Palestine Journal of Mathematics

Vol. 9(1)(2020) , 561–568

$\mathcal{I}_2$-Cesàro Summability of Double Sequences of Sets

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 40A05; Secondary 40A35.

Keywords and phrases: Cesàro summability, statistical convergence, lacunary sequence, $\mathcal{I}_2$-convergence, double sequences of sets, Wijsman convergence.

Abstract. In this paper, we defined concept of Wijsman $\mathcal{I}_2$-Cesàro summability and investigate the relationships between the concepts of Wijsman strongly $\mathcal{I}_2$-Cesàro summability, Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman p-strongly $\mathcal{I}_2$-Cesàro summability and Wijsman $\mathcal{I}_2$-statistical convergence of double sequences of sets.

1 Introduction

The concept of convergence of sequences of real numbers $\mathbb{R}$ has been extended to statistical convergence independently by Fast [10] and Schoenberg [20]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence.

Freedman et al. [11] established the connection between the strongly Cesàro summable sequences space and the strongly lacunary summable sequences space. Connor [5] gave the relationships between the concepts of strongly p-Cesàro convergence and statistical convergence of sequences.

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [2, 3, 4, 24, 25]). Nuray and Rhoades [15] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [21] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Also, Ulusu and Nuray [22] introduced the concept of Wijsman strongly lacunary summability of sequences of sets.

Kişi and Nuray [13] introduced a new convergence notion for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence by using ideal. Recently, Ulusu and Kişi [23] studied concept of Wijsman $\mathcal{I}$-Cesàro summability for sequences of sets. Nuray et al. [16] studied the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-Cauchy double sequences of sets. Also, Nuray et al. [17] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigate the relationships between them.

2 Definitions and Notations

Now, we recall the basic definitions and concepts (see, [1, 2, 7, 6, 8, 9, 12, 14, 17, 16, 18, 19, 23]).

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper, we take (X, ρ) be a separable metric space and A, A_{kj} be non-empty closed subsets of X.

A double sequence $\{A_{kj}\}$ is said to be bounded if for each $x \in X$,

$$\sup_{k,j} |d(x, A_{kj})| < \infty.$$

The set of all bounded set sequences is denoted by L_∞.

A double sequence $\{A_{kj}\}$ is said to be Wijsman convergent to A if for each $x \in X$,

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \quad \text{or} \quad \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A).$$

It is denoted by $W_2 - \lim A_{kj} = A$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman Cesàro summable to A if $\big(d(x, A_{kj})\big)$ is Cesàro summable to $d(x, A)$, that is, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj}) = d(x, A).$$

It is denoted by $A_{kj} \stackrel{(W_2\sigma_1)}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman strongly Cesàro summable to A if $\big(d(x, A_{kj})\big)$ is strongly Cesàro summable to $d(x, A)$, that is, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| = 0.$$

It is denoted by $A_{kj} \stackrel{[W_2\sigma_1]}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman strongly p-Cesàro summable to A if $\big(d(x, A_{kj})\big)$ is strongly p-Cesàro summable to $d(x, A)$, that is, for each p positive real number and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)|^p = 0.$$

It is denoted by $A_{kj} \stackrel{[W_2\sigma_p]}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman statistically convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \Big| \big\{ k \le m, j \le n : |d(x, A_{kj}) - d(x, A)| \ge \varepsilon \big\} \Big| = 0,$$

that is,

$$|d(x, A_{kj}) - d(x, A)| < \varepsilon, \quad \text{a.a. } (k, j).$$

It is denoted by $st_2 - \lim_W A_{kj} = A$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:
(i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a non-trivial ideal if $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Let $X \neq \emptyset$. A non-empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:
(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}$, $B \supset A$ implies $B \in \mathcal{F}$.

Lemma 2.1 ([14]). *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \big\{ M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A) \big\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Throughout the paper we take $\mathcal{I}_2$ as an admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-Cesàro summable to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\left\{n \in \mathbb{N} : \Big|\frac{1}{n}\sum_{k=1}^{n} d(x, A_k) - d(x, A)\Big| \geq \varepsilon\right\} \in \mathcal{I}.$$

It is denoted by $A_k \overset{C_1(\mathcal{I}_W)}{\longrightarrow} A$.

A sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}$-Cesàro summable to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} |d(x, A_k) - d(x, A)| \geq \varepsilon\right\} \in \mathcal{I}.$$

It is denoted by $A_k \overset{C_1[\mathcal{I}_W]}{\longrightarrow} A$.

A sequence $\{A_k\}$ is said to be Wijsman p-strongly $\mathcal{I}$-Cesàro summable to A if for every $\varepsilon > 0$, each p positive real number and each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p \geq \varepsilon\right\} \in \mathcal{I}.$$

It is denoted by $A_k \overset{C_p[\mathcal{I}_W]}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is said to be $\mathcal{I}_{W_2}$-convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\big\{(k, j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\big\} \in \mathcal{I}_2.$$

It is denoted by $\mathcal{I}_{W_2} - \lim A_{kj} = A$.

A double sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-statistical convergent to A or $S\left(\mathcal{I}_{W_2}\right)$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and each $x \in X$,

$$\left\{(k, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\Big|\big\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\big\}\Big| \geq \delta\right\} \in \mathcal{I}_2.$$

It is denoted by $A_{kj} \to A\big(S\left(\mathcal{I}_{W_2}\right)\big)$.

A double sequence $\theta = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \ \text{ as } \ r \to \infty \ \text{ and } \ j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \ \text{ as } \ u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \quad \text{and} \quad j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \quad \text{and} \quad q_u = \frac{j_u}{j_{u-1}}.$$

A double sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta\left[\mathcal{I}_{W_2}\right]$-convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$A(\varepsilon, x) = \left\{(r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u}\sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

It is denoted by $A_{kj} \to A\left(N_\theta\left[\mathcal{I}_{W_2}\right]\right)$.

3 Main Results

In this section, we defined concepts of Wijsman $\mathcal{I}_2$-Cesàro summability, Wijsman strongly $\mathcal{I}_2$-Cesàro summability and Wijsman p-strongly $\mathcal{I}_2$-Cesàro summability for double sequences of sets. Also, we investigate the relationships between the concepts of Wijsman strongly $\mathcal{I}_2$-Cesàro summability, Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman p-strongly $\mathcal{I}_2$-Cesàro summability and Wijsman $\mathcal{I}_2$-statistical convergence of double sequences of sets.

Definition 3.1. A double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \Big| \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj}) - d(x, A) \Big| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{C_1(\mathcal{I}_{W_2})}{\longrightarrow} A$.

Definition 3.2. A double sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{C_1[\mathcal{I}_{W_2}]}{\longrightarrow} A$.

Theorem 3.3. *Let* $\theta = \{(k_r, j_u)\}$ *be a double lacunary sequence. If* $\liminf_r q_r > 1$, $\liminf_u q_u > 1$, *then*

$$A_{kj} \overset{C_1[\mathcal{I}_{W_2}]}{\longrightarrow} A \Rightarrow A_{kj} \overset{N_\theta[\mathcal{I}_{W_2}]}{\longrightarrow} A.$$

Proof. Let $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$. Then, there exist $\lambda, \mu > 0$ such that $q_r \geq 1 + \lambda$ and $q_u \geq 1 + \mu$ for all $r, u \geq 1$, which implies that

$$\frac{k_r j_u}{h_r \bar{h}_u} \leq \frac{(1+\lambda)(1+\mu)}{\lambda\mu} \quad \text{and} \quad \frac{k_{r-1} j_{u-1}}{h_r \bar{h}_u} \leq \frac{1}{\lambda\mu}.$$

Let $\varepsilon > 0$ and for each $x \in X$ we define the set

$$S = \left\{(k_r, j_u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x, A_{is}) - d(x, A)| < \varepsilon \right\}.$$

We can easily say that $S \in \mathcal{F}(\mathcal{I}_2)$, which is a filter of the ideal $\mathcal{I}_2$. Then, we have

$$\begin{aligned}
\tfrac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| &= \tfrac{1}{h_r \bar{h}_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x, A_{is}) - d(x, A)| \\
&\quad - \tfrac{1}{h_r \bar{h}_u} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x, A_{is}) - d(x, A)| \\
&= \tfrac{k_r j_u}{h_r \bar{h}_u} \cdot \left(\tfrac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x, A_{is}) - d(x, A)| \right) \\
&\quad - \tfrac{k_{r-1} j_{u-1}}{h_r \bar{h}_u} \cdot \left(\tfrac{1}{k_{r-1} j_{u-1}} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x, A_{is}) - d(x, A)| \right) \\
&\leq \left(\tfrac{(1+\lambda)(1+\mu)}{\lambda\mu} \right) \varepsilon - \left(\tfrac{1}{\lambda\mu} \right) \varepsilon'
\end{aligned}$$

for each $x \in X$ and each $(k_r, j_u) \in S$.

Choose $\eta = \left(\frac{(1+\lambda)(1+\mu)}{\lambda\mu}\right)\varepsilon + \left(\frac{1}{\lambda\mu}\right)\varepsilon'$. Therefore,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r\overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \eta\right\} \in \mathcal{F}(\mathcal{I}_2)$$

and it completes the proof. □

Theorem 3.4. *Let θ be a double lacunary sequence. If* $\limsup_r q_r < \infty$, $\limsup_u q_u < \infty$ *then,*

$$A_{kj} \overset{N_\theta[\mathcal{I}_{W_2}]}{\longrightarrow} A \Rightarrow A_{kj} \overset{C_1[\mathcal{I}_{W_2}]}{\longrightarrow} A.$$

Proof. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there exists $M, N > 0$ such that $q_r < M$ and $q_u < N$ for all $r, u \geq 1$. Let $A_{kj} \overset{N_\theta[\mathcal{I}_{W_2}]}{\longrightarrow} A$ and for $\varepsilon_1, \varepsilon_2 > 0$ we define the sets T and R such that

$$T = \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r\overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \varepsilon_1\right\}$$

and

$$R = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| < \varepsilon_2\right\},$$

for each $x \in X$. Let

$$a_{tv} = \frac{1}{h_t\overline{h}_v} \sum_{(i,s)\in I_{tv}} |d(x, A_{is}) - d(x, A)| < \varepsilon_1$$

for all $(t, v) \in T$. It is obvious that $T \in \mathcal{F}(\mathcal{I}_2)$. Choose m, n is any integer with $k_{r-1} < m < k_r$ and $j_{u-1} < n < j_u$, where $(r, u) \in T$. Then, for each $x \in X$ we have

$$\begin{aligned}
\frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| \;&\leq\; \frac{1}{k_{r-1}j_{u-1}} \sum_{i,s=1,1}^{k_r,j_u} |d(x, A_{is}) - d(x, A)| \\
&= \frac{1}{k_{r-1}j_{u-1}} \left(\sum_{(i,s)\in I_{11}} |d(x, A_{is}) - d(x, A)| \right. \\
&\quad + \sum_{(i,s)\in I_{12}} |d(x, A_{is}) - d(x, A)| \\
&\quad + \sum_{(i,s)\in I_{21}} |d(x, A_{is}) - d(x, A)| \\
&\quad + \sum_{(i,s)\in I_{22}} |d(x, A_{is}) - d(x, A)| \\
&\quad \left. + \cdots + \sum_{(i,s)\in I_{ru}} |d(x, A_{is}) - d(x, A)| \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k_1 j_1}{k_{r-1} j_{u-1}} \left(\frac{1}{h_1 \bar{h}_1} \sum_{(i,s)\in I_{11}} |d(x, A_{is}) - d(x, A)| \right) \\
&\quad + \frac{k_1 (j_2 - j_1)}{k_{r-1} j_{u-1}} \left(\frac{1}{h_1 \bar{h}_2} \sum_{(i,s)\in I_{12}} |d(x, A_{is}) - d(x, A)| \right) \\
&\quad + \frac{(k_2 - k_1) j_1}{k_{r-1} j_{u-1}} \left(\frac{1}{h_2 \bar{h}_1} \sum_{(i,s)\in I_{21}} |d(x, A_{is}) - d(x, A)| \right) \\
&\quad + \frac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1} j_{u-1}} \left(\frac{1}{h_2 \bar{h}_2} \sum_{(i,s)\in I_{22}} |d(x, A_{is}) - d(x, A)| \right) \\
&\quad + \cdots + \frac{(k_r - k_{r-1})(j_u - j_{u-1})}{k_{r-1} j_{u-1}} \left(\frac{1}{h_r \bar{h}_u} \sum_{(i,s)\in I_{ru}} |d(x, A_{is}) - d(x, A)| \right) \\
&= \frac{k_1 j_1}{k_{r-1} j_{u-1}} a_{11} + \frac{k_1 (j_2 - j_1)}{k_{r-1} j_{u-1}} a_{12} + \frac{(k_2 - k_1) j_1}{k_{r-1} j_{u-1}} a_{21} + \frac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1} j_{u-1}} a_{22} \\
&\quad + \cdots + \frac{(k_r - k_{r-1})(j_u - j_{u-1})}{k_{r-1} j_{u-1}} a_{ru} \\
&\leq \left(\sup_{(t,v)\in T} a_{tv} \right) \frac{k_r j_u}{k_{r-1} j_{u-1}} \\
&< \varepsilon_1 \cdot M \cdot N.
\end{aligned}
$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{M \cdot N}$ and in view of the fact that

$$\bigcup \big\{ (m,n) : k_{r-1} < m < k_r, j_{u-1} < n < j_u, \, (r,u) \in T \big\} \subset R,$$

where $T \in \mathcal{F}(\mathcal{I}_2)$, it follows from our assumption on $\theta = \{(k_r, j_u)\}$ that the set R also belongs to $F(\mathcal{I}_2)$ and this completes the proof of the theorem. □

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let $\theta = \{(k_r, j_u)\}$ be a double lacunary sequence. If*

$$1 < \liminf_r q_r < \limsup_r q_r < \infty \quad \text{and} \quad 1 < \liminf_u q_u < \limsup_u q_u < \infty,$$

then

$$A_{kj} \overset{C_1[\mathcal{I}_{W_2}]}{\longrightarrow} A \Leftrightarrow A_{kj} \overset{N_\theta[\mathcal{I}_{W_2}]}{\longrightarrow} A.$$

Definition 3.6. A double sequence $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$, each p positive real number and each $x \in X$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)|^p \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{C_p[\mathcal{I}_{W_2}]}{\longrightarrow} A$.

Theorem 3.7. *If a double sequence $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A, then the double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A.*

Proof. Let $A_{kj} \overset{C_p[\mathcal{I}_{W_2}]}{\longrightarrow} A$ and $\varepsilon > 0$ be given. Then, for each $x \in X$ we have

$$\begin{aligned}
\sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)|^p &\geq \sum_{\substack{k,j=1,1 \\ \left|d(x,A_{kj})-d(x,A)\right| \geq \varepsilon}}^{m,n} |d(x,A_{kj}) - d(x,A)|^p \\
&\geq \varepsilon^p . \left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right|
\end{aligned}$$

and so

$$\frac{1}{\varepsilon^p \cdot mn} \sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)|^p \geq \frac{1}{mn} \left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| .$$

Hence, for a given $\delta > 0$ and each $x \in X$,

$$\begin{aligned}
&\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\} \right| \geq \delta \right\} \\
&\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)|^p \geq \varepsilon^p \cdot \delta \right\} \in \mathcal{I}_2.
\end{aligned}$$

Therefore, $A_k \overset{S(\mathcal{I}_{W_2})}{\longrightarrow} A$. □

Theorem 3.8. *Let $\{A_{kj}\} \in L_\infty$. If a double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A, then the double sequence $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A.*

Proof. Suppose that $\{A_{kj}\}$ is bounded and $A_{kj} \overset{S(\mathcal{I}_{W_2})}{\longrightarrow} A$. Then, there is a $M > 0$ such that

$$|d(x,A_{kj}) - d(x,A)| \leq M$$

for each $x \in X$ and all k, j. Hence, for given $\varepsilon > 0$ and each $x \in X$ we have

$$\begin{aligned}
\frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)|^p &= \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ \left|d(x,A_{kj})-d(x,A)\right| \geq \varepsilon}}^{m,n} |d(x,A_{kj}) - d(x,A)|^p \\
&\quad + \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ \left|d(x,A_{kj})-d(x,A)\right| < \varepsilon}}^{m,n} |d(x,A_{kj}) - d(x,A)|^p \\
&\leq \frac{1}{mn} M^p \cdot \left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| \\
&\quad + \frac{1}{mn} \varepsilon^p \cdot \left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| < \varepsilon\right\}\right| \\
&\leq \frac{M^p}{mn} \left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| + \varepsilon^p.
\end{aligned}$$

Then, for any $\delta > 0$ and each $x \in X$,

$$\begin{aligned}
&\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)|^p \geq \delta \right\} \\
&\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| \geq \frac{\delta}{M^p} \right\} \in \mathcal{I}_2.
\end{aligned}$$

Therefore, $A_k \overset{C_p[\mathcal{I}_{W_2}]}{\longrightarrow} A$. □

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Received: October 21, 2017.
Accepted: March 21, 2018