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# Oscillatory behavior of fourth-order differential equations with delay argument

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**Abstract**. This paper deals with the oscillation of the fourth-order delay differential equation. New oscillation criteria are obtained by employing a refinement of the generalized Riccati transformations and new comparison principles .An example is included to illustrate the main results.

## 1 Introduction

In this work, we are concerned with the oscillation and the asymptotic behavior of solutions of the fourth order nonlinear differential equations with delayed argument

$$\left[b\left(t\right)\left(x'''\left(t\right)\right)^{\gamma}\right]'+q\left(t\right)f\left(x\left(\tau\left(t\right)\right)\right)=0,\quad t\geq t_0. \tag{1.1}$$

We assume that $\gamma$ is a quotient of odd positive integers, $b\in C^1[t_0,\,\infty)$, $b'\left(t\right)\geq 0,\ b\left(t\right)>0$, $q,\,\tau\in C[t_0,\,\infty)$, $f\in C\left(\mathbb{R},\,\mathbb{R}\right),\ q>0$, $\tau\left(t\right)\leq t$, $\lim\limits_{t\to\infty}\tau\left(t\right)=\infty$, there exist constants $k>0$ such that $f\left(u\right)/u^{\gamma}\geq k$, for $u\neq 0$ and under the condition

$$\int_{t_0}^{\infty}\frac{1}{b^{\frac{1}{\gamma}}\left(t\right)}d\acute{t}<\infty. \tag{1.2}$$

By a solution of (1.1) we mean a function $x\in C'''[T_x,\,\infty)$, $T_x\geq t_\circ$, which has the property $b\left(t\right)\left[x'''\left(t\right)\right]^{\gamma}\in C^1[T_x,\,\infty)$, and satisfies (1.1) on $[T_x,\,\infty)$.We consider only those solutions $x$ of (1.1) which satisfy $\sup\{|x\left(t\right)|:t\geq T\}>0$, for all $T\ >T_x$. We assume that (1.1) possesses such a solution. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\,\infty)$, and other wiseit is called to be nonoscillatory. (1.1) is said to be oscillatory if all its solutions are oscillatory.

As is well known, the fourth-order differential equations are derived from many different areas of applied mathematics and physics, for instance, deflection of buckling beam with a fixed or variable cross-section, three-layer beam, electromagnetic waves, gravity-driven flows, etc. In recent years, the oscillation theory of fourth-order differential equations has received a great deal of attention since it has been widely applied in research of physical sciences, mechanics, radio technology, lossless high-speed computer network, control system, life sciences, and population growth.

The oscillations of fourth-order differential equations have been studied by several authors and several techniques have been proposed for obtaining oscillatory criteria for higher and fourth order differential equations. For treatments on this subject, we refer the reader to the texts [[2, 6, 18], [14]-[16]] and the articles [[1], [3]-[13], [17]-[25]]. In what follows, we review some results that have provided the background and the motivation, for the present work.

Zhang, et al.[23] consider the oscillation of a fourth-order quasilinear delay differential equation

$$\left[b\left(t\right)\left(x'''\left(t\right)\right)^{\gamma}\right]'+q\left(t\right)x^{\gamma}\left(\tau\left(t\right)\right)=0,\quad t\geq t_0.$$

Grace et al.[13] studied the oscillation behavior of the fourth-order nonlinear differential equation

$$\left[r(t)\left(x'(t)\right)^{\gamma}\right]''' + q(t) f(x(g(t))) = 0, \qquad t \geq t_0.$$

Moaaz et al. [18] considered with the oscillatory behavior of solutions of non-linear fourth order differential equations of the type

$$\left(r(t)\left(x'''(t)\right)^{\alpha}\right)' + \int_a^b q(t,\xi) f(x(g(t,\xi))) d\sigma(\xi) = 0,$$

Chatzarakis et al. [11] studied the oscillation behavior of the fourth-order differential equation

$$\left[r(t)\left(\left[x(t) + p(t) x(\tau(t))\right]'''\right)^{\alpha}\right]' + \int_a^b q(t,\xi) f(x(g(t,\xi))) d\xi = 0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt = \infty.$$

Agarwal et al.[3] and Zhang et al.[24] consider the oscillatory properties of the higher-order differential equation

$$\left[b(t)\left(x^{(n-1)}(t)\right)^{\gamma}\right]' + q(t) x^{\gamma}(\tau(t)) = 0, \qquad t \geq t_0.$$

Our aim in the present paper is to employ the Riccatti technique to establish some new conditions for the oscillation of all solutions of (1.1). The results obtained in the paper a generalized some results from [23]. One example are presented to illustrate our main results.

The proof of our main results are essentially based on the following lemmas.

**Lemma 1.1.** *Let* $z \in (C^n[t_0, \infty], \mathbb{R}^+)$ *and assume that* $z^{(n)}$ *is of fixed sign and not identically zero on a subray of* $[t_0, \infty]$. *If moreover,* $z(t) > 0$, $z^{(n-1)}(t) z^{(n)}(t) \leq 0$ *and* $\lim_{t\to\infty} z(t) \neq 0$, *then, for every* $\lambda \in (0,1)$, *there exists* $t_\lambda \geq t_\circ$ *such that*

$$z(t) \geq \frac{\lambda}{(n-1)} t^{n-1} \left|z^{(n-1)}(t)\right|, \text{ for } t \in [t_\lambda, \infty).$$

**Lemma 1.2.** *Let* $\gamma \geq 1$ *be a ratio of two numbers, where* $C$ *and* $D$ *are constants. Then*

$$Cy - Dy^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{C^{\gamma+1}}{D^{\gamma}}, \ D > 0.$$

**Lemma 1.3.** *If the function* $z$ *satisfies* $z^{(i)} > 0$, $i = 0, 1, ..., n$, *and* $z^{(n+1)} < 0$, *then*

$$\frac{z(t)}{t^n/n!} \geq \frac{z'(t)}{t^{n-1}/(n-1)!}.$$

## 2 MAIN RESULTS

In this section, we shall establish some oscillation criteria for equation (1.1). We are now ready to state and prove the main results. For convenience, we denote

$$\pi(s) := \int_{t_0}^{\infty} \frac{1}{b(s)} ds, \ \delta'_+(t) := \max\{0, \ \delta'(t)\} \text{ and } \rho'_+(t) := \max\{0, \ \rho'(t)\}.$$

**Theorem 2.1.** *Let (1.2) hold. Assume that there exists a positive function* $\delta \in C^1[t_0, \infty)$ *such that*

$$\int_{t_0}^{\infty} \left[kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^{\gamma} \delta(s) - \frac{2^{\gamma} b(s)\left(\delta'(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\lambda_1 \delta(s) s^2\right)^{\gamma}}\right] ds = \infty, \tag{2.1}$$

*for some constant* $\lambda_1 \in (0,\ 1)$ *. Assume further that there exists a positive function* $\rho \in C^1[t_0,\ \infty)$ *such that*

$$\int_{t_0}^{\infty}\left[\rho(s)\int_s^{\infty}\left[\frac{1}{b(\vartheta)}\int_{\varpi}^{\infty}kq(\nu)^{\gamma}\,d\nu\right]^{\frac{1}{\gamma}}d\vartheta-\frac{(\rho'(s))^2}{4\rho(s)}\right]ds=\infty. \tag{2.2}$$

*If*

$$\int_{t_0}^{\infty}\left[kq(s)\left(\int_s^{\infty}\int_u^{\infty}\pi(\chi)\,d\chi du\right)^{\gamma}-\frac{\gamma^{\gamma+1}\int_s^{\infty}\pi(\chi)d\chi}{(\gamma+1)^{\gamma+1}\int_s^{\infty}\int_u^{\infty}\pi(\chi)d\chi du}\right]ds=\infty, \tag{2.3}$$

*and*

$$\int_{t_0}^{\infty}\left[kq(s)\left(\frac{\lambda_2}{2}\tau^2(s)\right)^{\gamma}\pi^{\gamma}(s)-\frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}\,\pi(s)\,b^{\frac{1}{\gamma}}(s)}\right]ds=\infty, \tag{2.4}$$

*for some constant* $\lambda_2 \in (0,\ 1)$ *, then every solution of (1.1) is oscillatory.*

*Proof.* Assume that (1.1) has a nonoscillatory solution $x$. Without loss of generality, we can assume that $x(t) > 0$. It follows from (1.1) that there exist four possible cases for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large:

Case 1 : $x'(t) > 0, x''(t) > 0, x'''(t) > 0, x^{(4)}(t) \leq 0, (b(x''')^{\gamma})'(t) \leq 0.$

Case 2 : $x'(t) > 0, x''(t) < 0, x'''(t) > 0, x^{(4)}(t) \leq 0, (b(x''')^{\gamma})'(t) \leq 0.$

Case 3 : $x'(t) < 0, x''(t) > 0, x'''(t) < 0, (b(x''')^{\gamma})'(t) \leq 0.$

Case 4 : $x'(t) > 0, x''(t) > 0, x'''(t) < 0, (b(x''')^{\gamma})'(t) \leq 0.$

Assume that we have Case1 .Define.

$$\omega(t) := \delta(t)\frac{b(t)(x''')^{\gamma}(t)}{x^{\gamma}(t)}. \tag{2.5}$$

Then $\omega(t) > 0$ for $t \geq t_1$ and

$$\omega'(t)=\delta'(t)\frac{b(t)(x''')^{\gamma}(t)}{x^{\gamma}(t)}+\delta(t)\frac{(b(x''')^{\gamma})'(t)}{x^{\gamma}(t)}$$

$$-\gamma\delta(t)\frac{x^{\gamma-1}(t)(x)'(t)\,b(t)(x''')^{\gamma}(t)}{x^{2\gamma}(t)}, \tag{2.6}$$

from Lemma (1.1) that

$$x'(t) \geq \frac{\lambda}{2}t^2x'''(t), \tag{2.7}$$

for every $\lambda \in (0,\ 1)$ , and all sufficiently large $t$. By Lemma (1.3), we find $x(t) \geq (t/3)\,x'(t)$ and, hence

$$\frac{x(\tau_i(t))}{x(t)} \geq \frac{\tau^3(t)}{t^3}. \tag{2.8}$$

Hence, by (2.7) and (2.8), we obtain

$$\omega'(t) \leq \delta'(t)\frac{b(t)(x''')^{\gamma}(t)}{x^{\gamma}(t)}+\delta(t)\frac{(b(x''')^{\gamma})'(t)}{x^{\gamma}(t)}-\frac{\gamma\lambda}{2}t^2\delta(t)\frac{x'''(t)\,b(t)(x''')^{\gamma}(t)}{(x)^{\gamma+1}(t)}. \tag{2.9}$$

In view of (1.1), we get

$$\omega'(t) \leq -kq(t)\left(\frac{\tau^3(t)}{t^3}\right)^{\gamma}\delta(t)+\frac{\delta'(t)}{\delta(t)}\omega(t)-\frac{\gamma\lambda t^2}{2(b(t)\delta(t))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t). \tag{2.10}$$

Define now

$$C := \frac{\gamma\lambda t^2}{2(b(t)\delta(t))^{\frac{1}{\gamma}}},\ D := \frac{\delta'(t)}{\delta(t)},\ y := \omega(t).$$

Applying Lemma (1.2), we find

$$\frac{\delta'(t)}{\delta(t)}\omega(t)-\frac{\gamma\lambda t^2}{2(b(t)\delta(t))^{\frac{1}{\gamma}}}\omega(t)^{\frac{\gamma+1}{\gamma}} \leq \frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{b(t)(\delta'(t))^{\gamma+1}}{(\lambda\delta(t)t^2)^{\gamma}}. \tag{2.11}$$

Hence, we obtain

$$\omega'(t) \leq -kq(t)\left(\frac{\tau^3(t)}{t^3}\right)^\gamma \delta(t) + \frac{2^\gamma}{(\gamma+1)^{\gamma+1}} \frac{b(t)(\delta'(t))^{\gamma+1}}{(\lambda\delta(t)t^2)^\gamma}.$$

Integrating from $t_1$ to $t$, we get

$$\int_t^t \left[-kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma \delta(s) - \frac{2^\gamma b(s)(\delta'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda_1\delta(s)s^2)^\gamma}\right] ds \leq \omega(t_1),$$

for every $\lambda \in (0, 1)$, and all sufficiently large $t$ but this contradicts (2.1).

Assume that we have Case 2. Define

$$\psi(t) := \rho(t)\frac{x'(t)}{x(t)}, \ t \geq t_1. \tag{2.12}$$

Then $\psi(t) > 0$ for $t \geq t_1$ and

$$\psi'(t) = \rho'(t)\frac{x'(t)}{x(t)} - \rho(t)\frac{x''(t)x(t) - (x')^2(t)}{x^2(t)}, \tag{2.13}$$

$$\psi'(t) = \rho(t)\frac{x''(t)}{x(t)} + \frac{\rho'(t)}{\rho(t)}\psi(t) - \frac{\psi^2(t)}{\rho(t)}.$$

Integrating (1.1) from $t$ to $u$ we find

$$\begin{aligned} b(u)(x''')^\gamma(u) - b(t)(x''')^\gamma(t) + \int_t^u kq(s)x^\gamma(\tau_i(s))\,ds &\leq 0, \\ b(u)(x''')^\gamma(u) - b(t)(x''')^\gamma(t) + x^\gamma(t)\int_t^u kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma ds &\leq 0. \end{aligned}$$

Letting $u \to \infty$, we arrive at the inequality

$$x''(t) + x(t)\int_t^\infty \left[\frac{1}{b(\vartheta)}\int_y^\infty kq(t)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma ds\right]^{\frac{1}{\gamma}} d\vartheta \leq 0. \tag{2.14}$$

Hence, by (2.14) in (2.13), we find

$$\begin{aligned} \psi'(t) \leq\ & -\rho(t)\int_t^\infty \left[\frac{1}{b(\vartheta)}\int_\vartheta^\infty kq(t)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma ds\right]^{\frac{1}{\gamma}} d\vartheta \\ & + \frac{\rho'(t)}{\rho(t)}\psi(t) - \frac{\psi^2(t)}{\rho(t)}. \end{aligned}$$

Thus, we have

$$\psi'(t) \leq -\rho(t)\int_t^\infty \left[\frac{1}{b(\vartheta)}\int_y^\infty kq(s)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma ds\right]^{\frac{1}{\gamma}} d\vartheta + \frac{(\rho'(t))^2}{4\rho(t)}. \tag{2.15}$$

Integrating from $t_1$ to $t$, we get

$$\int_{t_1}^t \left[\rho(s)\int_s^\infty \left[\frac{1}{b(\vartheta)}\int_\varpi^\infty kq(\nu)\left(\frac{\tau^3(\nu)}{\nu^3}\right)^\gamma d\nu\right]^{\frac{1}{\gamma}} d\vartheta - \frac{(\rho'(s))^2}{4\rho(s)}\right] ds \leq \psi(t_1),$$

which contradicts (2.2).

Assume that we have Case 3. Recalling that $r\ (x''')^\gamma$ is nonincreasing, we obtain

$$\begin{aligned} b^{\frac{1}{\gamma}}(s)x'''(s) &\leq b^{\frac{1}{\gamma}}(t)x'''(t),\ s \geq t \geq t_1, \\ x'''(s) &\leq b^{\frac{1}{\gamma}}(t)x'''(t)b^{\frac{-1}{\gamma}}(s). \end{aligned}$$

Integrating again from $t$ to $v$, we get

$$x''(t) - x''(v) \geq -b^{\frac{1}{\gamma}}(t)\, x'''(t) \int_t^v b^{\frac{-1}{\gamma}}(s)\, ds.$$

Letting $v \to \infty$, we obtain

$$x''(t) \geq -b^{\frac{1}{\gamma}}(t)\, x'''(t)\, \pi(t). \tag{2.16}$$

Integrating from $t$ to $\infty$, we get

$$-x'(t) \geq -b^{\frac{1}{\gamma}}(t)\, x'''(t) \int_t^\infty \pi(s)\, ds. \tag{2.17}$$

Integrating again from $t$ to $\infty$, we get

$$x'(t) \geq -b^{\frac{1}{\gamma}}(t)\, x'''(t) \int_t^\infty \int_u^\infty \pi(s)\, dsdu. \tag{2.18}$$

We define

$$\xi(t) := \frac{b(t)\,(x''')^{\gamma}(t)}{x^{\gamma}(t)}. \tag{2.19}$$

Then $\xi(t) < 0$ for $t \geq t_1$ and by (2.19), we conclude that

$$\xi'(t) = \frac{\left(b\,(x''')^{\gamma}\right)'(t)}{x^{\gamma}(t)} - \gamma \frac{(x)'(t)\, b(t)\,(x''')^{\gamma}(t)}{x^{\gamma+1}(t)}.$$

$$\xi'(t) \leq -kq(t)\left(\frac{x(\tau(t))}{x(t)}\right)^{\gamma} - \gamma \frac{b^{\frac{\gamma+1}{\gamma}}(t)\,(x''')^{\gamma+1}(t)}{x^{\gamma+1}(t)} \int_t^\infty \pi(s)\, ds. \tag{2.20}$$

Hence, by (2.20) and (2.19), we obtain

$$\xi'(t) \leq -kq(t) - \gamma \xi^{\frac{\gamma+1}{\gamma}}(t) \int_t^\infty \pi(s)\, ds,$$

from (2.6), we get

$$\xi(t)\left(\int_t^\infty \int_u^\infty \pi(s)\, dsdu\right)^{\gamma} \geq -1. \tag{2.21}$$

Multiplying (2.20) by $\left(\int_t^\infty \int_u^\infty \pi(s)\, dsdu\right)^{\gamma}$ and integrating the resulting inequality from $t_1$ to $t$, we get

$$\begin{aligned}
&\left(\int_t^\infty \int_u^\infty \pi(s)\, dsdu\right)^{\gamma} \xi(t) - \left(\int_{t_1}^\infty \int_u^\infty \pi(s)\, dsdu\right)^{\gamma} \xi(t_1) \\
&+\gamma \int_{t_1}^t \int_s^\infty \pi(\chi)\, d\chi \left(\int_s^\infty \int_u^\infty \pi(\chi)\, d\chi du\right)^{\gamma-1} \xi(s)\, ds \\
&+\int_{t_1}^t kq(s)\left(\int_s^\infty \int_u^\infty \pi(\chi)\, d\chi du\right)^{\gamma} ds \\
&+\gamma \int_{t_1}^t \xi^{\frac{\gamma+1}{\gamma}}(s)\left(\int_s^\infty \int_u^\infty \pi(\chi)\, d\chi du\right)^{\gamma} \int_s^\infty \pi(\chi)\, d\chi ds \leq 0.
\end{aligned}$$

We set

$$D := \int_s^\infty \pi(\chi)\, d\chi \left(\int_s^\infty \int_u^\infty \pi(\chi)\, d\chi du\right)^{\gamma-1},$$

$$C := \left(\int_s^\infty \int_u^\infty \pi(\chi)\, d\chi du\right)^{\gamma} \int_s^\infty \pi(\chi)\, d\chi, \; y := -\xi(s).$$

From Lemma (1.2), we conclude that

$$\begin{aligned}
& \int_s^\infty \pi(\chi)\,d\chi \left(\int_s^\infty \int_u^\infty \pi(\chi)\,d\chi du\right)^{\gamma-1} \xi(s) \\
& +\xi^{\frac{\gamma+1}{\gamma}}(s)\left(\int_s^\infty \int_u^\infty \pi(\chi)\,d\chi du\right)^{\gamma} \int_s^\infty \pi(\chi)\,d\chi \\
\geq\ & \frac{\gamma^\gamma \int_s^\infty \pi(\chi)\,d\chi}{(\gamma+1)^{\gamma+1} \int_s^\infty \int_u^\infty \pi(\chi)\,d\chi du}.
\end{aligned}$$

Hence, by (2.21), and integrating the resulting inequality from $t_1$ to $t$, we obtain

$$\begin{aligned}
& \int_{t_1}^t \left[kq(s)\left(\int_s^\infty \int_u^\infty \pi(\chi)\,d\chi du\right)^{\gamma} - \frac{\gamma^{\gamma+1}\int_s^\infty \pi(\chi)d\chi}{(\gamma+1)^{\gamma+1}\int_s^\infty \int_u^\infty \pi(\chi)d\chi du}\right] ds \\
\geq\ & \left(\int_{t_1}^\infty \int_u^\infty \pi(s)\,dsdu\right)^{\gamma} \xi(t_1)+1,
\end{aligned}$$

which contradicts (2.3).

Assume that we have Case 4 . In view of the proof of Case 3, we have (2.6). On the other hand, by Lemma (1.1), we get

$$x(t) \geq \frac{\lambda}{2} t^{\,2} x''(t), \tag{2.22}$$

for every $\lambda \in (0,\ 1)$ and all sufficiently large $t$. We now define

$$\sigma(t) := \frac{b(t)\left(x'''\right)^\gamma(t)}{\left(x''\right)^\gamma(t)}. \tag{2.23}$$

Then $\sigma(t) < 0$ for $t \geq t_1$ and, by virtue of (2.22) and (2.23), we conclude that

$$\sigma'(t) = -kq(t)\frac{x^\gamma(\tau(t))}{\left(x''(\tau(t))\right)^\gamma}\frac{\left(x''(\tau(t))\right)^\gamma}{\left(x''\right)^\gamma(t)} - \gamma\frac{\sigma^{\frac{\gamma+1}{\gamma}}(t)}{b^{\frac{1}{\gamma}}(t)}.$$

$$\sigma'(t) \leq -kq(t)\left(\frac{\lambda}{2}\tau^2(t)\right)^\gamma - \gamma\frac{\sigma^{\frac{\gamma+1}{\gamma}}(t)}{b^{\frac{1}{\gamma}}(t)}. \tag{2.24}$$

Multiplying this inequality by $\pi^\gamma(t)$ and integrating the resulting inequality from $t_1$ to $t$, we get

$$\begin{aligned}
& \pi^\gamma(t)\ \sigma(t) - \pi^\gamma(t_1)\sigma(t_1) + \gamma\int_{t_1}^t b^{\frac{-1}{\gamma}}(s)\,\pi^{\gamma-1}(s)\,\sigma(s)\,ds \\
\leq\ & -\int_{t_1}^t kq(s)\left(\frac{\lambda}{2}\tau_i^2(s)\right)^\gamma \pi^\gamma(s)\,ds - \gamma\int_{t_1}^t \frac{\sigma^{\frac{\gamma+1}{\gamma}}(t)}{b^{\frac{1}{\gamma}}(t)}\pi^\gamma(s)\,ds.
\end{aligned}$$

We set

$$D := b^{\frac{-1}{\gamma}}(s)\,\pi^{\gamma-1}(s),\ C := \frac{\pi^\gamma(s)}{b^{\frac{1}{\gamma}}(t)},\ y := -\sigma(s).$$

Applying Lemma (1.2) and (2.16), for every $\lambda \in (0,\ 1)$, and all sufficiently large$t$. we obtain

$$\int_{t_1}^t \left[kq(s)\left(\frac{\lambda}{2}\tau^2(s)\right)^\gamma \pi^\gamma(s) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}\pi(s)b^{\frac{1}{\gamma}}(s)}\right] ds \leq \pi^\gamma(t_1)\,\sigma(t_1)+1,$$

but this contradicts (2.4).

Theorem (2.1) is proved. □

It is well known (see[2]) that the differential equation

$$\left[a(t)\left(x'(t)\right)^\alpha\right]' + q(t)\,x^\alpha(\tau(t)) = 0, \qquad t \geq t_0, \tag{2.25}$$

where $\alpha > 0$ is the ratio of odd positive integers, $a$ , $q \in C[t_\circ, \infty)$, is nonoscillatory if and only if there exist a number $T \geq t_\circ$, and a function $v \in C^1[T, \infty)$, satisfying the inequality

$$v'(t) + \alpha a^{\frac{-1}{\alpha}}(t)(v(t))^{\frac{(1+\alpha)}{\alpha}} + q(t) \leq 0, \quad \text{on } [T, \infty).$$

In what follows, we compare the oscillatory behavior of (1.1) with the second-order half-linear equations of type (2.28). There are numerous results concerning the oscillation of (2.28), which in clued Hille and Nehari types, Philos type, etc.

**Theorem 2.2.** *Let* $(A_1)$, $(A_2)$ , $(A_3)$ *and (1.2) hold. Assume that the equation*

$$\left[\frac{b(t)}{t^{2\gamma}}(x'(t))^\gamma\right]' + kq(t)\left(\frac{\lambda_1\tau^3(t)}{2t^3}\right)^\gamma x^\gamma(t) = 0, \tag{2.26}$$

*is oscillatory for some constant* $\lambda_1 \in (0,\ 1)$ , *the equation*

$$x''(t) + x(t)\int_t^\infty \left[\frac{1}{b(\vartheta)}\int_\vartheta^\infty kq(t)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma ds\right]^{\frac{1}{\gamma}} d\vartheta = 0, \tag{2.27}$$

*is oscillatory, the equation*

$$\left[\left(\int_t^\infty \pi(s)\,ds\right)^{-1}(x'(t))^\gamma\right]' + kq(t)\,x^\gamma(\tau(t)) = 0, \tag{2.28}$$

*is oscillatory, and the equation*

$$[b(t)(x'(t))^\gamma]' + kq(t)\left(\frac{\lambda_2\tau^2(t)}{2}\right)^\gamma x^\gamma(t) = 0, \tag{2.29}$$

*is oscillatory for some constant* $\lambda_2 \in (0, 1)$ . *Then every solution of (1.1) is oscillatory.*

*Proof.* Proceeding as in the proof of Theorem (2.1). If we set $\delta(t) = 1$ in (2.9), then we get

$$\omega'(t) + \frac{\gamma\lambda t^2}{2(b(t))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(t) + kq(t)\left(\frac{\tau^3(t)}{t^3}\right)^\gamma \leq 0,$$

for every constant $\lambda \in (0,\ 1)$ . Thus, we can see that equation (2.26) is nonoscillatory for every constant $\lambda_1 \in (0,\ 1)$ , which is a contradiction. If we now set $\rho(t) = 1$ in (2.14), then we find

$$\psi'(t) + \psi^2(t) + \rho(t)\int_t^\infty \left[\frac{1}{b(\vartheta)}\int_\vartheta^\infty kq(t)\left(\frac{\tau^3(s)}{s^3}\right)^\gamma ds\right]^{\frac{1}{\gamma}} d\vartheta \leq 0.$$

Hence, equation (2.27) is nonoscillatory, which is a contradiction. Thus, it follows from (2.20) that

$$\xi'(t) + \gamma\xi^{\frac{\gamma+1}{\gamma}}(t)\int_t^\infty \pi(s)\,ds + kq(t) \leq 0.$$

Therefore, we conclude that equation (2.28) is nonoscillatory, which is a contradiction. From (2.23), we get

$$\sigma'(t) + \gamma\frac{\sigma^{\frac{\gamma+1}{\gamma}}(t)}{b^{\frac{1}{\gamma}}(t)} + kq(t)\left(\frac{\lambda}{2}\tau^2(t)\right)^\gamma \leq 0,$$

for every constant $\lambda \in (0,\ 1)$ . Thus, we can see that equation (2.29) is nonoscillatory for every constant $\lambda_2 \in (0,\ 1)$ , which is a contradiction.

Theorem (2.2) is proved. □

It is well known (see[21]) that if

$$\int_{t_0}^{\infty}\frac{1}{a(t)}dt=\infty,\ \text{and}\ \liminf_{t\to\infty}\left(\int_{t_0}^{t}\frac{1}{a(s)}ds\right)\int_{t}^{\infty}q(s)\,ds>\frac{1}{4},$$

then equation (2.25) with $\gamma=1$ is oscillatory. It is also well known that if

$$\int_{t_0}^{\infty}\frac{1}{a(t)}dt<\infty,\ \text{and}\ \liminf_{t\to\infty}\left(\int_{t}^{\infty}\frac{1}{a(s)}ds\right)^{-1}\int_{t}^{\infty}\left(\int_{s}^{\infty}\frac{1}{a(\upsilon)}d\upsilon\right)^{2}q(s)\,ds>\tfrac{1}{4}.$$

then equation (2.25) with $\gamma=1$ is oscillatory.

Based on the above results and Theorem (2.2), we can easily obtain the following Hille and Nehari type oscilla-tion criteria for (1.1) with $\gamma=1$.

**Theorem 2.3.** *Let* $\gamma=1$ *and (1.2) hold. Assume that*

$$\int_{t_0}^{\infty}\frac{t^2}{b(t)}dt=\infty,\ \text{and}\ \liminf_{t\to\infty}\left(\int_{t_0}^{t}\frac{s^2}{b(s)}ds\right)k\int_{t}^{\infty}q(s)\left(\frac{\tau^3(s)}{s^3}\right)ds>\frac{1}{2\lambda_1},$$

for some constant $\lambda_1\in(0,\ 1)$,

$$\liminf_{t\to\infty}\int_{t}^{\infty}\int_{\theta}^{\infty}\frac{1}{b(\vartheta)}\int_{\vartheta}^{\infty}kq(s)\left(\frac{\tau^3(s)}{s^3}\right)ds\,d\vartheta\, d\theta>\frac{1}{4},\tag{2.30}$$

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\pi(s)\,dtds=\infty\ \text{and}\ \liminf_{t\to\infty}\left(\int_{t_0}^{t}\int_{s}^{\infty}\pi(\upsilon)\,d\upsilon ds\right)\int_{t}^{\infty}kq(s)\,ds>\tfrac{1}{4},$$

and

$$\liminf_{t\to\infty}\left(\int_{t}^{\infty}\frac{1}{b(s)}ds\right)^{-1}\int_{t}^{\infty}\left(\int_{s}^{\infty}\frac{1}{b(\upsilon)}d\upsilon\right)^{2}kq(t)\,\tau^3(s)\,ds>\frac{1}{2\lambda_2},\tag{2.31}$$

for some constant $\lambda_2\in(0,\ 1)$. Then every solution of equation (1.1) with $\gamma=1$ is oscillatory.

**Theorem 2.4.** *Let* $\gamma=1$ *and (2.30) hold. Assume that*

$$\int_{t_0}^{\infty}\frac{t^2}{b(t)}dt<\infty,$$

$$\liminf_{t\to\infty}\left(\int_{t}^{\infty}\frac{s^2}{b(s)}ds\right)^{-1}\int_{t}^{\infty}\left(\int_{s}^{\infty}\frac{\upsilon^2}{b(\upsilon)}d\upsilon\right)^{2}kq(t)\left(\frac{\tau^3(s)}{s^3}\right)ds>\frac{1}{2\lambda_1},$$

*for some constant* $\lambda_1\in(0,\ 1)$,

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\pi(s)\,dtds<\infty,$$

$$\liminf_{t\to\infty}\left(\int_{t}^{\infty}\int_{s}^{\infty}\pi(\upsilon)\,d\upsilon ds\right)^{-1}\int_{t}^{\infty}\left(\int_{s}^{\infty}\int_{u}^{\infty}\pi(\upsilon)\,d\upsilon du\right)^{2}kq(t)\,ds>\tfrac{1}{4},$$

*and (2.31) holds for some constant* $\lambda_2\in(0,\ 1)$ *.Then every solution of equation (1.1) with* $\gamma=1$ *is oscillatory.*

## 3 EXAMPLE

In this section, we give the following example to illustrate our main results.

**Example 3.1.** Consider a differential equation

$$\left(t^6\left(x'''(t)\right)\right)'+\eta t^2\left(x^3(t)+x(t)\right)=0,\ t\geq 1, \tag{3.1}$$

where $\eta>0$ is a constant. Let

$$\begin{aligned}\gamma &= 1,\ b(t)=t^6>0,\ b'(t)=6t^5\geq 0,\ b\in C^1[t_0,\infty),\\ q(t) &= \eta t^2>0,\ q\in C[t_0,\infty),\ f(x(t))=x^3(t)+x(t),\\ \tau(t) &= t,\ \lim_{t\to\infty} t=\infty,\ \tau(t)\in C[t_0,\infty).\end{aligned}$$

Thus, we get

$$\pi(t)=\frac{1}{5t^5},\ \int_s^\infty \pi(\chi)\,d\chi=\frac{1}{20s^4},\ \int_s^\infty\int_u^\infty \pi(\chi)\,d\chi du=\frac{1}{60s^3}.$$

If we now set $\delta(t)=\rho(t)=1$ and $k=1$, then we conclude that (2.1) and (2.2) are satisfied. As a result of calculations, we see that (2.3) and (2.4) hold for $\eta>45$.

$$\int_{t_0}^\infty\left[kq(s)\left(\tfrac{\tau^3(s)}{s^3}\right)^\gamma\delta(s)-\frac{2^\gamma b(s)\left(\delta'(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\lambda_1\delta(s)s^2)^\gamma}\right]ds=\infty,$$

$$\int_{t_0}^\infty \eta s^2 ds=\infty$$

$$\int_{t_0}^\infty\left[\rho(s)\int_s^\infty\left[\tfrac{1}{b(\vartheta)}\int_\varpi^\infty kq(\nu)^\gamma\,d\nu\right]^{\frac{1}{\gamma}}d\vartheta-\frac{\left(\rho'(s)\right)^2}{4\rho(s)}\right]ds=\infty,$$

$$\int_{t_0}^\infty\left[kq(s)\left(\int_s^\infty\int_u^\infty\pi(\chi)\,d\chi du\right)^\gamma-\frac{\gamma^{\gamma+1}\int_s^\infty\pi(\chi)d\chi}{(\gamma+1)^{\gamma+1}\int_s^\infty\int_u^\infty\pi(\chi)d\chi du}\right]ds=\infty,$$

$$\int_{t_0}^\infty\left(\frac{\eta s^2}{60s^3}-\frac{60s^3}{4\left(20s^4\right)}\right)ds=\int_{t_\circ}^\infty\frac{1}{s}\left(\frac{\eta}{60}-\frac{3}{4}\right)ds=\infty,\ \text{for}\ \eta>45.$$

$$\int_{t_0}^\infty\left[kq(s)\left(\frac{\lambda_2}{2}\tau^2(s)\right)^\gamma\pi^\gamma(s)-\frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}\pi(s)\,b^{\frac{1}{\gamma}}(s)}\right]ds=\infty,$$

$$\int_{t_0}^\infty\frac{1}{s}\left(\frac{\eta\lambda_2}{10}-\frac{5}{4}\right)ds=\infty,\ \text{for}\ \eta>45,\ \lambda_2=\frac{5}{18}\in(0,\ 1).$$

Hence, by Theorem(2.1), every solution of equation (3.1) is oscillatory for $\eta>45$.

**Remark 3.2.** The results of this example can not apply to results of [25].

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