Palestine Journal of Mathematics

Vol. 9(1)(2020) , 557–560

ON THE COSET PRESERVING SUBCENTRAL AUTOMORPHISMS OF FINITE GROUPS

Mohammad Mehdi Nasrabadi and Parisa Seifizadeh

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20D45; Secondary 20D15.

Keywords and phrases: Subcentral automorphism, Finite p-groups, Frattini subgroup, Inner automorphism.

Abstract. Let G be a group and M be a characteristic subgroup of G. We denote by $Aut_M^M(G)$ the set of all automorphisms of G which centralize G/M and M. In this paper, we give the necessary and sufficient conditions for equality $Aut_M^M(G)$ with $Aut^M(G)$. Also we study equivalent conditions for equality $Aut_M^M(G)$ with Inn(G).

1 Introduction

In this paper p denotes a prime number. Let us denote $\Phi(G)$, $G^{'}$, $Z(G)$, Aut(G) and Inn(G), respectively the Frattini subgroup, commutator subgroup, the center, the full automorphism group and the inner automorphisms group of G. An automorphism α of G is called a central automorphism if $x^{-1}\alpha(x) \in Z(G)$ for $x \in G$. All the elements of the central automorphism group of G, denoted by $Aut^Z(G)$, is a normal subgroup of $Aut(G)$.
There has been a number of results on the central automorphisms of a group. Curran and McCaughan [4] proved that for any non-abelian finite group G, $Aut_Z^Z(G) \cong Hom(G/G^{'}Z(G), Z(G))$ where $Aut_Z^Z(G)$ is group of all those central automorphisms which preserve the center Z(G) elementwise. Adney and Yen [1] proved that if a finite group G has no abelian direct factor, then there is a one-one and onto map between $Aut^Z(G)$ and Hom$(G, Z(G))$. Ghumed and Ghate [5] proved that for a finite group G, $Aut_M^M(G) \cong$ Hom$(G/KM, M)$. Also he proved that if G is a purely non-abelian finite group, then $|Aut^M(G)| = |$Hom$(G, M)|$. Shabani Attar [10] characterized all finite p-groups G for which the equality $Aut^Z(G) = Aut_Z^Z(G)$ holds. Let IA(G) be the subgroup of Aut(G) which consists of those automorphisms α for which $g^{-1}\alpha(g) \in G^{'}$ for each $g \in G$. A group G is called semicomplete if $IA(G) = Inn(G)$. Shabani Attar [9] gave some necessary conditions for finite p-groups to be semicomplete. In this paper, we give necessary and sufficient conditions for G such that $Aut_M^M(G) = Aut^M(G)$ and show some equivalent conditions for equality Aut_M^M with Inn(G).

2 Preliminaries

Let M be a characteristic subgroup of G. By $Aut^M(G)$, we mean the subgroup of $Aut(G)$ consisting of all automorphisms which induce identity on G/M. By $Aut_M(G)$, we mean the subgroup of $Aut(G)$ consisting of all automorphisms which restrict to the identity on M. Let $Aut_M^M(G) = Aut^M(G) \cap Aut_M(G)$. From now onwards, M will be a characteristic central subgroup and elements of $Aut^M(G)$ will be called subcentral automorphisms of G (with respect to subcentral subgroup M). It can be seen that $Aut^M(G)$ is a normal subgroup of $Aut^Z(G)$. We further let A^* be the set $\{\alpha \in Aut_M(G) : \alpha\beta = \beta\alpha, \forall\ \beta \in Aut^M(G)\ \}$. Clearly A^* is a normal subgroup of Aut(G). Since every inner automorphism commutes with elements of $Aut^Z(G)$, Inn(G) $\leq A^*$. Let

$$P = \langle [g, \alpha] : g \in G, \alpha \in A^* \rangle, \text{ where } [g, \alpha] = g^{-1}\alpha(g).$$

It is easy to check that P is a characteristic subgroup of G.

We first state some results that will be used in the proof of the main theorems.

Proposition 2.1. *[5, Proposition2.1]. $Aut^M(G)$ acts trivially on P.*

Let A^{**} be any normal subgroup of Aut(G) contained in A^*, and $K = \langle [g,\alpha]$:g $\in G, \alpha \in A^{**}\rangle$. In particular, when $A^{**} = Inn(G)$, we get $K = G^{'}$. Since K is a subgroup of P, it is invariant under the action of $Aut^M(G)$. It is easy to see that K is a characteristic subgroup of G, and hence it is a normal subgroup of G.

Theorem 2.2. *[5, Theorem A]. For a finite group G, $Aut_M^M(G) \cong Hom(G/KM, M)$.*

Theorem 2.3. *[5, Theorem C]. If G is a purely non-abelian finite group, then $|Aut^M(G)| = |Hom(G,M)|$.*

Proposition 2.4. *[5, Proposition 2.4]. Let G be a purely non-abelian finite group, then for each $\alpha \in Hom(G,M)$ and each $x \in K$, we have $\alpha(x) = 1$. Further $Hom(G/K,M) \cong Hom(G,M)$.*

3 Main Results

We note that in this section M is a central characteristic subgroup.

Theorem 3.1. *Let G be a finite group. Then G/M is abelian if and only if $Inn(G) \leq Aut^M(G)$.*

Proof. Suppose G/M is abelian. Thus $G^{'} \leq M$. Let $x \in G$. Then for the inner automorphism θ_x induced by x and every $g \in G$ we have, $g^{-1}\theta_x(g) = [g,x] \in G^{'} \subseteq M$. So for every $\alpha \in Inn(G)$, $g^{-1}\alpha(g) \in M$. This means $Inn(G) \subseteq Aut^M(G)$. Hence $Inn(G) \leq Aut^M(G)$.
Conversely, suppose $Inn(G) \leq Aut^M(G)$. Hence it is clear $G^{'} \subseteq M$ and so G/M is abelian. □

Here we give a basic lemmas that will be used in the proof of the results.

Lemma 3.2. *[3, Lemma E]. Suppose H is an abelian p-group of exponent p^c, and K is cyclic group of order divisible by p^c. Then $Hom(H,K)$ is isomorphic to H.*

Lemma 3.3. *[3, Lemma H]. Let G be a purely non-abelian p-group, of nilpotent class 2. Then $|Hom(G/Z(G), G^{'})| \geq |G/Z(G)|p^{r(s-1)}$, where $r = rank(G/Z(G))$ and $s = rank(G^{'})$.*

Theorem 3.4. *Suppose G is a non-abelian finite p-group for which G/M is abelian. Then $|Aut^M(G) : Inn(G)| \geq p^{r(s-1)}$, where r and s are as defined before.*

Proof. By Theorem 2.3, $|Aut^M(G)| = |\text{Hom}(G,M)|$. Now we have

$$|\text{Hom}(G,M)| \geq |\text{Hom}(G/Z(G),M)| \geq |\text{ Hom}(G/Z(G),G^{'})| \geq |G/Z(G)|p^{r(s-1)}.$$

Hence $|Aut^M(G)| \geq |G/Z(G)|p^{r(s-1)}$, thus $|Aut^M(G) : Inn(G)| = |Aut^M(G)|/|G/Z(G)| \geq p^{r(s-1)}$. □

Find necessary and sufficient conditions on a finite p-group G such that $Aut_M^M(G) = Aut^M(G)$
.

Let G be a non-abelian finite p-group. Let

$$G/K = C_{p^{a_1}} \times C_{p^{a_2}} \times ... \times C_{p^{a_k}},$$

where $C_{p^{a_i}}$ is a cyclic group of order p^{a_i}, $1 \leq i \leq k$, and $a_1 \geq a_2 \geq ... \geq a_k \geq 1$. Let

$$G/KM = C_{p^{b_1}} \times C_{p^{b_2}} \times ... \times C_{p^{b_l}}$$

and

$$M = C_{p^{c_1}} \times C_{p^{c_2}} \times ... \times C_{p^{c_m}},$$

where $b_1 \geq b_2 \geq ... \geq b_l \geq 1$ and $c_1 \geq c_2 \geq ... \geq c_m \geq 1$.

Since G/KM is a quotient of G/K, by [2, Section 25], we have $l \leq k$ and $b_i \leq a_i$ for all $1 \leq i \leq l$.

Theorem 3.5. *Let G be a purely non-abelian finite p-group (p odd). Then $Aut_M^M(G) = Aut^M(G)$ if and only if $M \leq K$ or $M \leq \Phi(G)$, k= l and $c_1 \leq b_t$ where t is the largest integer between 1 and k such that $a_t > b_t$.*

Proof. Let $M \leq K$, by Proposition 2.1 and since $K \leq P$, every $\alpha \in Aut^M(G)$ fixes M and so $Aut^M(G) \leq Aut_M^M(G)$, since $Aut_M^M(G) \leq Aut^M(G)$. Thus $Aut_M^M(G) = Aut^M(G)$. Now suppose that $M \leq \Phi(G)$, $k = l$ and $c_1 \leq b_t$. Since G is purely non-abelian and by Theorem 2.3 and Proposition 2.4, we have

$$|Aut^M(G)| = |Hom(G, M)| = |Hom(G/K, M)| = \prod_{1\leq i\leq k, 1\leq j\leq m} p^{min\{a_i,c_j\}}$$

On the other hand, by Theorem 2.2, we have

$$|Aut_M^M(G)| = |Hom(G/KM, M)| = \prod_{1\leq i\leq l, 1\leq j\leq m} p^{min\{b_i,c_j\}}$$

Since $b_t \geq c_1$, we have

$$b_1 \geq b_2 \geq ... \geq b_{t-1} \geq b_t \geq c_1 \geq c_2 \geq ... \geq c_m \geq 1.$$

Therefore, $c_j \leq b_i \leq a_i$ for all $1 \leq j \leq m$ and $1 \leq i \leq t$, whence $\min\{a_i, c_j\} = c_j = \min\{b_i, c_j\}$ for all $1 \leq j \leq m$ and $1 \leq i \leq t$. Since $a_i = b_i$ for all $i > t$, we have $\min\{a_i, c_j\} = \min\{b_i, c_j\}$ for all $1 \leq j \leq m$ and $t + 1 \leq i \leq k$. Thus $\min\{b_i, c_j\} = \min\{a_i, c_j\}$ for all $1 \leq j \leq m$ and $1 \leq i \leq k$. Therefore, $Aut_M^M(G) = Aut^M(G)$.
Conversely if $Aut_M^M(G) = Aut^M(G)$ and $M \nleq K$. We claim that $M \leq \Phi(G)$. Assume contrarily that M is not contained in $\Phi(G)$. Then there exists a maximal subgroup D of G such that $M \nleq D$. The maximality of D implies that $G = DM$ and $D \leq G$. Hence we assume that $|G/D| = p$, where p is a prime number. Now we consider the following two cases.

Case 1: $p||M \cap D|$. Choose $z \in M \cap D$ such that $o(z) = p$ and fix $g \in M\backslash D$. It is clear that $G = D < g >$. Then the map α defined on G by $\alpha(dg^i) = dg^iz^i$ for every $d \in D$ and every $i \in \{0, 1, 2, ..., p - 1\}$, $\alpha \in Aut^M(G)$. By the given hypothesis $g = \alpha(g) = gz$, whence $z = 1$, which is a contradiction. Hence $M \leq \Phi(G)$.

Case 2: $p \nmid |M \cap D|$. In this case, since

$$p = |G/D| = |DM/D| = |M/M \cap D|,$$

we see that p divides $|M|$ and we may choose $z \in M$ such that $o(z) = p$ and $z \notin D$. Hence $G = \langle D, z\rangle = D \times \langle z\rangle$. Consider the map $\alpha : G \to G$ where $\alpha(dz^i) = dz^{2i}$ for every $d \in D$ and every $i \in \{0, 1, 2, ..., p - 1\}$, then $\alpha \in Aut^M(G)$. By the given hypothesis and since $z \in M$ it is clear that $z = \alpha(z) = \alpha(1.z^1) = z^2$, a contradiction. The proof of the theorem is complete. □

Lemma 3.6. *Let G be a finite p-group, then $Aut_M^M(G) \cong Hom(G/M, M)$.*

Proof. Consider the map $\phi : Aut_M^M(G) \longrightarrow$ Hom$(G/M, M)$ defined by $\phi(\alpha) = \alpha^*$, where $\alpha^*(gM) = g^{-1}\alpha(g)$ for all $g \in G$. It is clear that α^* is a well defined homomorphism. We show that ϕ is an isomorphism. If $\alpha_1, \alpha_2 \in Aut_M^M(G)$ and $\alpha_1 = \alpha_2$, then $g^{-1}\alpha_1(g) = g^{-1}\alpha_2(g)$, thus $\alpha_1^*(gM) = \alpha_2^*(gM)$, hence $\alpha_1^* = \alpha_2^*$, therefore $\phi(\alpha_1) = \phi(\alpha_2)$, so ϕ is a well defined homomorphism. It is easy to check that ϕ is a monomorphism. We show that ϕ is onto. For given any $f \in$ Hom$(G/M, M)$, define $\alpha : G \longrightarrow G$ by $\alpha(g) = gf(gM)$. Clearly $\alpha \in Aut(G)$. We show that $\alpha \in Aut_M^M(G)$. Since $g^{-1}\alpha(g) = f(gM) \in M$, then $\alpha \in Aut^M(G)$. Also for each $m \in M$, we have $\alpha(m) = mf(mM) = m$, thus $\alpha \in Aut_M(G)$. So $\alpha \in Aut_M^M(G)$ and $\phi(\alpha) = f$. Therefore ϕ is an isomorphism and $Aut_M^M(G) \cong$ Hom $(G/M, M)$. □

Theorem 3.7. *Let G be a finite p-group such that G/M is abelian. Then the following are equivalent:*
(1) $Aut_M^M(G) = Inn(G)$.
(2) $Hom(G/M, M) \cong G/Z(G)$.
(3) *M is cyclic and* $Hom(G/M, M) \cong Hom(G/Z(G), M)$.

Proof. (1) $\Rightarrow$ (2) By Lemma 3.6, and since $Aut_M^M(G) = Inn(G)$ we have, $Hom(G/M, M) \cong Inn(G) \cong G/Z(G)$.
(2) $\Rightarrow$ (1) By Lemma 3.6, and since $Hom(G/M, M) \cong G/Z(G)$, thus $Aut_M^M(G) \cong Inn(G)$, also since G/M is abelian we have $G^{'} \leq M$ and so $Inn(G) \leq Aut^M(G)$, also for every $\alpha \in Inn(G)$ and $m \in M$, we have $\alpha(m) = m$, therefore $Inn(G) \leq Aut_M^M(G)$ and so $Aut_M^M(G) = Inn(G)$.
(1) $\Rightarrow$ (3) Since $Aut_M^M(G) = Inn(G)$, every $f \in Aut_M^M(G)$ is an inner one, and so fixes each element of Z(G). Therefore, for every $f \in Aut_M^M(G)$, the map $\sigma_f : G/Z(G) \longrightarrow M$ defined by $\sigma_f(gZ(G)) = g^{-1}f(g)$ is well defined. Now consider the map $\sigma : f \longmapsto \sigma_f$. It is easy to check that σ is an isomorphism from $Aut_M^M(G)$ onto $Hom(G/Z(G), M)$. Thus $Hom(G/Z(G), M) \cong G/Z(G)$. Now we show that M is cyclic. Assume contrarily that M isnot cyclic and exp(M)= p^e, then $M = C_{p^e} \times N$ where C_{p^e} is cyclic subgroup of M and N is a nontrivial proper subgroup of M. We have $|G/Z(G)| = |Hom(G/Z(G), M)| = |Hom(G/Z(G), C_{p^e} \times N)| = |Hom(G/Z(G), C_{p^e})||Hom(G/Z(G), N)|$. Since G/M is abelian we have $G^{'} \leq M \leq Z(G)$, so G is nilpotent of class 2, and exp($G/Z(G)$) =exp($G^{'}$). Now by Lemma 3.2, $|Hom(G/Z(G), C_{p^e})| = |G/Z(G)|$. Therefore

$$|G/Z(G)| = |Hom(G/Z(G), M)| = |G/Z(G)||Hom(G/Z(G), N)|,$$

which is a contradiction. Hence M is cyclic.
(3) $\Rightarrow$ (1)Since M is cyclic and $G/Z(G)$ is an abelian p-group of exponent $|G^{'}|$ and $G^{'}$ is cyclic, it follows that $Hom(G/Z(G), M) \cong G/Z(G)$. By Lemma 3.6, $Aut_M^M(G) \cong Hom(G/M, M)$ and also $Inn(G) \leq Aut_M^M(G)$.Thus $Aut_M^M(G) = Inn(G)$. □

References

[1] J. E. Adney and T. Yen , Automorphisms of a p-group, *Illinois J. Math.* **9**, 137–143 (1965).

[2] R. D. Carmichael, Groups of Finite Order, *Dover Publications, New York* (1965).

[3] M. J. Curran and D. J. McCaughan, Central automorphisms that are almost inner, *Comm. Algebra.* **29**, 2081–2087 (2001).

[4] M. J. Curran and D. J. McCaughan, Finite groups with central automorphism group of minimal order, *Math. Proc. R. Ir. Acad.* **104**, 223–229 (2004).

[5] R. G. Ghumde and S. H. Ghate, Group of automorphisms preserving cosets of a central charactristic subgroup and related results, *Acta Math. Univ. Comenianae.* **2**, 181–189 (2016).

[6] M. Hangyang and G. Xiuyun, The absolute center of finite groups, *J. Group Theory.* **18**, 887–904 (2015).

[7] Z. Kaboutari Farimani and M. M. Nasrabadi, On absolute central automorphisms fixing the center elementwise, *Journal of Algebraic Systems.* **2**, 127–131 (2016).

[8] M. M. Nasrabadi and Z. Kaboutary Farimani, Absolute central automorphism that are inner, *Indagationes Mathematica.* **26**, 137–141 (2015).

[9] M. Shabani Attar, Semicomplete Finite p-Groups, *Algebra Colloquium.* **18**, 937–944 (1974).

[10] M. Shabani Attar, finite p-groups in which each central automorphism fixes centre elementwise, *Communi. cations in Algebra.* **40** , 1096–1102 (2012).

Author information

Mohammad Mehdi Nasrabadi and Parisa Seifizadeh, Department of Mathematics, University of Birjand, South Khorasan, Birjand, Iran.
E-mail: (Corresponding Author) paris.seifizadeh@birjand.ac.ir

Received: November 11, 2018.
Accepted: January 30, 2019.