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SOME INEQUALITIES FOR LOGARITHM WITH APPLICATIONS TO WEIGHTED MEANS

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Abstract In this paper we establish several inequalities for logarithm and apply them to obtain some new inequalities involving weighted arithmetic mean, geometric mean and harmonic mean of n-tuples of positive sequences. The case of two positive numbers and an analysis of which bound is better and when are also considered.

1 Introduction

There are a number of inequalities for logarithm, see for instance

http : //functions.wolfram.com/ElementaryFunctions/Log/29/

and [5] that are well know and widely used in literature, such as:

$$\frac{x-1}{x} \leq \ln x \leq x-1 \text{ for } x>0, \tag{1.1}$$

$$\frac{2x}{2+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}} \text{ for } x \geq 0, \tag{1.2}$$

$$x \leq -\ln(1-x) \leq \frac{x}{1-x}, \text{ for } x<1,$$

$$\ln x \leq n\left(x^{1/n}-1\right) \text{ for } n>0 \text{ and } x>0,$$

$$\ln(1-|x|) \leq \ln(x+1) \leq -\ln(1-|x|) \text{ for } |x|<1,$$

and

$$-\frac{3}{2}x \leq \ln(1-x) \leq \frac{3}{2}x \text{ for } 0<x \leq 0.5838.$$

A simple proof of the first inequality in (1.2) may be found, for instance, in [6], see also [7] where the following rational bounds are provided as well:

$$\frac{x\left(1+\frac{5}{6}x\right)}{(1+x)\left(1+\frac{1}{3}x\right)} \leq \ln(1+x) \leq \frac{x\left(1+\frac{1}{6}x\right)}{1+\frac{2}{3}x} \text{ for } x \geq 0.$$

In the recent paper [3] we established the following result:

$$(0 \leq)\,(1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \leq \nu(1-\nu)(b-a)(\ln b-\ln a) \tag{1.3}$$

for any $a, b>0$ and $\nu \in (0,1)$.

If we take in (1.3) $b=x+1$, $x>0$ and $a=1$, then we get

$$\ln(x+1) \geq \frac{1-\nu+\nu(x+1)-(x+1)^{\nu}}{\nu(1-\nu)x}\,(\geq 0) \tag{1.4}$$

for any $\nu \in (0,1)$ and, in particular

$$\ln(x+1) \geq \frac{2\left(\sqrt{x+1}-1\right)^2}{x} \ (\geq 0) \tag{1.5}$$

for any $x > 0$ and $\nu \in (0,1)$.

In this paper we establish some inequalities for the quantity

$$\frac{b-a}{a} - \ln b + \ln a$$

when $a, b > 0$ and apply them to obtain some new inequalities involving weighted arithmetic mean, geometric mean and harmonic mean of n-tuples of positive numbers. The case of two positive numbers and an analysis of which bound is better and when are also considered.

2 Logarithmic Inequalities

The following theorem is well known in the literature as Taylor's theorem with the integral remainder.

Theorem 2.1. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on I, then for each $x \in I$*

$$f(x) = T_n(f;a,x) + R_n(f;a,x), \tag{2.1}$$

where $T_n(f;a,x)$ is Taylor's polynomial, i.e.,

$$T_n(f;a,x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

(Note that $f^{(0)} := f$ and $0! := 1$), and the remainder is given by

$$R_n(f;a,x) := \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt.$$

The following result holds [2]:

Lemma 2.2. *For any $a, b > 0$ we have for $n \geq 1$ that*

$$\ln b - \ln a + \sum_{k=1}^{n} \frac{(-1)^k (b-a)^k}{ka^k} = (-1)^n \int_a^b \frac{(b-t)^n}{t^{n+1}}dt. \tag{2.2}$$

Proof. Consider the function $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = \ln x$, then

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}, \ n \geq 1, \ x > 0,$$

$$T_n(f;a,x) = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k}, \ a > 0$$

and

$$R_n(f;a,x) = (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}}dt.$$

Now, using (2.1) we have the equality,

$$\ln x = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k} + (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}}dt,$$

i.e.,

$$\ln x - \ln a + \sum_{k=1}^{n} \frac{(-1)^{k}(x-a)^k}{ka^k} = (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}}dt, \quad x, a > 0.$$

Choosing in the last equality $x = b$, we get (2.2). □

Theorem 2.3. *For any $a, b > 0$ we have*

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 = \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2. \tag{2.3}$$

Proof. For $n = 1$ we get from (2.2) that

$$\int_a^b \frac{b-t}{t^2}dt = \frac{b-a}{a} - \ln b + \ln a \tag{2.4}$$

for any $a, b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.5}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \ge \int_b^a \frac{t-b}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.6}$$

Therefore, by (2.5) and (2.6) we have for any $a, b > 0$ that

$$\int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (2.4) we then get the desired result (2.3). □

When some bounds for a, b are provided, then we have:

Corollary 2.4. *Assume that $a, b \in [m, M] \subset (0, \infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{m^2} \tag{2.7}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{2.8}$$

Remark 2.5. If we take in (2.3) $a = 1$ and $b = x \in (0, \infty)$, then we get

$$\frac{1}{2}\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2 = \frac{1}{2}\frac{(x-1)^2}{\max^2\{1,x\}} \le x - 1 - \ln x \le \frac{1}{2}\frac{(x-1)^2}{\min^2\{1,x\}} = \frac{1}{2}\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2 \tag{2.9}$$

and if we take $a = x$ and $b = 1$, then we also get

$$\frac{1}{2}\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2 = \frac{1}{2}\frac{(x-1)^2}{\max^2\{1,x\}} \le \ln x - \frac{x-1}{x} \le \frac{1}{2}\frac{(x-1)^2}{\min^2\{1,x\}} = \frac{1}{2}\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2. \tag{2.10}$$

If $x \in [k,K] \subset (0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\} \le \min\{1,x\} \le \min\{1,K\}$$

and

$$\max\{1,k\} \le \max\{1,x\} \le \max\{1,K\}.$$

By (2.9) and (2.10) we get the *local bounds*

$$\frac{1}{2}\frac{(x-1)^2}{\max^2\{1,K\}} \le x-1-\ln x \le \frac{1}{2}\frac{(x-1)^2}{\min^2\{1,k\}} \tag{2.11}$$

and

$$\frac{1}{2}\frac{(x-1)^2}{\max^2\{1,K\}} \le \ln x - \frac{x-1}{x} \le \frac{1}{2}\frac{(x-1)^2}{\min^2\{1,k\}} \tag{2.12}$$

for any $x \in [k,K]$.

We have by (2.11) and (2.12):

Corollary 2.6. *Let $a, b > 0$ and such that $\frac{b}{a} \in [k,K] \subset (0,\infty)$. Then we have*

$$\frac{1}{2}\frac{(b-a)^2}{a^2\max^2\{1,K\}} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{a^2\min^2\{1,k\}} \tag{2.13}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{a^2\max^2\{1,K\}} \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2}\frac{(b-a)^2}{a^2\min^2\{1,k\}}. \tag{2.14}$$

If we assume that $a, b \in [m,M] \subset (0,\infty)$, then by taking $k = \frac{m}{M} < 1 < \frac{M}{m} = K$ in (2.13) and (2.14) we get

$$\frac{1}{2}\frac{m^2}{M^2}\left(\left(\frac{b}{a}\right)^2 - 2\frac{b}{a} + 1\right) \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{M^2}{m^2}\left(\left(\frac{b}{a}\right)^2 - 2\frac{b}{a} + 1\right) \tag{2.15}$$

and

$$\frac{1}{2}\frac{m^2}{M^2}\left(\left(\frac{b}{a}\right)^2 - 2\frac{b}{a} + 1\right) \le \ln b - \ln a - \frac{b-a}{b} \le \frac{1}{2}\frac{M^2}{m^2}\left(\left(\frac{b}{a}\right)^2 - 2\frac{b}{a} + 1\right). \tag{2.16}$$

Observe also that for $x \in [k,K]$ we have

$$1 - \frac{\min\{1,x\}}{\max\{1,x\}} \ge 1 - \frac{\min\{1,K\}}{\max\{1,k\}} \ge 0$$

and

$$0 \leq \frac{\max\{1,x\}}{\min\{1,x\}} - 1 \leq \frac{\max\{1,K\}}{\min\{1,k\}} - 1.$$

Now, by (2.9) and (2.10) we get the *global bounds*

$$\frac{1}{2}\left(1 - \frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \leq x - 1 - \ln x \leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}} - 1\right)^2 \tag{2.17}$$

and

$$\frac{1}{2}\left(1 - \frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \leq \ln x - \frac{x-1}{x} \leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}} - 1\right)^2 \tag{2.18}$$

for any $x \in [k, K]$.

By (2.17) and (2.18) we have:

Corollary 2.7. *Let $a, b > 0$ and such that $\frac{b}{a} \in [k, K] \subset (0, \infty)$. Then we have*

$$\frac{1}{2}\left(1 - \frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}} - 1\right)^2 \tag{2.19}$$

and

$$\frac{1}{2}\left(1 - \frac{\min\{1,K\}}{\max\{1,k\}}\right)^2 \leq \ln b - \ln a - \frac{b-a}{b} \leq \frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}} - 1\right)^2. \tag{2.20}$$

We observe that from (2.19) we actually have

$$\frac{1}{2}\begin{cases} (1-K)^2 \text{ if } K < 1, \\ 0 \text{ if } k \leq 1 \leq K, \\ \left(1 - \frac{1}{k}\right)^2 \text{ if } 1 < k, \end{cases} \tag{2.21}$$

$$\leq \frac{b-a}{a} - \ln b + \ln a$$

$$\leq \frac{1}{2}\begin{cases} \left(\frac{1}{k} - 1\right)^2 \text{ if } K < 1, \\ \left(\frac{K}{k} - 1\right)^2 \text{ if } k \leq 1 \leq K, \\ (K-1)^2 \text{ if } 1 < k \end{cases}$$

and the same bounds for $\ln b - \ln a - \frac{b-a}{b}$.

We also have:

Theorem 2.8. *For any $a, b > 0$ we have*

$$(0 \leq) \frac{b-a}{a} - \ln b + \ln a \leq \frac{(b-a)^2}{ab} \tag{2.22}$$

and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{(b-a)^2}{ab}. \tag{2.23}$$

Proof. If $b > a$, then

$$\int_a^b \frac{b-t}{t^2} dt \leq (b-a) \int_a^b \frac{1}{t^2} dt = (b-a)\frac{b-a}{ab} = \frac{(b-a)^2}{ab}.$$

If $a > b$, then

$$\int_a^b \frac{b-t}{t^2} dt = \int_b^a \frac{t-b}{t^2} dt \leq (a-b) \int_b^a \frac{1}{t^2} dt = (a-b)\frac{a-b}{ab} = \frac{(b-a)^2}{ab}.$$

Therefore,

$$\int_a^b \frac{b-t}{t^2} dt \leq \frac{(b-a)^2}{ab}$$

for any $a, b > 0$ and by the representation (2.4) we get the desired result (2.22). □

It is natural to ask, which of the upper bounds for the quantity

$$\frac{b-a}{a}-\ln b+\ln a$$

as provided by (2.3) and (2.22) is better?

Consider the difference

$$\Delta(a,b):=\frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}}-\frac{(b-a)^2}{ab},\ a,\ b>0.$$

We observe that for $b>a$ we get

$$\Delta(a,b):=\frac{1}{2}\frac{(b-a)^2}{a^2}-\frac{(b-a)^2}{ab}=\frac{(b-a)^2}{2a^2b}(b-2a).$$

Therefore $\Delta(a,b)>0$ if $b>2a$ and $\Delta(a,b)<0$ if $a<b<2a$, meaning that neither of the upper bounds in (2.3) and (2.22) is always best.

If we take in (2.22) and (2.23) $a=1$ and $b=x\in(0,\infty)$, then we get

$$(0\le)\,x-1-\ln x\le\frac{(x-1)^2}{x} \tag{2.24}$$

and

$$(0\le)\,\ln x-\frac{x-1}{x}\le\frac{(x-1)^2}{x} \tag{2.25}$$

for any $x>0$.

Corollary 2.9. *Let $a,b>0$ and such that $\frac{b}{a}\in[k,K]\subset(0,\infty)$. Then we have*

$$\frac{b-a}{a}-\ln b+\ln a\le U(k,K) \tag{2.26}$$

and

$$\ln b-\ln a-\frac{b-a}{b}\le U(k,K), \tag{2.27}$$

where

$$U(k,K):=\begin{cases}\frac{(k-1)^2}{k} & \text{if } K<1,\\ \max\left\{\frac{(k-1)^2}{k},\frac{(K-1)^2}{K}\right\} & \text{if } k\le 1\le K,\\ \frac{(K-1)^2}{K} & \text{if } 1<k.\end{cases}$$

Proof. Consider the function $f(x)=\frac{(x-1)^2}{x}$, $x>0$. We observe that

$$f'(x)=\frac{x^2-1}{x^2}\text{ and }f''(x)=\frac{2}{x^3},$$

which shows that f is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ and strictly convex for $x>0$. We also have $f\left(\frac{1}{x}\right)=f(x)$ for $x>0$.

By (2.24) and by the properties of f we then have that for any $x\in[k,K]$

$$\begin{aligned} x-1-\ln x &\le \max_{x\in[k,K]}\frac{(x-1)^2}{x} \\ &= \begin{cases}\frac{(k-1)^2}{k} & \text{if } K<1,\\ \max\left\{\frac{(k-1)^2}{k},\frac{(K-1)^2}{K}\right\} & \text{if } k\le 1\le K,\\ \frac{(K-1)^2}{K} & \text{if } 1<k.\end{cases} \\ &= U(k,K). \end{aligned} \tag{2.28}$$

Now, put $x=\frac{b}{a}\in[k,K]$ in (2.28) to get the desired inequality (2.26).

Let $y=\frac{1}{x}$ with $x=\frac{b}{a}\in[k,K]$. Then $y\in\left[\frac{1}{K},\frac{1}{k}\right]$ and we have like in (2.28) that

$$\begin{aligned} y-1-\ln y &\le \max_{y\in[K^{-1},k^{-1}]}\frac{(y-1)^2}{y} \\ &= \begin{cases} \frac{\left(K^{-1}-1\right)^2}{K^{-1}} \text{ if } k^{-1}<1, \\ \max\left\{\frac{\left(K^{-1}-1\right)^2}{K^{-1}},\frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}}\right\} \text{ if } k\le 1\le K^{-1}, \\ \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}} \text{ if } 1<\frac{1}{K^{-1}}, \end{cases} \\ &= U(k,K), \end{aligned}$$

which implies (2.27). □

Now, by Corollary 2.4 we have the *global upper bound*

$$\frac{b-a}{a}-\ln b+\ln a\le\frac{1}{2}\frac{(M-m)^2}{m^2}, \tag{2.29}$$

for any $a,b\in[m,M]$. Moreover, if $a,b\in[m,M]$, then $K=\frac{M}{m}$ and $k=\frac{m}{M}$ and by Corollary 2.9 we also get

$$\frac{b-a}{a}-\ln b+\ln a\le\frac{(M-m)^2}{mM}, \tag{2.30}$$

which implies that

$$(0\le)\,\frac{b-a}{a}-\ln b+\ln a\le\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\} \tag{2.31}$$

for any $a,b\in[m,M]$.

We observe that, for $m<M<2m$, the inequality (2.29) is better than (2.30). If $M\ge 2m$, then the conclusion is the other way around.

From the above consideration, we can conclude that the following inequality is also valid

$$(0\le)\,\ln b-\ln a-\frac{b-a}{b}\le\frac{(M-m)^2}{mM}\min\left\{\frac{M}{2m},1\right\} \tag{2.32}$$

for any $a,b\in[m,M]$.

3 Applications for Weighted AM-GM Inequality

Define the *weighted arithmetic mean* of the positive n-tuple $x=(x_1,...,x_n)$ with the *probability distribution* $w=(w_1,...,w_n)$ by

$$A_n(w,x):=\sum_{i=1}^{n}w_ix_i$$

and the *weighted geometric mean* of the same n-tuple, by

$$G_n(w,x):=\left({}_{i=1}^{n}x_i^{w_i}\right).$$

It is well know that the following arithmetic mean-geometric mean inequality holds

$$A_n(w,x)\ge G_n(w,x).$$

Define also

$$A_{n,2}(w,x):=\sum_{i=1}^{n}w_i\,x_i^2,$$

the *weighted harmonic mean*

$$H_n(w,x) := \frac{1}{\sum_{i=1}^{n}\frac{w_i}{x_i}} = A_n^{-1}\left(w,x^{-1}\right),$$

and the *dispersion*

$$D_n^2(w,x) := A_{n,2}(w,x) - A_n^2(w,x).$$

We have the following result:

Theorem 3.1. *Assume that the n-tuple $x=(x_1,...,x_n)$ satisfies the condition*

$$0 < m \le x_i \le M < \infty \tag{3.1}$$

for any $i \in \{1,...,n\}$, then for any probability distribution $w=(w_1,...,w_n)$ we have

$$\begin{aligned}&\exp\left[A_n(w,x)H_n^{-1}(w,x) - 1 - \frac{1}{2m^2}D_n^2(w,x)\right] \\ &\le \frac{A_n(w,x)}{G_n(w,x)} \\ &\le \exp\left[A_n(w,x)H_n^{-1}(w,x) - 1 - \frac{1}{2M^2}D_n^2(w,x)\right]\end{aligned} \tag{3.2}$$

and

$$\exp\left[\frac{1}{2M^2}D_n^2(w,x)\right] \le \frac{A_n(w,x)}{G_n(w,x)} \le \exp\left[\frac{1}{2m^2}D_n^2(w,x)\right]. \tag{3.3}$$

Proof. We have that $A_n(w,x) \in [m,M]$ and by (2.7) we obtain

$$\begin{aligned}\frac{1}{2}\frac{(A_n(w,x)-a)^2}{M^2} &\le \frac{A_n(w,x)-a}{a} - \ln A_n(w,x) + \ln a \\ &\le \frac{1}{2}\frac{(A_n(w,x)-a)^2}{m^2}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\frac{1}{2}\frac{(b-A_n(w,x))^2}{M^2} &\le \frac{b-A_n(w,x)}{A_n(w,x)} - \ln b + \ln A_n(w,x) \\ &\le \frac{1}{2}\frac{(b-A_n(w,x))^2}{m^2}\end{aligned} \tag{3.5}$$

for any $a, b \in [m,M]$.

Take in (3.4) $a = x_i$, multiply the obtained inequality by w_i and sum over $i \in \{1,...,n\}$ to get

$$\begin{aligned}&\frac{1}{2M^2}\sum_{i=1}^{n} w_i\,(A_n(w,x)-x_i)^2 \\ &\le A_n(w,x)\sum_{i=1}^{n}\frac{w_i}{x_i} - 1 - \ln A_n(w,x) + \sum_{i=1}^{n} w_i\,\ln x_i \\ &\le \frac{1}{2m^2}\sum_{i=1}^{n} w_i\,(A_n(w,x)-x_i)^2.\end{aligned} \tag{3.6}$$

Since

$$\sum_{i=1}^{n} w_i\,(A_n(w,x)-x_i)^2 = A_{n,2}(w,x) - (A_n(w,x))^2 = D_n^2(w,x),$$

$$\sum_{i=1}^{n} \frac{w_i}{x_i} = H_n^{-1}(w,x)$$

and

$$\sum_{i=1}^{n} w_i \ln x_i = \ln G_n(w,x),$$

hence by (3.6) we have

$$\begin{aligned} &\frac{1}{2M^2} D_n^2(w,x) \\ &\le A_n(w,x) H_n^{-1}(w,x) - 1 - \ln A_n(w,x) + \ln G_n(w,x) \\ &\le \frac{1}{2m^2} D_n^2(w,x) \end{aligned} \tag{3.7}$$

that is equivalent to

$$\begin{aligned} &A_n(w,x) H_n^{-1}(w,x) - 1 - \frac{1}{2m^2} D_n^2(w,x) \\ &\le \ln A_n(w,x) - \ln G_n(w,x) \\ &\le A_n(w,x) H_n^{-1}(w,x) - 1 - \frac{1}{2M^2} D_n^2(w,x) \end{aligned}$$

and by taking the exponential, we get (3.2).

Further, take in (3.4) $b = x_i$, multiply the obtained inequality by w_i and sum over $i \in \{1, ..., n\}$ to get

$$\begin{aligned} \frac{1}{2M^2} \sum_{i=1}^{n} w_i \left(A_n(w,x) - x_i\right)^2 &\le \ln A_n(w,x) - \ln G_n(w,x) \\ &\le \frac{1}{2m^2} \sum_{i=1}^{n} w_i \left(A_n(w,x) - x_i\right)^2 \end{aligned} \tag{3.8}$$

and by taking the exponential, we deduce (3.3). □

Remark 3.2. Choose $n = 2$ and let $w_1 = 1 - \nu$, $w_2 = \nu$, $x_1 = a$, $x_2 = b$ with $\nu \in [0,1]$ and a, $b > 0$. Then

$$A_2(w,x) = (1-\nu)a + \nu b,$$

$$H_2^{-1}(w,x) = (1-\nu)\frac{1}{a} + \nu\frac{1}{b} = \frac{(1-\nu)b + \nu a}{ab}$$

and

$$\begin{aligned} D_2^2(w,x) &= (1-\nu)a^2 + \nu b^2 - \left((1-\nu)a + \nu b\right)^2 \\ &= (1-\nu)a^2 + \nu b^2 - (1-\nu)^2 a^2 - 2(1-\nu)\nu ab - \nu^2 b^2 \\ &= (1-\nu)\nu(b-a)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} &A_2(w,x) H_2^{-1}(w,x) - 1 \\ &= \frac{[(1-\nu)a + \nu b][(1-\nu)b + \nu a]}{ab} - 1 \\ &= \frac{(1-\nu)^2 ab + \nu(1-\nu)b^2 + \nu(1-\nu)a^2 + \nu^2 ab - ab}{ab} \\ &= \frac{\nu(1-\nu)(b-a)^2}{ab}. \end{aligned}$$

Then

$$A_2(w,x)\,H_2^{-1}(w,x)-1-\frac{1}{2m^2}D_2^2(w,x)$$
$$=\frac{\nu(1-\nu)(b-a)^2}{ab}-\frac{(1-\nu)\nu(b-a)^2}{2m^2}$$
$$=\nu(1-\nu)(b-a)^2\left(\frac{1}{ab}-\frac{1}{2m^2}\right)$$

and

$$A_2(w,x)\,H_2^{-1}(w,x)-1-\frac{1}{2M^2}D_2^2(w,x)$$
$$=\frac{\nu(1-\nu)(b-a)^2}{ab}-\frac{(1-\nu)\nu(b-a)^2}{2M^2}$$
$$=\nu(1-\nu)(b-a)^2\left(\frac{1}{ab}-\frac{1}{2M^2}\right).$$

Then by (3.2) and (3.3) we get

$$\exp\left[\nu(1-\nu)(b-a)^2\left(\frac{1}{ab}-\frac{1}{2m^2}\right)\right] \le \frac{A_\nu(a,b)}{G_\nu(a,v)} \le \exp\left[\nu(1-\nu)(b-a)^2\left(\frac{1}{ab}-\frac{1}{2M^2}\right)\right] \tag{3.9}$$

and

$$\exp\left[\frac{1}{2M^2}(1-\nu)\nu(b-a)^2\right] \le \frac{A_\nu(a,b)}{G_\nu(a,v)} \le \exp\left[\frac{1}{2m^2}(1-\nu)\nu(b-a)^2\right] \tag{3.10}$$

where

$$A_\nu(a,b):=(1-\nu)a+\nu b$$

is the weighted arithmetic mean of (a,b) and

$$G_\nu(a,b):=a^{1-\nu}b^\nu$$

is the weighted geometric mean of (a,b).

The inequality (3.10) has been obtained in different ways in either of the recent papers [1] and [4].

In order to compare the upper and lower bounds for the quotient $\frac{A_\nu(a,b)}{G_\nu(a,v)}$ provided by (3.9) and (3.10) we consider the difference

$$D_{m,M}(a,b):=\frac{1}{ab}-\frac{1}{2M^2}-\frac{1}{2m^2}$$

where $a,b\in[m,M]$.

We observe that

$$\lim_{a,b\to m}D_{m,M}(a,b):=\frac{1}{m^2}-\frac{1}{2M^2}-\frac{1}{2m^2}=\frac{M^2-m^2}{2m^2M^2}>0$$

and

$$\lim_{a,b\to M}D_{m,M}(a,b)=\frac{1}{M^2}-\frac{1}{2M^2}-\frac{1}{2m^2}=\frac{m^2-M^2}{2m^2M^2}<0,$$

which show that neither of the lower or upper bounds in (3.9) and (3.10) is always best.

We also have:

Theorem 3.3. *Assume that the n-tuple $x = (x_1, ..., x_n)$ satisfies the condition (3.1) for any $i \in \{1, ..., n\}$ then for any probability distribution $w = (w_1, ..., w_n)$ we have*

$$\frac{\exp\left[A_n(w,x)\, H_n^{-1}(w,x) - 1\right]}{\frac{A_n(w,x)}{G_n(w,x)}} \le \exp\left[\frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\}\right] \tag{3.11}$$

and

$$\frac{A_n(w,x)}{G_n(w,x)} \le \exp\left[\frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\}\right]. \tag{3.12}$$

Proof. From the inequalities (2.31) and (2.32) we have

$$\frac{A_n(w,x) - a}{a} - \ln A_n(w,x) + \ln a \le \frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\} \tag{3.13}$$

and

$$\frac{b - A_n(w,x)}{A_n(w,x)} - \ln b + \ln A_n(w,x) \le \frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\} \tag{3.14}$$

for any $a, b \in [m, M]$.

By a similar argument to the one in the proof of Theorem 3.1 we get

$$A_n(w,x)\, H_n^{-1}(w,x) - 1 - \ln A_n(w,x) + \ln G_n(w,x) \le \frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\}$$

and

$$\ln A_n(w,x) - \ln G_n(w,x) \le \frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\}$$

that are equivalent to the desired results (3.11) and (3.12). □

Now, we observe that since $\nu(1-\nu) \le \frac{1}{4}$ for any $\nu \in [0,1]$, then by (3.10) we have

$$\frac{A_\nu(a,b)}{G_\nu(a,v)} \le \exp\left[\frac{1}{8m^2}(M-m)^2\right] \tag{3.15}$$

while from (3.12) we get

$$\frac{A_\nu(a,b)}{G_\nu(a,v)} \le \exp\left[\frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\}\right] \tag{3.16}$$

for any $\nu \in [0,1]$ and any $a, b \in [m, M]$.

Now, if $m < M < 2m$, then $\frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\} = \frac{(M-m)^2}{2m^2}$, which shows that the upper bound from (3.15) is better than the one from (3.16). If $2m < M < 8m$ then $\frac{(M-m)^2}{mM} \min\left\{\frac{M}{2m}, 1\right\} = \frac{(M-m)^2}{mM}$, which shows that still the upper bound from (3.15) is better than the one from (3.16). If $8m \le M$, then the bound in (3.16) is better than the one in (3.15).

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