)x$; $f_3(x) = ((r+1)m-1)x$; $f_4(x) = (2rm-1)x$; $L_1(x) = 2(r^3-r)$; $L_3(x) = L_4(x) = r$, $L_4(x) = 1$ for all $x \in X$

Moreover , for every $\xi, \mu \in Y^X$ and $x \in X$, we have

$$\|\mathcal{F}_m\xi(x) - \mathcal{F}_m\mu(x)\| \leq \sum_{i=1}^{4} L_i(x)\|(\xi - \mu)(f_i(x))\|$$

So,H_2 is valid. Now, we can find $m_0 \in N^* - \{1; 2\}$ such that

$$2(r^3 - r)m^p + r((r-1)m+1)^p + r((r+1)m-1)^p + (2rm-1)^p < 1$$

for all $m_0 \leq m$

Therefore, we obtain that

$$\epsilon^*(x) := \sum_{n=0}^{\infty} \triangle^n \epsilon(x) = c(m^p + (rm-1)^p)\|x\|^p \sum_{n=0}^{\infty}(2(r^3-r)m^p + r((1-r)m+1)^p +$$

$$r((r+1)m-1)^p + (2rm-1)^p)^n = \frac{c(m^p + (rm-1)^p)}{1 - ((2(r^3-r)m^p + r(3m+1)^p + r(m-1)^p + (2rm-1)^p)}$$

for all$x \in X$ and $m \geq m_0$. The rest of the proof is similar to the proof of Theorem 2.1.

Corollary 2.5.. *Assume that X is that a nonempty symmetric with respect to 0 subset of a normed space such that $0 \notin X$ and Y be a banach space. Let $F : X^2 \to Y$ be a mapping such that $F(x_O, y_0) \neq 0$ for some $x_0, y_0 \in X$ and*

$$\|F(x,y)\| \leq c\|x\|^p\|y\|^q \tag{2.6}$$

Or

$$\|F(x,y)\| \leq c(\|x\|^p + \|y\|^p) \tag{2.7}$$

For all $x, y \in X$, where $c \geq 0$ and $p, q \in R$. Assume that the numbers $p; q$ satisfy $p + q < 1$ and $p + q \neq 1$ In the case (2.8) and $p < 0$ in the case (2.7), Then the functional equation:

$$h(rx+y) + h(rx-y) + F(x,y) = rh(x+y) + rh(x-y) + ((2(r^3-r)h(x) \tag{2.8}$$

$x, y \in X$ Has no solution in the class of functions $h : X \to Y$

Proof. Suppose that $h : X \to Y$ is a solution to (2.8); Then(2.1)or (2.3)holds, and consequently, according to above theorems, h is cubic on X, which means that $F(x_0, y_0) = 0$. This is contradiction.

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Received: January 23, 2018.
Accepted: July 29, 2018.