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ON SOME GROWTH ANALYSIS OF ENTIRE AND MEROMORPHIC FUNCTIONS IN THE LIGHT OF THEIR INTEGER TRANSLATION

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Abstract. Using relative $(p,q,t)L$-th order, relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type, in the paper we establish some results depending on the comparative growth properties of entire and meromorphic functions on the basis of integer translation applied upon them.

1 Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [6, 9, 13, 14]. We also use the standard notations and definitions of the theory of entire functions which are available in [15] and therefore we do not explain those in details. Now let $f(z)$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to $f(z)$ is defined on $|z| = r$ as $M_f(r) = \max|z| = r\,|f(z)|$. In this connection the following definition is relevant:

Definition 1.1. $\{$[1]$\}$ A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [1].

Two entire functions $g(z)$ and $h(z)$ are said to be asymptotically equivalent if there exists A $(0 < A < \infty)$ such that

$$\frac{M_g(r)}{M_h(r)} \to A \text{ as } r \to \infty$$

and in this case we write $g \sim h$. Clearly if $g \sim h$ then $h \sim g$.

When $f(z)$ is meromorphic, one may define a different function $T_f(r)$ termed as Nevanlinna's Characteristic function of $f(z)$, playing same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)\left(\overline{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t}dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t}dt + \overline{n}_f(0,a)\log r\right),$$

moreover we denote by $n_f(r,a)\,(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of $f(z)$. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

Further, the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of $f(z)$ is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If $f(z)$ is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of $f(z)$ is defined as

$$T_f(r) = m_f(r) .$$

Moreover, if $f(z)$ is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exist and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$

Further let $f(z)$ be a meromorphic function and $n \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, then the translation of $f(z)$ be denoted by $f(z+n)$. For each $n \in \mathbb{N}$, one may obtain a function with some properties. Let us consider this family by $f_n(z)$ where

$$f_n(z) = \{f(z+n) : n \in \mathbb{N}\} .$$

We should recall that if α is a regular point of an analytic function $f(z)$ and if $f(\alpha) = 0$ then α is called a zero of $f(z)$. The point $z = \alpha$ is called a zero of $f(z)$ of order or multiplicity m (m being a positive integer) if in some neighborhood of α, $f(z)$ can be expanded in a Taylor's series of the form $f(z) = \sum_{x=m}^{\infty} a_x (z-\alpha)^x$ where $a_m \neq 0$.

It is clear that the number of zeros of $f(z)$ may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$ i.e.,

$$N_{f(z+n)}(r) = N_f(r) + e_n, \tag{1.1}$$

where e_n is a residue term such that $e_n \to 0$ as $r \to \infty$.

Also

$$m_{f(z+n)}(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}+n\right)\right| d\theta$$

$$i.e.,\ m_{f(z+n)}(r) = m_f(r) + e_n^{'}, \tag{1.2}$$

where $e_n^{'}$ (may be distinct from e_n) be such that $e_n^{'} \to 0$ as $r \to \infty$.

Therefore from (1.1) and (1.2), one may obtain that

$$\begin{aligned} N_{f(z+n)}(r) + m_{f(z+n)}(r) &= N_f(r) + e_n + m_f(r) + e_n^{'} \\ i.e.,\ T_{f(z+n)}(r) &= T_f(r) + e_n + e_n^{'} . \end{aligned}$$

Now if n varies then the Nevanlinna's Characteristic function for the family $f_n(z)$ is

$$T_{f_n}(r) = nT_f(r) + {}_n\left(e_n + e_n^{'}\right) . \tag{1.3}$$

For any two meromorphic functions $f(z)$ and $g(z)$ the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of $f(z)$ with respect to $g(z)$ in terms of their Nevanlinna's Characteristic functions. However let us consider that $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers. We define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper $a, b, d, p, q, i, j, m, n, l$ always denote positive integers and $t \in \mathbb{N} \cup \{-1, 0\}$. Now considering this, we introduce the definition of the (p,q)-th order and (p,q)-th lower order of an entire or meromorphic function which are as follows:

Definition 1.2. The (p,q)-th order and (p,q)-th lower order of an entire function $f(z)$ are defined as:

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

If $f(z)$ is a meromorphic function, then

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r},$$

Definition 1.2 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [7]. Moreover for entire and meromorphic functions when $p < q$, then Definition 1.2 is a special case of Proposition 1.2 and Definition 1.6 of [12] respectively for $\varphi(r) = \log^{[l]} r$ where $l > p - q$. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho_f^{[l]}$ and $\lambda^{(l,1)}(f) = \lambda_f^{[l]}$ where $\rho_f^{[l]}$ and $\lambda_f^{[l]}$ are respectively known as generalized order and generalized lower order of f. Also for $p = 2$ and $q = 1$ we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by ρ_f and λ_f where ρ_f and λ_f are the classical growth indicator known as order and lower order of $f(z)$. In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [7]) :

Definition 1.3. An entire function $f(z)$ is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots . \end{cases}$$

Analogously one can easily verify that Definition 1.3 of index-pair can also be applicable to a meromorphic function $f(z)$.

However, the function $f(z)$ is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of $f(z)$ are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

In order to compare the growth of entire functions having the same (p,q)-th order, Juneja, Kapoor and Bajpai [8] also introduced the concepts of (p,q)-th type and (p,q)-th lower type of entire function. Next we recall the definitions of (p,q)-th type and (p,q)-th lower type of entire and meromorphic function where we will give a minor modification to the original definition (see e.g. [8]):

Definition 1.4. The (p,q)-th type and the (p,q)-th lower type of entire function $f(z)$ having non-zero finite positive (p,q)-th order $\rho_f(p,q)$ are defined as :

$$\sigma^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}},$$

$$0 \leq \overline{\sigma}^{(p,q)}(f) \leq \sigma^{(p,q)}(f) \leq \infty .$$

If $f(z)$ is meromorphic function with $0 < \rho^{(p,q)}(f) < \infty$, then

$$\sigma^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}},$$

$$0 \leq \overline{\sigma}^{(p,q)}(f) \leq \sigma^{(p,q)}(f) \leq \infty .$$

Likewise, to compare the growth of entire functions having the same (p,q)-th lower order, one can also introduced the concept of (p,q)-th weak type in the following manner :

Definition 1.5. The (p,q)-th weak type of entire function $f(z)$ having non-zero finite positive (p,q)-th lower order $\lambda_f(p,q)$ is defined as :

$$\tau^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}.$$

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function $f(z)$ in the following way :

$$\overline{\tau}^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}\ ,\ 0 < \lambda^{(p,q)}(f) < \infty.$$

If $f(z)$ is meromorphic function with $0 < \lambda^{(p,q)}(f) < \infty$, then

$$\tau^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}} \text{ and } \overline{\tau}^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}},$$

where $0 < \lambda^{(p,q)}(f) < \infty$. It is also obvious that $0 \le \tau^{(p,q)}(f) \le \overline{\tau}^{(p,q)}(f) \le \infty$.

Somasundaram and Thamizharasi [11] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". The more generalized concept of L-order and L-type of meromorphic functions are L^*-order and L^*-type (resp. L^*- lower type) respectively which are as follows:

Definition 1.6. [11] The L^*-order $\rho^{L^*}(f)$ and the L^*-lower order $\lambda^{L^*}(f)$ of an entire function $f(z)$ are defined by

$$\rho^{L^*}(f) = \varlimsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda^{L^*}(f) = \varliminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

If $f(z)$ is a meromorphic function, then

$$\rho^{L^*}(f) = \varlimsup_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda^{L^*}(f) = \varliminf_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]}.$$

Extending the notion of Somasundaram and Thamizharasi [11], one may introduce the definition of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of a meromorphic function $f(z)$ in the following way:

$$\rho^{(p,q,t)L}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)} \text{ and } \lambda^{(p,q,t)L}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r + \exp^{[t]} L(r)}\ .$$

However Lahiri and Banerjee [10] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 1.7. [14] Let $f(z)$ be meromorphic and $g(z)$ be entire. The relative order of $f(z)$ with respect to $g(z)$ denoted by $\rho_g(f)$ is defined as

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \varlimsup_{r\to\infty} \frac{\log T_g^{-1}(T_f(r))}{\log r}.\end{aligned}$$

The definition coincides with the classical one [10] if $g(z) = \exp z$. Similarly one can define the relative lower order of a meromorphic function $f(z)$ with respect to an entire $g(z)$ denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \varliminf_{r\to\infty} \frac{\log T_g^{-1}(T_f(r))}{\log r}.$$

Debnath et al. [5] introduced the definitions of relative (p,q)-th order and relative (p,q)-th lower order of a meromorphic function with respect to another entire function in the light of index-pair. In order to keep accordance with Definition 1.2 and Definition 1.3, we will give a minor modification to the original definition of relative (p,q)-th order and relative (p,q)-th lower order of entire and meromorphic function (see e.g. [5]).

Definition 1.8. Let $f(z)$ be any meromorphic function and $g(z)$ be any entire function with index-pairs (m,q) and (m,p) respectively. Then the relative (p,q)-th order and relative (p,q)-th lower order of $f(z)$ with respect to $g(z)$ are defined as

$$\rho_g^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r}.$$

Further a meromorphic function $f(z)$, for which relative (p,q)-th order and relative (p,q)-th lower order with respect to an entire function $g(z)$ are the same is called a function of regular relative (p,q) growth with respect to $g(z)$. Otherwise, $f(z)$ is said to be irregular relative (p,q) growth with respect to $g(z)$.

Now in order to refine the above growth scale, one may introduce the definitions of an another growth indicators, such as relative (p,q)-th type and relative (p,q)-th lower type of entire or meromorphic functions with respect to another entire function in the light of their index-pair which are as follows:

Definition 1.9. Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with index-pairs (m,q) and (m,p) respectively. The relative (p,q)-th type and the relative (p,q)-th lower type of $f(z)$ with respect to $g(z)$ when $0 < \rho_g^{(p,q)}(f) < \infty$ are defined as:

$$\sigma_g^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}} \text{ and } \overline{\sigma}_g^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}},$$

where $0 < \rho_g^{(p,q)}(f) < \infty$.

Analogously, to determine the relative growth of $f(z)$ having same non zero finite relative (p,q)-th lower order with respect to $g(z)$, one can introduced the definition of relative (p,q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ of finite positive relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 1.10. Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with index-pairs (m,q) and (m,p) respectively. The relative (p,q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ when $0 < \lambda_g^{(p,q)}(f) < \infty$ are defined as:

$$\tau_g^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}} \text{ and } \overline{\tau}_g^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}},$$

where $0 < \lambda_g^{(p,q)}(f) < \infty$.

In order to make some progress in the study of relative order, now we introduce relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of a meromorphic function with respect to an entire function in the following way:

Definition 1.11. Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function. Then relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ are define as

$$\rho_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)} \text{ and } \lambda_g^{(p,q,t)L}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r + \exp^{[t]} L(r)}.$$

Further to compare the relative growth of two meromorphic functions having same non zero finite relative $(p,q,t)L$-th order with respect to another entire function, one may introduce the definitions of relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th lower type in the following manner:

Definition 1.12. The relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th lower type denoted respectively by $\sigma_g^{(p,q,t)L}(f)$ and $\overline{\sigma}_g^{(p,q,t)L}(f)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ such that $0 < \rho_g^{(p,q,t)L}(f) < \infty$ are respectively defined as follows:

$$\sigma_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_g^{(p,q,t)L}(f)}}$$

and

$$\overline{\sigma}_g^{(p,q,t)L}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_g^{(p,q,t)L}(f)}}.$$

Analogously to determine the relative growth of two meromorphic functions having same non zero finite relative $(p,q,t)L$-th lower order with respect to an entire function one may introduce the definition of relative $(p,q,t)L$-th weak type in the following way:

Definition 1.13. The relative $(p,q,t)L$-th weak type denoted by $\tau_g^{(p,q,t)L}(f)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ such that $0 < \lambda_g^{(p,q,t)L}(f) < \infty$ is defined as follows:

$$\tau_g^{(p,q,t)L}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_g^{(p,q,t)L}(f)}}.$$

Similarly one may define the growth indicator $\overline{\tau}_g^{(p,q,t)L}(f)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ in the following manner :

$$\overline{\tau}_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\lambda_g^{(p,q,t)L}(f)}}, \ 0 < \lambda_g^{(p,q,t)L}(f) < \infty.$$

In the paper we establish the relationship between the relative $(p,q,t)L$-th order, relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ and that of integer translation applied upon $f(z)$ and entire $g(z)$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *If $f(z)$ be a meromorphic function of regular (p,q) growth i.e., $\rho_f(p,q) = \lambda_f(p,q)$ then*

$$\sigma_f(p,q) = \overline{\sigma}_f(p,q) = \tau_f(p,q) = \overline{\tau}_f(p,q).$$

We omit the proof of Lemma 2.1 because it can be carried out in the line of Theorem 6 of [3].

Lemma 2.2. *[2] Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$ then*

$$\lim_{r\to\infty} \frac{T_{f_n}(r)}{T_f(r)} = n.$$

Lemma 2.3. *[1] Let g be an entire function and $\alpha > 1, 0 < \beta < \alpha$. Then*

$$M_g(\alpha r) > \beta M_g(r) \text{ for all large } r.$$

Lemma 2.4. *[1] Let f be an entire function which satisfies the property (A) then for any positive integer n, and for all large r,*

$$[M_f(r)]^n \le M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 2.5. *[6] Let g be entire. Then for sufficiently large values of r*

$$T_g(r) \le \log M_g(r) \le 3T_g(2r).$$

3 Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with $0 < \lambda_g(l,p) \le \rho_g(l,p) < \infty$ where $l > 1$. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$, then*

$$\frac{\lambda^{(l,p)}(g)}{\rho^{(l,p)}(g)} \le \varliminf_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \le \varlimsup_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p]} T_g^{-1}(T_f(r))} \le \frac{\rho^{(l,p)}(g)}{\lambda^{(l,p)}(g)}.$$

Proof. For any $\varepsilon(>0)$, we get from Lemma 2.2 for all sufficiently large values of r that

$$T_{f_n}(r) \le (n+\varepsilon)T_f(r) \tag{3.1}$$

and

$$T_{f_n}(r) \ge (n-\varepsilon)T_f(r). \tag{3.2}$$

Also from Lemma 2.2, we get for all sufficiently large values of r that

$$T_{g_m}(r) \ge (m-\varepsilon)T_g(r)$$

$$i.e.,\ r \ge T_{g_m}^{-1}((m-\varepsilon)T_g(r))$$

$$i.e.,\ T_g^{-1}\left(\frac{r}{m-\varepsilon}\right) \ge T_{g_m}^{-1}(r) \tag{3.3}$$

and

$$T_{g_m}(r) \le (m+\varepsilon)T_g(r)$$

$$i.e.,\ r \le T_{g_m}^{-1}((m+\varepsilon)T_g(r))$$

$$i.e.,\ T_g^{-1}\left(\frac{r}{m+\varepsilon}\right) \le T_{g_m}^{-1}(r). \tag{3.4}$$

Now from (3.1) and (3.3) it follows for all sufficiently large values of r that

$$T_{g_m}^{-1}(T_{f_n}(r)) \le T_{g_m}^{-1}((n+\varepsilon)T_f(r))$$

$$i.e.,\ T_{g_m}^{-1}(T_{f_n}(r)) \le T_g^{-1}\left(\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right). \tag{3.5}$$

Again from (3.2) and (3.4), it follows for all sufficiently large values of r that

$$T_{g_m}^{-1}\left(T_{f_n}(r)\right) \geq T_{g_m}^{-1}\left((n-\varepsilon) T_f(r)\right)$$

$$i.e.,\ T_{g_m}^{-1}\left(T_{f_n}(r)\right) \geq T_g^{-1}\left(\left(\frac{n-\varepsilon}{m+\varepsilon}\right) T_f(r)\right). \tag{3.6}$$

Now from (3.5) and (3.6), we get for all sufficiently large values of r that

$$\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right) \leq \log^{[p]} T_g^{-1}\left(\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)\right) \tag{3.7}$$

and

$$\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right) \geq \log^{[p]} T_g^{-1}\left(\left(\frac{n-\varepsilon}{m+\varepsilon}\right) T_f(r)\right). \tag{3.8}$$

Now for the definition of (l,p)-th order and (l,p)-th lower order of g, we get for all sufficiently large values of r that

$$T_g\left(\exp^{[p-1]}\left[\log^{[l-2]} T_f(r)\right]^{\frac{1}{\rho^{(l,p)}(g)+\varepsilon}}\right) \leq T_f(r)$$

$$i.e.,\ \log^{[p]} T_g^{-1}\left(T_f(r)\right) \geq \frac{1}{\left(\rho^{(l,p)}(g)+\varepsilon\right)} \log^{[l-1]} T_f(r) \tag{3.9}$$

and

$$T_g\left(\exp^{[p-1]}\left[\log^{[l-2]}\left[\left(\frac{n+\varepsilon}{(m-\varepsilon)}\right) T_f(r)\right]\right]^{\frac{1}{\lambda^{(l,p)}(g)-\varepsilon}}\right) \geq \left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)\right]$$

$$i.e.,\ \exp^{[p-1]}\left[\log^{[l-2]}\left[\left(\frac{n+\varepsilon}{(m-\varepsilon)}\right) T_f(r)\right]\right]^{\frac{1}{\lambda^{(l,p)}(g)-\varepsilon}} \geq T_g^{-1}\left(\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)\right)$$

$$i.e.,\ \frac{1}{\left(\lambda^{(l,p)}(g)-\varepsilon\right)} \log^{[l-1]} T_f(r)+O(1) \geq \log^{[p]} T_g^{-1}\left(\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)\right). \tag{3.10}$$

Therefore from (3.7) and (3.10), it follows for all sufficiently large values of r that

$$\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right) \leq \frac{1}{\left(\lambda^{(l,p)}(g)-\varepsilon\right)} \log^{[l-1]} T_f(r)+O(1). \tag{3.11}$$

Therefore from (3.9) and (3.11), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)} \leq \left(\frac{\rho^{(l,p)}(g)+\varepsilon}{\lambda^{(l,p)}(g)-\varepsilon}\right) \cdot \frac{\log^{[l-1]} T_f(r)+O(1)}{\log^{[l-1]} T_f(r)}$$

$$i.e.,\ \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)} \leq \frac{\rho^{(l,p)}(g)}{\lambda^{(l,p)}(g)}. \tag{3.12}$$

Similarly, from (3.8) it can be shown for all sufficiently large values of r that

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)} \geq \frac{\lambda^{(l,p)}(g)}{\rho^{(l,p)}(g)}. \tag{3.13}$$

Therefore from (3.12) and (3.13), we obtain that

$$\frac{\lambda^{(l,p)}(g)}{\rho^{(l,p)}(g)} \leq \underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)} \leq \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)} \leq \frac{\rho^{(l,p)}(g)}{\lambda^{(l,p)}(g)}.$$

Thus the theorem follows from above.

Corollary 3.2. *Under the same conditions of Theorem 3.1 if $g(z)$ is of regular (l,p) growth where $l>1$, then one may get that*

$$\lim_{r\to\infty}\frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)}=1.$$

As an application of Corollary 3.2, we prove the following theorems.

Theorem 3.3. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular (l,p) growth where $l>1$. If $f_n(z)=f(z+n)$ and $g_m(z)=g(z+m)$, then the relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of $f_n(z)$ with respect to $g_m(z)$ are same as those of $f(z)$ with respect to $g(z)$.*

Proof. In view of Corollary 3.2, we obtain that

$$\begin{aligned}\rho_{g_m}^{(p,q,t)L}(f_n)&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[q]} r+\exp^{[t]} L(r)}\\&=\overline{\lim_{r\to\infty}}\frac{\log^{[p]} T_g^{-1}\left(T_f(r)\right)}{\log^{[q]} r+\exp^{[t]} L(r)}\cdot\lim_{r\to\infty}\frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_g^{-1}\left(T_f(r)\right)}\\&=\rho_g^{(p,q,t)L}(f)\cdot 1=\rho_g^{(p,q,t)L}(f).\end{aligned}$$

In a similar manner, $\lambda_{g_m}^{(p,q,t)L}(f_n)=\lambda_g^{(p,q,t)L}(f)$.

Thus the theorem follows.

Using the definition of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of meromorphic function with respect to another entire function and in view of Theorem 3.3, we easily get the following result:

Theorem 3.4. *Let $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0<\lambda_h^{(l,q,t)L}(f)\le\rho_h^{(l,q,t)L}(f)<\infty$ and $0<\lambda_k^{(d,q,t)L}(g)\le\rho_k^{(d,q,t)L}(g)<\infty$. Also let $h(z)$ and $k(z)$ be of regular (a,l)-growth and regular (b,d)-growth respectively where $a>1$ and $b>1$. If $f_n(z)=f(z+n)$, $g_m(z)=g(z+m)$, $h_i(z)=h(z+i)$ and $k_j(z)=k(z+j)$, then*

$$\begin{aligned}\frac{\lambda_h^{(l,q,t)L}(f)}{\rho_k^{(d,q,t)L}(g)}&\le\underline{\lim_{r\to\infty}}\frac{\log^{[l]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)}{\log^{[d]} T_{k_j}^{-1}\left(T_{g_m}(r)\right)}\le\min\left\{\frac{\lambda_h^{(l,q,t)L}(f)}{\lambda_k^{(d,q,t)L}(g)},\frac{\rho_h^{(l,q,t)L}(f)}{\rho_k^{(d,q,t)L}(g)}\right\}\\&\le\max\left\{\frac{\lambda_h^{(l,q,t)L}(f)}{\lambda_k^{(d,q,t)L}(g)},\frac{\rho_h^{(l,q,t)L}(f)}{\rho_k^{(d,q,t)L}(g)}\right\}\le\overline{\lim_{r\to\infty}}\frac{\log^{[l]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)}{\log^{[d]} T_{k_j}^{-1}\left(T_{g_m}(r)\right)}\le\frac{\rho_h^{(l,q,t)L}(f)}{\lambda_k^{(d,q,t)L}(g)}.\end{aligned}$$

The proof is omitted.

Theorem 3.5. *Let $f(z)$ be meromorphic function and $g(z)$, $h(z)$ be any two entire functions of regular (l,p)-growth where $l>1$. Also let $g(z)$ and $h(z)$ have the property (A) and $g\sim h$. If $f_n(z)=f(z+n)$, $g_m(z)=g(z+m)$, $h_i(z)=g(z+i)$, then*

$$\rho_{g_m}^{(p,q,t)L}(f_n)=\rho_{h_i}^{(p,q,t)L}(f_n)\ \text{and}\ \lambda_{g_m}^{(p,q,t)L}(f_n)=\lambda_{h_i}^{(p,q,t)L}(f_n).$$

Proof. Let $\varepsilon>0$ is arbitrary. Since $g(z)\sim h(z)$, we get from Lemma 2.3 for all sufficiently large values of r that

$$M_g(r)<(A+\varepsilon)M_h(r)\le M_h(\alpha r), \tag{3.14}$$

where $\alpha>1$ is such that $A+\varepsilon<\alpha$. Further from Lemma 2.5 and in view of the definition of relative $(p,q,t)L$-th order, we obtain for all sufficiently large values of r that

$$\begin{aligned}T_f(r)&\le T_g\left(\exp^{[p-1]}\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\left(\rho_g^{(p,q,t)L}(f)+\varepsilon\right)}\right)\\ i.e.,\ T_f(r)&\le\log M_g\left(\exp^{[p-1]}\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]^{\left(\rho_g^{(p,q,t)L}(f)+\varepsilon\right)}\right).\end{aligned}$$

Therefore in view of (3.14), Lemma 2.4 and Lemma 2.5, it follows from above for any $\delta > 1$ that

$$T_f(r) \leq \frac{1}{3}\log\left(M_h\left(\alpha\left(\exp^{[p-1]}\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]\right)^{\left(\rho_g^{(p,q,t)L}(f)+\varepsilon\right)}\right)\right)^3$$

$$i.e.,\ T_f(r) \leq \frac{1}{3}\log M_h\left(\alpha^\delta\left(\exp^{[p-1]}\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]\right)^{\delta\left(\rho_g^{(p,q,t)L}(f)+\varepsilon\right)}\right)$$

$$i.e.,\ T_f(r) \leq T_h\left(2\alpha^\delta\left(\exp^{[p-1]}\left[\log^{[q-1]} r\cdot\exp^{[t+1]} L(r)\right]\right)^{\delta\left(\rho_g^{(p,q,t)L}(f)+\varepsilon\right)}\right)$$

$$i.e.,\ \frac{\log^{[p]} T_h^{-1}(T_f(r))}{\log^{[q]} r+\exp^{[t]} L(r)} \leq$$

$$\delta\left(\rho_g^{(p,q,t)L}(f)+\varepsilon\right)\frac{\log^{[q]} r+\exp^{[t]} L(r)}{\log^{[q]} r+\exp^{[t]} L(r)}+\frac{O(1)}{\log^{[q]} r+\exp^{[t]} L(r)}.$$

Letting $\delta \to 1+$ we get from above that

$$\rho_h^{(p,q,t)L}(f) \leq \rho_g^{(p,q,t)L}(f). \tag{3.15}$$

Since $h(z) \sim g(z)$, we also obtain that

$$\rho_g^{(p,q,t)L}(f) \leq \rho_h^{(p,q,t)L}(f). \tag{3.16}$$

Now in view of Theorem 3.3 we obtain from (3.15) and (3.16) that

$$\rho_{g_m}^{(p,q,t)L}(f_n) = \rho_{h_i}^{(p,q,t)L}(f_n).$$

Similarly we have

$$\lambda_{g_m}^{(p,q,t)L}(f_n) = \lambda_{h_i}^{(p,q,t)L}(f_n).$$

Thus the theorem follows.

Theorem 3.6. *Let $f(z)$ be meromorphic function and $g(z)$, $h(z)$ be any two entire functions of regular (l,p)-growth where $l > 1$ such that $0 < \rho_g^{(p,q,t)L}(f) < \infty$ and $0 < \rho_h^{(p,q,t)L}(f) < \infty$. Also let $g(z)$ and $h(z)$ have the property (A) and $g(z) \sim h(z)$. If $f_n(z) = f(z+n)$, $g_m(z) = g(z+m)$, $h_i(z) = h(z+i)$, then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p]} T_{h_i}^{-1}(T_{f_n}(r))} \leq 1 \leq \varlimsup_{r\to\infty}\frac{\log^{[p]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p]} T_{h_i}^{-1}(T_{f_n}(r))}.$$

Proof. From the definition of $\rho_{g_m}^{(p,q,t)L}(f_n)$ we get for all large values of r that

$$\log^{[p]} T_{g_m}^{-1}(T_{f_n}(r)) \leq \left(\rho_{g_m}^{(p,q,t)L}(f_n)+\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right) \tag{3.17}$$

and for a sequence of values of r tending to infinity, it follows that

$$\log^{[p]} T_{g_m}^{-1}(T_{f_n}(r)) \geq \left(\rho_{g_m}^{(p,q,t)L}(f_n)-\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right). \tag{3.18}$$

Further from the definition of $\rho_{h_i}^{(p,q,t)L}(f_n)$, we obtain for a sequence of values of r tending to infinity that

$$\log^{[p]} T_{h_i}^{-1}(T_{f_n}(r)) \geqslant \left(\rho_{h_i}^{(p,q,t)L}(f_n)-\varepsilon\right)\left[\left(\log^{[q]} r+\exp^{[t]} L(r)\right)\right] \tag{3.19}$$

and for all large values of r, it follows that

$$\log^{[p]} T_{h_i}^{-1}(T_{f_n}(r)) \leq \left(\rho_{h_i}^{(p,q,t)L}(f_n)+\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right). \tag{3.20}$$

Now from (3.17) and (3.19) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \leq \frac{\left(\rho_{g_m}^{(p,q,t)L}(f_n)+\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right)}{\left(\rho_{h_i}^{(p,q,t)L}(f_n)-\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \leq \frac{\rho_{g_m}^{(p,q,t)L}(f_n)}{\rho_{h_i}^{(p,q,t)L}(f_n)}. \tag{3.21}$$

Since $g(z)$ and $h(z)$ have the property (A) and $g(z)\sim h(z)$, in view of Theorem 3.5, we obtain from (3.21) that

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \leq 1. \tag{3.22}$$

Again combining (3.18) and (3.20) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \geqslant \frac{\left(\rho_{g_m}^{(p,q,t)L}(f_n)-\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right)}{\left(\rho_{h_i}^{(p,q,t)L}(f_n)+\varepsilon\right)\left(\log^{[q]} r+\exp^{[t]} L(r)\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \geqslant \frac{\rho_{g_m}^{(p,q,t)L}(f_n)}{\rho_{h_i}^{(p,q,t)L}(f_n)}. \tag{3.23}$$

Now as $g(z)$ and $h(z)$ have the property (A) and $g\sim h$, in view of Theorem 3.5 we obtain from (3.23) that

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \geqslant 1. \tag{3.24}$$

Thus the theorem follows from (3.22) and (3.24) .

In view of Theorem 3.5, one can derived the following theorem with the help Theorem 3.6 and therefore its proof is omitted.

Theorem 3.7. *Let $f(z)$ be meromorphic function and $g(z)$, $h(z)$ be any two entire functions of regular (l,p) growth where $l>1$ such that $0<\lambda_g^{(p,q,t)L}(f)<\infty$ and $0<\lambda_h^{(p,q,t)L}(f)<\infty$. Also let $g(z)$ and $h(z)$ have the property (A) and $g(z)\sim h(z)$. If $f_n(z)=f(z+n)$, $g_m(z)=g(z+m)$, $h_i(z)=h(z+i)$, then*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)} \leq 1 \leq \varlimsup_{r\to\infty} \frac{\log^{[p]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p]} T_{h_i}^{-1}\left(T_{f_n}(r)\right)}.$$

Theorem 3.8. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with $0<\tau^{(l,p)}(g)\leq\overline{\tau}^{(l,p)}(g)<\infty$ and $0<\overline{\sigma}^{(l,p)}(g)\leq\sigma^{(l,p)}(g)<\infty$ where $l>2$. If $f_n(z)=f(z+n)$ and $g_m(z)=g(z+m)$, then*

$$\max\left\{\left(\frac{\tau^{(l,p)}(g)}{\overline{\tau}^{(l,p)}(g)}\right)^{\frac{1}{\lambda^{(l,p)}(g)}}, \left(\frac{\overline{\sigma}^{(l,p)}(g)}{\sigma^{(l,p)}(g)}\right)^{\frac{1}{\rho^{(l,p)}(g)}}\right\} \leq \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)}$$

$$\leq \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_{g_m}^{-1}\left(T_{f_n}(r)\right)}{\log^{[p-1]} T_g^{-1}\left(T_f(r)\right)} \leq \min\left\{\left(\frac{\overline{\tau}^{(l,p)}(g)}{\tau^{(l,p)}(g)}\right)^{\frac{1}{\lambda^{(l,p)}(g)}}, \left(\frac{\sigma^{(l,p)}(g)}{\overline{\sigma}^{(l,p)}(g)}\right)^{\frac{1}{\rho^{(l,p)}(g)}}\right\}.$$

Proof. From the definition of (l,p)-th type and (l,p)-th lower type, we get for all sufficiently large values of r that

$$T_g\left(\exp^{[p-1]}\left\{\frac{\log^{[l-2]} T_f(r)}{\left(\sigma^{(l,p)}(g)+\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}}\right) \leq T_f(r)$$

$$i.e.,\ \log^{[p-1]} T_g^{-1}(T_f(r)) \geq \left\{\frac{\log^{[l-2]} T_f(r)}{\left(\sigma^{(l,p)}(g)+\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}} \tag{3.25}$$

and

$$T_g\left(\exp^{[p-1]}\left\{\frac{\log^{[l-2]}\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)}{\left(\overline{\sigma}^{(l,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}}\right) \geq \left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)\right]$$

$$i.e.,\ \exp^{[p-1]}\left\{\frac{\log^{[l-2]}\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)}{\left(\overline{\sigma}^{(l,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}} \geq T_g^{-1}\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)\right]. \tag{3.26}$$

Therefore from (3.5) and (3.26) , it follows for all sufficiently large values of r that

$$T_{g_m}^{-1}(T_{f_n}(r)) \leq \exp^{[p-1]}\left\{\frac{\log^{[l-2]}\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)}{\left(\overline{\sigma}^{(l,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}}$$

$$i.e.,\ \log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r)) \leq \left\{\frac{\log^{[l-2]}\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)}{\left(\overline{\sigma}^{(l,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}}. \tag{3.27}$$

Therefore from (3.25) and (3.27) , it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \leq \frac{\left\{\frac{\log^{[l-2]}\left(\frac{n+\varepsilon}{m-\varepsilon}\right) T_f(r)}{\left(\overline{\sigma}^{(l,p)}(g)-\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}}}{\left\{\frac{\log^{[l-2]} T_f(r)}{\left(\sigma^{(l,p)}(g)+\varepsilon\right)}\right\}^{\frac{1}{\rho^{(l,p)}(g)}}}$$

$$i.e.,\ \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \leq \left(\frac{\sigma^{(l,p)}(g)+\varepsilon}{\overline{\sigma}^{(l,p)}(g)-\varepsilon}\right)^{\frac{1}{\rho^{(l,p)}(g)}} \cdot \left(\frac{\log^{[l-2]} T_f(r)+O(1)}{\log^{[l-2]} T_f(r)}\right)^{\frac{1}{\rho^{(l,p)}(g)}}$$

$$i.e.,\ \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \leq \left(\frac{\sigma^{(l,p)}(g)}{\overline{\sigma}^{(l,p)}(g)}\right)^{\frac{1}{\rho^{(l,p)}(g)}}. \tag{3.28}$$

Similarly from (3.6) , it can be shown for all sufficiently large values of r that

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \geq \left(\frac{\overline{\sigma}^{(l,p)}(g)}{\sigma^{(l,p)}(g)}\right)^{\frac{1}{\rho^{(l,p)}(g)}}. \tag{3.29}$$

Therefore from (3.28) and (3.29) , we obtain that

$$\left(\frac{\overline{\sigma}^{(l,p)}(g)}{\sigma^{(l,p)}(g)}\right)^{\frac{1}{\rho^{(l,p)}(g)}} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \leq \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))}$$

$$\leq \left(\frac{\sigma^{(l,p)}(g)}{\overline{\sigma}^{(l,p)}(g)}\right)^{\frac{1}{\rho^{(l,p)}(g)}}. \tag{3.30}$$

Similarly, using the weak type one can easily verify that

$$\left(\frac{\tau^{(l,p)}(g)}{\overline{\tau}^{(l,p)}(g)}\right)^{\frac{1}{\lambda^{(l,p)}(g)}} \leq \lim_{r\to\infty} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \leq \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))}$$
$$\leq \left(\frac{\overline{\tau}^{(l,p)}(g)}{\tau^{(l,p)}(g)}\right)^{\frac{1}{\lambda^{(l,p)}(g)}}. \tag{3.31}$$

Thus the theorem follows from (3.30) and (3.31).

Corollary 3.9. *Under the same conditions of Theorem 3.8, if $g(z)$ is of regular (l,p) growth then by Lemma 2.1 one can easily obtain that*

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} = 1.$$

Theorem 3.10. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular (l,p) growth and non zero finite (l,p)-th type where $l > 2$. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$, then the relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th lower type of $f_n(z)$ with respect to $g_m(z)$ are same as those of $f(z)$ with respect to $g(z)$ if $\rho_g^{(p,q,t)L}(f)$ is positive finite.*

Proof. From Theorem 3.3 and Corollary 3.9, we get that

$$\sigma_{g_m}^{(p,q,t)L}(f_n) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_{g_m}^{(p,q,t)L}(f_n)}}$$
$$= \lim_{r\to\infty} \frac{\log^{[p-1]} T_{g_m}^{-1}(T_{f_n}(r))}{\log^{[p-1]} T_g^{-1}(T_f(r))} \cdot \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_g^{-1}(T_f(r))}{\left[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)\right]^{\rho_g^{(p,q,t)L}(f)}}$$
$$= 1 \cdot \sigma_g^{(p,q,t)L}(f) = \sigma_g^{(p,q,t)L}(f).$$

Similarly, $\overline{\sigma}_{g_m}^{(p,q,t)L}(f_n) = \overline{\sigma}_g^{(p,q,t)L}(f)$.

This proves the theorem.

Theorem 3.11. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular (l,p) growth and non-zero finite (l,p)-th type where $l > 2$. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$, then $\tau_{g_m}^{(p,q,t)L}(f_n)$ and $\overline{\tau}_{g_m}^{(p,q,t)L}(f_n)$ are same as those of $f(z)$ with respect to $g(z)$ i.e.,*

$$\tau_{g_m}^{(p,q,t)L}(f_n) = \tau_g^{(p,q,t)L}(f) \text{ and } \overline{\tau}_{g_m}^{(p,q,t)L}(f_n) = \overline{\tau}_g^{(p,q,t)L}(f).$$

when $\lambda_g^{(p,q,t)L}(f)$ is positive finite.

We omit the proof of Theorem 3.11 because it can be carried out in the line of Theorem 3.10.

Using the definition of relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th lower type of meromorphic function with respect to another entire function and in view of Theorem 3.10, we easily get the following result:

Theorem 3.12. *Let $f(z), g(z)$ be any two meromorphic functions and $h(z), k(z)$ be any two entire functions such that $0 < \overline{\sigma}_h^{(l,q,t)L}(f) \leq \sigma_h^{(l,q,t)L}(f) < \infty$, $0 < \overline{\sigma}_k^{(d,q,t)L}(g) \leq \sigma_k^{(d,q,t)L}(g) < \infty$ and $\rho_h^{(l,q,t)L}(f) = \rho_k^{(d,q,t)L}(g)$. Also let $h(z)$ be of regular (a,l) growth having non-zero finite (a,l)-th type and $k(z)$ be of regular (b,d) growth having non zero finite (b,d)-th type where $a > 2$ and $b > 2$. If $f_n(z) = f(z+n)$, $g_m(z) = g(z+m)$, $h_i(z) = h(z+i)$ and*

$k_j(z) = k(z+j)$, *then*

$$\frac{\overline{\sigma}_h^{(l,q,t)L}(f)}{\sigma_k^{(d,q,t)L}(g)} \le \varliminf_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \min\left\{\frac{\overline{\sigma}_h^{(l,q,t)L}(f)}{\overline{\sigma}_k^{(d,q,t)L}(g)}, \frac{\sigma_h^{(l,q,t)L}(f)}{\sigma_k^{(d,q,t)L}(g)}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_h^{(l,q,t)L}(f)}{\overline{\sigma}_k^{(d,q,t)L}(g)}, \frac{\sigma_h^{(l,q,t)L}(f)}{\sigma_k^{(d,q,t)L}(g)}\right\} \le \varlimsup_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \frac{\sigma_h^{(l,q,t)L}(f)}{\overline{\sigma}_k^{(d,q,t)L}(g)}.$$

The proof is omitted.

Now in the line of Theorem 3.12 and with the help of Theorem 3.11, one can easily prove the following theorem using the notion of relative $(p,q,t)L$-th weak type and therefore its proof is omitted.

Theorem 3.13. *Let $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0 < \tau_h^{(l,q,t)L}(f) \le \overline{\tau}_h^{(l,q,t)L}(f) < \infty$, $0 < \tau_k^{(d,q,t)L}(g) \le \overline{\tau}_k^{(d,q,t)L}(g) < \infty$ and $\lambda_h^{(mlq,t)L} = \lambda_k^{(d,q,t)L}(g)$. Also let $h(z)$ be of regular (a,l) growth having non-zero finite (a,l)-th type and $k(z)$ be of regular (b,d) growth having non zero finite (b,d)-th type where $a > 2$ and $b > 2$. If $f_n(z) = f(z+n)$, $g_m(z) = g(z+m)$, $h_i(z) = h(z+i)$ and $k_j(z) = k(z+j)$, then*

$$\frac{\tau_h^{(l,q,t)L}(f)}{\overline{\tau}_k^{(d,q,t)L}(g)} \le \varliminf_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \min\left\{\frac{\tau_h^{(l,q,t)L}(f)}{\tau_k^{(d,q,t)L}(g)}, \frac{\overline{\tau}_h^{(l,q,t)L}(f)}{\overline{\tau}_k^{(d,q,t)L}(g)}\right\}$$
$$\le \max\left\{\frac{\tau_h^{(l,q,t)L}(f)}{\tau_k^{(d,q,t)L}(g)}, \frac{\overline{\tau}_h^{(l,q,t)L}(f)}{\overline{\tau}_k^{(d,q,t)L}(g)}\right\} \le \varlimsup_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \frac{\overline{\tau}_h^{(l,q,t)L}(f)}{\tau_k^{(d,q,t)L}(g)}.$$

We may now state the following two theorems without their proofs based on relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type:

Theorem 3.14. *Let $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0 < \overline{\sigma}_h^{(l,q,t)L}(f) \le \sigma_h^{(l,q,t)L}(f) < \infty$, $0 < \tau_k^{(d,q,t)L}(g) \le \overline{\tau}_k^{(d,q,t)L}(g) < \infty$ and $\rho_h^{(l,q,t)L}(f) = \lambda_k^{(d,q,t)L}(g)$. Also let $h(z)$ be of regular (a,l) growth having non-zero finite (a,l)-th type and $k(z)$ be of regular (b,d) growth having non zero finite (b,d)-th type where $a > 2$ and $b > 2$. If $f_n(z) = f(z+n)$, $g_m(z) = g(z+m)$, $h_i(z) = h(z+i)$ and $k_j(z) = k(z+j)$, then*

$$\frac{\overline{\sigma}_h^{(l,q,t)L}(f)}{\overline{\tau}_k^{(d,q,t)L}(g)} \le \varliminf_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \min\left\{\frac{\overline{\sigma}_h^{(l,q,t)L}(f)}{\tau_k^{(d,q,t)L}(g)}, \frac{\sigma_h^{(l,q,t)L}(f)}{\overline{\tau}_k^{(d,q,t)L}(g)}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_h^{(l,q,t)L}(f)}{\tau_k^{(d,q,t)L}(g)}, \frac{\sigma_h^{(l,q,t)L}(f)}{\overline{\tau}_k^{(d,q,t)L}(g)}\right\} \le \varlimsup_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \frac{\sigma_h^{(l,q,t)L}(f)}{\tau_k^{(d,q,t)L}(g)}.$$

Theorem 3.15. *Let $f(z)$, $g(z)$ be any two meromorphic functions and $h(z)$, $k(z)$ be any two entire functions such that $0 < \tau_h^{(l,q,t)L}(f) \le \overline{\tau}_h^{(l,q,t)L}(f) < \infty$, $0 < \overline{\sigma}_k^{(d,q,t)L}(g) \le \sigma_k^{(d,q,t)L}(g) < \infty$ and $\lambda_h^{(l,q,t)L}(f) = \rho_k^{(d,q,t)L}(g)$. Also let $h(z)$ be of regular (a,l) growth having non-zero finite (a,l)-th type and $k(z)$ be of regular (b,d) growth having non zero finite (b,d)-th type where $a > 2$ and $b > 2$. If $f_n(z) = f(z+n)$, $g_m(z) = g(z+m)$, $h_i(z) = h(z+i)$ and $k_j(z) = k(z+j)$, then*

$$\frac{\tau_h^{(l,q,t)L}(f)}{\sigma_k^{(d,q,t)L}(g)} \le \varliminf_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \min\left\{\frac{\tau_h^{(l,q,t)L}(f)}{\overline{\sigma}_k^{(d,q,t)L}(g)}, \frac{\overline{\tau}_h^{(l,q,t)L}(f)}{\sigma_k^{(d,q,t)L}(g)}\right\}$$
$$\le \max\left\{\frac{\tau_h^{(l,q,t)L}(f)}{\overline{\sigma}_k^{(d,q,t)L}(g)}, \frac{\overline{\tau}_h^{(l,q,t)L}(f)}{\sigma_k^{(d,q,t)L}(g)}\right\} \le \varlimsup_{r\to\infty} \frac{\log^{[l-1]} T_{h_i}^{-1}(T_{f_n}(r))}{\log^{[d-1]} T_{k_j}^{-1}(T_{g_m}(r))} \le \frac{\overline{\tau}_h^{(l,q,t)L}(f)}{\overline{\sigma}_k^{(d,q,t)L}(g)}.$$

4 Concluding Remarks

The main aim of the paper is to extend and modify the notion of L-order to relative L-order of higher dimensions, and in this connection we have established some theorems depending on the comparative growth properties of entire and meromorphic functions on the basis of integer translation applied upon them. However recently Biswas [4] introduce the concepts of relative (p,q)-φ order and relative (p,q)-φ lower order of a meromorphic function with respect to another entire function where $p,q \in \mathbb{N}$ and $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. For detail about the relative (p,q)-φ order and the relative (p,q)-φ lower order, one may see [4]. The results presented in the present paper may further be estimated applying the concepts relative (p,q)-φ order and the relative (p,q)-φ lower order of a meromorphic function with respect to another entire function under some certain different conditions.

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