Palestine Journal of Mathematics

Vol. 9(1)(2020) , 523–526 

# COEFFICIENT INEQUALITIES FOR $q$-STARLIKE AND $q$-CONVEX FUNCTIONS OF RECIPROCAL ORDER

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Communicated by Jose Luis Lopez-Bonilla

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: $q$- starlike,Coefficient inequalities and Generalized q-Ruscheweyh derivative .

**Abstract**. In this article the two classes $\mathcal{S}_{q,*}(\alpha)$ and $\mathcal{K}_{q,*}(\alpha)$ of $q$-starlike and $q$-convex functions of reciprocal order $\alpha$ respectively are considered for the function $f(z)$ in the open unit disc $\mathcal{U}$.The coefficient inequalities for $f(z)$ in theses classes are discussed.

## 1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0. \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$.
Let $0 \leq \beta < 1$.The function $f \in \mathcal{A}$ is said to be $q$-starlike of order $\beta$ in $\mathcal{U}$ if it satisfies

$$Re\left(\frac{z\partial_q f(z)}{f(z)}\right) > \beta, (z \in \mathcal{U})$$

and the class of all such $q$-starlike functions of order $\beta$ is denoted by $\mathcal{S}_q^*(\beta)$.The function $f \in \mathcal{A}$ is said to be $q$- convex of order $\beta$ if

$$Re\left(\frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)}\right) > \beta, (z \in \mathcal{U}). \tag{1.2}$$

and the class of all such convex functions of order $\beta$ is denoted by $\mathcal{K}_q(\beta)$ As usual, let

$$\mathcal{S}_q^*(0) \equiv \mathcal{S}_q^* \; and \; \mathcal{K}_q(0) \equiv \mathcal{K}_q$$

.
A function $f \in \mathcal{S}_q^*(\beta)$ is $q$-starlike of reciprocal order $\alpha$ ,$(0 \leq \alpha < 1)$,in $\mathcal{U}$, if it satisfies

$$Re\left(\frac{f(z)}{z\partial_q f(z)}\right) > \alpha \; (z \in \mathcal{U}). \tag{1.3}$$

the class of such functions is denoted by $\mathcal{S}_{q,*}(\alpha)$.
Also, we define the subclass $\mathcal{K}_{q,*}(\alpha)$ of $\mathcal{A}$ which satisfy

$$Re\left(\frac{\partial_q f(z)}{\partial_q(z\partial_q f(z))}\right) > \alpha, \; (z \in \mathcal{U}) \tag{1.4}$$

the class of such functions are called $q$-convex of reciprocal order $\alpha$,$(0 \leq \alpha < 1)$, in $\mathcal{U}$
we also note that

$$f(z) \in \mathcal{K}_{q,*}(\alpha) \iff z\partial_q f(z) \in \mathcal{S}_{q,*}(\alpha)$$

In the present paper, we consider some coefficient inequalities for $f(z)$ in the classes $\mathcal{S}_{q,*}(\alpha)$ and $\mathcal{K}_{q,*}(\alpha)$.

## 2 Main Results

In order to consider some coefficient inequalities for $f(z)$, we have to recall the following definition.Let $p(z)$ be analytic in $\mathcal{U}$ with

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \tag{2.1}$$

If $p(z)$ given by (2.1) satisfies

$$Rep(z) > 0, (z \in \mathcal{U}), \tag{2.2}$$

then $p(z)$ is said to be caratheodory function in $\mathcal{U}$.It is well known that

$$|c_n| \leq 2, (n = 1, 2, 3, ...) \tag{2.3}$$

and the equality holds true for

$$p(z) = \frac{1+z}{1-z} \tag{2.4}$$

**Theorem 2.1.** *If* $f(z) \in \mathcal{S}_{q,*}(\alpha)$ *then*

$$|a_n| \leq \frac{2(1-\alpha)(1+[2]_q|a_2|)}{[n]_q - 1} \prod_{k=2}^{n-2} \left[1 + \frac{2(1-\alpha)[k+1]_q}{[k+1]_q - 1}\right] (n = 4, 5, 6, ...) \tag{2.5}$$

*with* $|a_2| \leq \dfrac{2(1-\alpha)}{[2]_q - 1}$ *and* $|a_3| \leq \dfrac{2(1-\alpha)}{[3]_q - 1}\left[1 + [2]_q|a_2|\right]$

If we define the function $p(z)$ by

$$p(z) = \frac{\dfrac{f(z)}{z\partial_q f(z)} - \alpha}{1-\alpha} = 1 + c_1 z + c_2 z^2 + ... \tag{2.6}$$

then

$$f(z) = z\partial_q f(z)[\alpha + (1-\alpha)p(z)]$$

that is,

$$z + \sum_{n=2}^{\infty} a_n z^n = (z + \sum_{n=2}^{\infty}[n]_q a_n z^n)(1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n).$$

which gives that

$$a_n = \frac{1-\alpha}{1-[n]_q}(c_{n-1} + [2]_q a_2 c_{n-2} + [3]_q a_3 c_{n-3} + ... + [n-1]_q a_{n-1} c_1)$$

applying $|c_n| \leq 2$ we obtain that

$$|a_n| \leq \frac{2(1-\alpha)}{[n]_q - 1}\left(1 + [2]_q|a_2| + [3]_q|a_3| + ... + [n-1]_q|a_{n-1}|\right)$$

this implies that

$$|a_2| \leq \frac{2(1-\alpha)}{[2]_q - 1}$$

and

$$|a_3| \leq \frac{2(1-\alpha)}{[3]_q - 1}\left(1 + [2]_q|a_2|\right)$$

For $n = 4$ we also see that

$$|a_4| \leq \frac{2(1-\alpha)}{[4]_q - 1}\left\{1 + [2]_q|a_2| + [3]_q|a_3|\right\}$$

$$\leq \frac{2(1-\alpha)}{[4]_q - 1}\left\{1 + [2]_q|a_2| + [3]_q\frac{2(1-\alpha)}{[3]_q - 1}\left[1 + [2]_q|a_2|\right]\right\}$$
$$\leq \frac{2(1-\alpha)}{[4]_q - 1}\left\{[1 + [2]_q|a_2|]\left[1 + [3]_q\frac{2(1-\alpha)}{[3]_q - 1}\right]\right\}$$

which proves that (2.5)holds true for $n = 4$.we suppose that that the coefficient inequality (2.5)holds true for $n = j$.Then we get

$$|a_{j+1}| \leq \frac{2(1-\alpha)}{[j+1]_q - 1}\left\{1 + [2]_q|a_2| + [3]_q|a_3| + ...[j]_q|a_j|\right\}.$$

$$|a_{j+1}| \leq \frac{2(1-\alpha)}{([j+1]_q - 1)}\left\{1 + [2]_q|a_2| + [3]_q\frac{2(1-\alpha)}{[3]_q - 1}[1 + [2]_q|a_2|] + [4]_q\frac{2(1-\alpha)}{[4]_q - 1}\left(1 + [2]_q|a_2|\left[1 + [3]_q\frac{2(1-\alpha)}{[3]_q - 1}\right]\right) + ..... + [j]_q\frac{2(1-\alpha)}{[j]_q - 1}\left\{(1 + [2]_q|a_2|)\prod_{k=2}^{j-2}\left(1 + \frac{2(1-\alpha)[k+1]_q}{[k+1]_q - 1}\right)\right\}\right\}$$

$$|a_{j+1}| \leq \frac{2(1-\alpha)(1 + [2]_q|a_2|)}{([j+1]_q - 1)}\prod_{k=2}^{j-1}\left(1 + \frac{2(1-\alpha)[k+1]_q}{[k+1]_q - 1}\right).$$

Thus (2.5)holds true for $n = j + 1$.Therefore applying the mathematical induction, we prove the coefficient inequality for $n = 4, 5, 6, ....$

**Corollary 2.2.** *If $f(z) \in \mathcal{S}_{q,*}$, then*

$$|a_n| \leq \frac{2(1 + [2]_q|a_2|)}{[n]_q - 1}\prod_{k=2}^{n-2}\left[\frac{3[k+1]_q - 1}{[k+1]_q - 1}\right] \quad (n = 4, 5, 6)$$

*with* $|a_2| \leq \frac{2}{[2]_q - 1}$ *and* $|a_3| \leq \frac{2}{[3]_q - 1}[1 + [2]_q|a_2|]$

For $f(z) \in \mathcal{K}_{q,*}(\alpha)$, we have

**Theorem 2.3.** *If $f(z) \in \mathcal{K}_{q,*}(\alpha)$, then*

$$|a_n| \leq \frac{2(1-\alpha)(1 + [2]_q^2|a_2|)}{[n]_q([n]_q - 1)}\prod_{k=2}^{n-2}\left[1 + \frac{2(1-\alpha)[k+1]_q}{[k+1]_q - 1}\right] \quad (n = 4, 5, 6, ...)$$

*with* $|a_2| \leq \frac{2(1-\alpha)}{[2]_q([2]_q - 1)]}$ *and* $|a_3| \leq \frac{2(1-\alpha)}{[3]_q([3]_q - 1)}\left\{1 + [2]_q^2|a_2|\right\}$

*Proof.* We note that $f(z) \in \mathcal{K}_{q.*}(\alpha)$ if and only if $z\partial_q f(z) \in \mathcal{S}_{q,*}(\alpha)$. This shows that

$$[n]_q|a_n| \leq \frac{2(1-\alpha)}{[n]_q - 1}(1 + [2]_q^2|a_2|)\prod_{k=2}^{n-2}\left[1 + \frac{2(1-\alpha)[k+1]_q}{[k+1]_q - 1}\right].$$

for $n = 4, 5, 6, ...,$
$|a_2| \leq \frac{2(1-\alpha)}{[2]_q([2]_q - 1)}$ and $|a_3 \leq \frac{2(1-\alpha)}{[3]_q([3]_q - 1)}\left\{1 + [2]_q^2|a_2|\right\}.$
This completes the proof of the theorem. □

**Corollary 2.4.** *If $f(z) \in \mathcal{K}_{q,*}$, then*

$$|a_n| \leq \frac{2}{[n]_q([n]_q - 1)}(1 + [2]_q^2|a_2|)\prod_{k=2}^{n-2}\left[1 + \frac{2[k+1]_q}{[k+1]_q - 1}\right].for \quad n = 4, 5, 6, ...,$$

$|a_2| \leq \frac{2}{[2]_q([2]_q - 1)}$ *and* $|a_3 \leq \frac{2}{[3]_q([3]_q - 1)}\left\{1 + [2]_q^2|a_2|\right\}.$

**Remark 2.5.** As $q \to 1$ we get the results of Junichi Nishiwaki and Shigeyoshi owa [2].

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Received: February 2, 2018.
Accepted: December 28, 2018.