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3-dimensional Golden almost contact metric manifolds

Gherici BELDJILALI

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Abstract. In this work, we have dealt with almost contact metric structures induced by Golden structures and simultaneously as one can be obtained from the other, and provided the following results besides some side ones. Various examples are discussed.

1 Introduction

The Golden section or Golden mean ϕ is the positive root of the polynomial equation $x^2-x-1=0$; i.e., $\phi=\frac{1+\sqrt{5}}{2}$. The negative root of the previous equation, usually denoted by ϕ^*, satisfies $\phi^*=\frac{1-\sqrt{5}}{2}=1-\phi$. In the last years the Golden mean can be found in many areas of mathematical and physical research.
In [6], Crasmareanu and Hretcanu introduced and studied the Golden structures and they gives relationships between it and other structures (almost product, almost tangent and almost complex). As generalization of the Golden mean appear the metallic means (see [7]), which are the positive root of the equation $x^2-px-q=0$, where p, q are positive integers.
Manifolds equipped with certain differential-geometric structures possess rich geometric structures and such manifolds and relations between them have been studied widely in differential geometry. Recently, the author [1] introduced the notion of Golden Riemannian manifolds of type (r,s) and starting from a Golden Riemannian structure, we have established many well-known structures on a Riemannian manifold. Also, he defined a new class of Golden manifolds [2].

Here we show that there is a correspondence between the Golden Riemannian structures and the almost contact metric structures. This text is organized in the following way:
Section 2 is devoted to the background of the structures which will be used in the sequel.

In **section** 3, starting from an almost contact metric structures we define a Golden Riemannian structures and we investigate conditions for those structures being integrable and parallel and we construct an example.
In **section** 4, we give the inverse study i.e. we define an almost contact metric structure induced by a Golden Riemannian structure. Finally, we prove the existence of such a manifold by a concrete example.

2 Review of needed notions

In this section, we give a brief information for Golden Riemannian manifolds and almost contact metric manifolds. We note that throughout this paper, all manifolds are three-dimentional and smooth and $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields on M.

2.1 Golden Riemannian manifold

Let (M, g) be a Riemannian manifold. A Golden structure on (M, g) is a non-null tensor field Φ of type $(1, 1)$ which satisfies the equation

$$\Phi^2 = \Phi + I, \tag{2.1}$$

where I is the identity transformation.

We say that the metric g is Φ compatible if

$$g(\Phi X, Y) = g(X, \Phi Y), \tag{2.2}$$

for all X, Y vectors fiels on M.

If we substitute ΦX into X in (2.2), equation (2.2) may also written as

$$g(\Phi X, \Phi Y) = g(\Phi^2 X, Y) = g((\Phi + I)X, Y) = g(\Phi X, Y) + g(X, Y).$$

The Riemannian metric (2.2) is called Φ-compatible and (M, Φ, g) is named a Golden Riemannian manifold [6].
Here, it's the occasion to show that each Golden structure on a Riemannian manifold generates a family of Riemannian metrics.

Proposition 2.1. *Any Golden manifold Φ on a paracompact manifold M admits a Riemannian metric Φ-compatible.*

Proof. Let h be any Riemannian metric on M and define g by

$$g(X, Y) = h(\Phi X, \Phi Y) + h(X, Y) = h(\Phi X, Y) + 2h(X, Y),$$

and check the details. □

Proposition 2.2. *If (M, g_0, Φ) is an almost Golden Riemannian manifold, then M admits a 1-parameter family of Riemannian metrics g_n defined by*

$$g_n = g_0 \circ (\Phi + 2I)^n, \qquad \forall n \in \mathbb{N}^*$$

where

$$(g_n \circ \Phi)(X, Y) = g_n(\Phi X, Y) = g_n(X, \Phi Y), \qquad \forall X, Y \in \mathfrak{X}(M).$$

Proof. Suppose that $g_n = g_0 \circ (\Phi + 2I)^n$, for any positve integer n. Then, using proposition (2.1) we get

$$\begin{aligned} g_{n+1} &= g_n \circ (\Phi + 2I) \\ &= (g_0 \circ (\Phi + 2I)^n) \circ (\Phi + 2I) \\ &= g_0 \circ (\Phi + 2I)^{n+1}. \end{aligned}$$

□

It is known that a Golden structure Φ is integrable if the Nijenhuis tensor N_Φ vanishes [6] i.e.

$$N_\Phi(X, Y) = \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] = 0. \tag{2.3}$$

We know that the integrability of Φ is equivalent to the existence of a torsion-free affine connection with respect to which the equation $\nabla \Phi = 0$ holds [8].

2.2 Almost contact metric manifold

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1,\ \varphi^2(X) = -X + \eta(X)\xi \quad and \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad and \quad \eta \circ \varphi = 0. \tag{2.5}$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Omega$, where $\Omega(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. If, in addition, ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M.
On the other hand, the almost contact metric structure of M is said to be normal if

$$N_\varphi(X,Y) = [\varphi,\varphi](X,Y) + 2d\eta\ (X,Y)\xi = 0, \tag{2.6}$$

for any X, Y, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

An almost contact metric structures (φ, ξ, η, g) on M is said to be:

$$\begin{cases} (a) : Sasakian \ \Leftrightarrow \ \Omega = \mathrm{d}\eta \text{ and } (\varphi,\xi,\eta) \text{ is normal}, \\ (b) : Cosymplectic \ \Leftrightarrow \ \mathrm{d}\Omega = \mathrm{d}\eta = 0 \text{ and } (\varphi,\xi,\eta) \text{ is normal}, \\ (c) : Kenmotsu \ \Leftrightarrow \ \mathrm{d}\eta = 0,\ \mathrm{d}\Omega = 2\eta \wedge \Omega \text{ and } (\varphi,\xi,\eta) \text{ is normal}. \end{cases} \tag{2.7}$$

where d denotes the exterior derivative.

For more background on almost contact metric manifolds, we recommend the reference [4] and [5].

3 From almost contact metric structure to Golden Riemannian structure

In this section, starting from an almost contact metric structures we define a Golden Riemannian structures and we investigate conditions for those structures being integrable and parallel.

Theorem 3.1. *Every almost contact metric structure (φ, ξ, η, g) on a Riemannian manifold (M, g) induces only two Golden Riemannian structures on (M, g), given as follows:*

$$\Phi_1 = \phi I - \sqrt{5}\,\eta \otimes \xi, \qquad \Phi_2 = \phi^* I + \sqrt{5}\,\eta \otimes \xi, \tag{3.1}$$

where ξ is the unique eigenvector of Φ_1 and Φ_2 associated with ϕ^* and ϕ respectively.

Proof. We try to write the Golden structure Φ defined on (M, g), using almost contact metric structure (φ, ξ, η, g), in the form $\Phi = aI + b\eta \otimes \xi$, where $a, b \in \mathbb{R}^*$. Thus

$$\Phi^2 = a^2 I + b(2a + b)\eta \otimes \xi,$$

and using formula (2.1) we obtain the formulas (3.1). Moreover, we have

$$g(\Phi X, Y) = g(X, \Phi Y) \Leftrightarrow g(\varphi X, Y) = -g(X, \varphi Y),$$

for every tangent vectors fields X and Y on M.
On the other hand, suppose that there exist another Golden Riemannian structure on M induces

by the almost contact metric structure (φ, ξ, η, g) denoted by Ψ and admits ξ as the unique eigenvector associated with ϕ (resp. ϕ^*) then, we have

$$\Psi^2 = \Psi + I, \qquad \Psi\xi = \phi\xi \quad (resp.\, \Psi\xi = \phi^*\xi). \tag{3.2}$$

First, note that for all $i \in \{1, 2\}$ we have

$$\Phi_i \Psi = \Psi \Phi_i,$$

and using (2.1) and (3.2) we get

$$\Psi^2 - \Phi_i^2 = \Psi - \Phi_i, \qquad i = \overline{1,2}$$

wich give

$$\Psi = \Phi_i \quad or \quad \Psi = I - \Phi_i \in \{\Phi_1, \Phi_2\},$$

which completes the demonstration. □

Remark 3.2.

$$\Phi_1 + \Phi_2 = I.$$

Proposition 3.3. *If (M, Φ, g) is a Golden Riemannian manifold, then (M, Φ, G) is also a Golden Riemannian manifold, where G is a Riemannian metric given by:*

$$G(X, Y) = g(\varphi X, \varphi Y),$$

for all vectors fields X,Y on M.

Proof. Obvious. □

In all the following, we deal with

$$\Phi = \phi I - \sqrt{5}\, \eta \otimes \xi. \tag{3.3}$$

and the results will be the same if we take $\Phi = \phi^* I + \sqrt{5}\, \eta \otimes \xi$.

We know that the Golden structure Φ is intégrable if and only if $N_\Phi = 0$. So, using (2.3) and (3.3), we can check that is very simply as follows:

$$\begin{aligned} N_\Phi(X, Y) &= -10\Big(\mathrm{d}\eta(X, Y) - \mathrm{d}\eta(X, \xi)\eta(Y) - \mathrm{d}\eta(\xi, Y)\eta(X)\Big)\xi, \\ &= -10\, \mathrm{d}\eta(\varphi^2 X, \varphi^2 Y)\xi, \end{aligned} \tag{3.4}$$

witch give the following proposition:

Proposition 3.4. *Let (M, Φ, g) be a Golden Riemannian manifold induced by the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. Then Φ is integrable if η is closed.*

Remark 3.5. If $(M, \varphi, \xi, \eta, g)$ is an almost cosymplectic or an almost Kenmotsu manifold then (M, Φ, g) is an integrable Golden Riemannian manifold.

Lemma 3.6. *If (Φ, g) is a Golden Riemannian structure induced by an almost contact metric structure (φ, ξ, η, g) on M then we have*

$$\varphi\Phi = \Phi\varphi = \phi\varphi. \tag{3.5}$$

Proof. Using formulas (3.3) and (2.5), the proof is direct. □

Lemma 3.7. *Let (M, Φ, g) be a Golden Riemannian manifold induced by the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. If ∇ is the Levi-Cevita connection then for all X and Y vectors fields on M we have*

$$(\nabla_X \Phi)Y = -\sqrt{5}\big(g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi\big). \tag{3.6}$$

Proof. Knowing that
$$(\nabla_X \Phi)Y = \nabla_X \Phi Y - \Phi \nabla_X Y,$$
and using formula (3.3), the proof is direct. □

In [8], the authors show that a Golden structure is integrable if $\nabla\Phi = 0$, where ∇ is the Levi-Civita connection of g. Now we shall introduce another possible sufficient condition of the integrability of Golden structures on Riemannian manifolds.

Proposition 3.8. *Let (M, Φ, g) be a Golden Riemannian manifold induced by the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. If ∇ is the Levi-Cevita connection of g then*
$$(\nabla_X \Phi)Y = 0 \Leftrightarrow \nabla_X \xi = 0,$$
for all X and Y vector field on M.

Proof. Suppose that $\nabla\Phi = 0$, from lemma (3.7) we have
$$g(\nabla_X \xi, Y)\xi + \eta(Y)\nabla_X \xi = 0,$$
taking $Y = \xi$ we obtain $\nabla_X \xi = 0$. The inverse is direct. □

Example 3.9. For this example, we rely on our example in [3]. We denote the Cartesian coordinates in a 3-dimensional Euclidean space E^3 by (x, y, z) and define a symmetric tensor field g by
$$g = \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$
where ρ and τ are functions on E^3 such that $\rho \neq 0$ everywhere.
Further, we define an almost contact metric (φ, ξ, η) on E^3 by
$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta = (-\tau, 0, 1).$$
Using the theorem (3.1) we ge
$$\Phi_1 = \begin{pmatrix} \phi & 0 & 0 \\ 0 & \phi & 0 \\ \sqrt{5}\,\tau & 0 & \phi^* \end{pmatrix}; \qquad \Phi_2 = \begin{pmatrix} \phi^* & 0 & 0 \\ 0 & \phi^* & 0 \\ -\sqrt{5}\,\tau & 0 & \phi \end{pmatrix},$$
where we can check that $\Phi^2 = \Phi + I$.
The fundamental 1-form η have the form,
$$\eta = dz - \tau dx,$$
and hence
$$d\eta = \tau_2 dx \wedge dy + \tau_3 dx \wedge dz.$$
where $\tau_i = \frac{\partial \tau}{\partial x_i}$.
With a straightforward computation yields one gets
$$\nabla_{\partial x}\xi = \frac{1}{\rho^2}\begin{pmatrix} \rho\rho_3 + \tau\tau_3 \\ \tau_2 \\ \tau\rho\rho_3 + \tau\tau_3 \end{pmatrix}; \qquad \nabla_{\partial y}\xi = \frac{1}{\rho^2}\begin{pmatrix} -\tau_2 \\ \rho_3 \\ -\tau\tau_2 \end{pmatrix}; \qquad \nabla_{\partial z}\xi = \frac{1}{\rho^2}\begin{pmatrix} -\tau_3 \\ 0 \\ -\tau\tau_3 \end{pmatrix},$$
where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.
On the other hand, according to the cases given in [3], the structure (φ, ξ, η, g) is a:

(1) Cosymplectic when $\rho_3 = \tau_2 = \tau_3 = 0$,

(2) Kenmotsu when $\rho_3 = \rho$, $\tau_2 = 0$ and $\tau_3 = 0$.

So,

(a) If $\tau_2 = \tau_3 = 0$ (i.e. $d\eta = 0$) then the Golden structures $\Phi_{1,2}$ is integrable.

(b) If $\rho_3 = \tau_2 = \tau_3 = 0$ (i. e. $\nabla_X \xi = 0$) then the Golden structure Φ is parallel.

4 From Golden Riemannian structure to almost contact metric structure

In this section, we give the inverse study i.e. starting from a Golden Riemannian structure we construct an almost contact metric structure and we deal some questions of the characterization of certain geometric structures.
Let (M, Φ, g) be a Golden Riemannian manifold of dimension 3 and ξ be the unique global unit eigenvector of Φ associated with ϕ^* (or ϕ) which give $\Phi\xi = \phi^*\xi$ (or $\Phi\xi = \phi\xi$) and let η be the g-dual of ξ i.e. $\eta(X) = g(X, \xi)$ for all vectors field X on M such that $\eta(\xi) = 1$.
Since the two cases are equivalent, we treat this section with the first case i.e. we only study the case where $\Phi\xi = \phi^*\xi$.
Now, for brevity we denote by ψ_1 and ψ_2 the two others global unit eigenvectors of Φ associated with ϕ. Then, we have the following

Proposition 4.1. *The Golden structure Φ admits the following two expressions:*

$$\Phi_1 = \phi I - \sqrt{5}\, \eta \otimes \xi, \qquad \Phi_2 = \phi^* I + \sqrt{5} \sum_{i=1}^{2} \omega_i \otimes \psi_i, \qquad i \in \{1, 2\} \tag{4.1}$$

where ω_i are the g-dual of ψ_i i.e. $\omega_i(X) = g(X, \psi_i)$ for all vectors field X on M such that $\omega_i(\psi_i) = 1$.

Proof. We try to write the Golden structure Φ in the form $\Phi = aI + b\eta \otimes \xi$, where $a, b \in \mathbb{R}^*$. Thus

$$\Phi^2 = a^2 I + b(a + \phi^*)\eta \otimes \xi,$$

on the other hand, we have

$$\Phi + I = (a + 1)I + b\, \eta \otimes \xi,$$

using formulas (2.1) with $\Phi\xi = \phi^*\xi$, we obtain the first formula. Knowing that for all vectors field X on M we have $X = \eta(X)\xi + \omega_1(X)\psi_1 + \omega_2(X)\psi_2$, then replacing $\eta(X)\xi$ in the first formula we get the second. □

Proposition 4.2. *Let (M, Φ, g) be a Golden Riemannian manifold and the set $\{\xi, \psi_1, \psi_2\}$ of vectors fields defined as above. Then we may easily check that $\{\xi, \psi_1, \psi_2\}$ is a global orthonormal basis on M.*

Proof. The proof is direct using formulas (2.2) and (4.1). □

We refer to this basis as Golden basis and we can state the following theorem:

Theorem 4.3. *Let (M, Φ, g) be a Golden Riemannian manifold. Then, $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold where $\varphi = \psi_2 \wedge_g \psi_1$, where*

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y, \qquad \forall X, Y, Z \in \mathfrak{X}(M).$$

Proof. To prove this theorem, it is enough to check the conditions (2.4). First recall that for all X vectors field on M,

$$\begin{aligned} (\psi_2 \wedge \psi_1)X &= g(\psi_1, X)\psi_2 - g(\psi_2, X)\psi_1 \\ &= \omega_1(X)\psi_2 - \omega_2(X)\psi_1. \end{aligned}$$

So, it is easy to see that $\varphi\xi = 0$, for the second condition we have

$$\begin{aligned}\varphi^2 X &= \varphi(\varphi X)\\ &= -\big(\omega_1(X)\psi_1 + \omega_2(X)\psi_2\big)\\ &= -X + \eta(X)\xi.\end{aligned}$$

Finally, for all X and Y vectors fields on M we have

$$\begin{aligned}g(\varphi X, \varphi Y) &= \omega_1(X)\omega_1(Y) + \omega_2(X)\omega_2(Y)\\ &= g\big(X, \omega_1(Y)\psi_1 + \omega_2(Y)\psi_2\big)\\ &= g\big(X, Y - \eta(Y)\xi\big)\\ &= g(X,Y) - \eta(X)\eta(Y).\end{aligned}$$

This completes the proof of the theorem. □

We refer to this induced manifold as Golden almost contact metric manifold.
On the other hand, the fundamental 2-form Ω of (φ, ξ, η, g) is :

$$\begin{aligned}\Omega(X,Y) &= g(X, \varphi Y)\\ &= \omega_1(Y)\omega_2(X) - \omega_1(X)\omega_2(Y),\end{aligned}$$

we can check that is very simply as follows:

$$\Omega = 2\,\omega_2 \wedge \omega_1, \tag{4.2}$$

Remark 4.4. Let $(M^3, \varphi = \psi_2 \wedge \psi_1, \xi, \eta, g)$ be a Golden almost contact metric manifold then the set $\{\xi, \psi_1, \varphi\psi_1\}$ called a Golden φ-basis.

In the presented paper, we are mainly interested in dimension 3. Below we recall certain results concerning this case. For an arbitrary 3-dimensional almost contact metric manifold M, we have

$$\mathrm{d}\Omega = 2\alpha\eta \wedge \Omega. \tag{4.3}$$

A 3-dimensional almost contact metric manifold M is normal if and only if for all X vectors field on M([12], Prop. 2)

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi. \tag{4.4}$$

or, equivalently,

$$\nabla_X\xi = -\alpha\varphi^2 X - \beta\varphi X, \tag{4.5}$$

and for a normal almost contact metric manifold M we have ([12], Corollary 1)

$$\nabla_\xi\xi = 0 \quad and \quad \mathrm{d}\eta = \beta\Phi. \tag{4.6}$$

where α and β are the functions defined by $2\alpha = div\xi$ and $2\beta = tr(\varphi\nabla\xi)$ and ∇ is the Levi-Civita connection on M.

Proposition 4.5. *A 3-dimensional Golden almost contact metric manifold is normal if and only if*

$$\nabla\xi = \alpha\sum_{i=1}^{2}\omega_i \otimes \psi_i + \beta\psi_1 \wedge \psi_2.$$

where ∇ is the Levi-Civita connection on M.

Proof. Knowing that $\varphi^2 = -I + \eta \otimes \xi = -\sum_{i=1}^{2}\omega_i \otimes \psi_i$ and using formula (4.5) the proof is direct. □

As a consequence of the above formulas (4.4), (4.3), (2.7) and proposition (4.5), we immediately obtain the following result:

Theorem 4.6. *A Golden almost contact metric manifold is:*

$$\begin{cases} (1): Golden\ Sasakian \ \Leftrightarrow \quad \nabla\xi = \psi_1 \wedge \psi_2, \\ (2): Golden\ cosymplectic \ \Leftrightarrow \quad \nabla\xi = 0, \\ (3): Golden\ Kenmotsu \ \Leftrightarrow \quad \nabla\xi = \sum_{i=1}^{2} \omega_i \otimes \psi_i. \end{cases} \tag{4.7}$$

Proof. According to the cases given in formulas (2.7) we have:
(1): A Golden almost contact metric manifold is Sasakian if and only if

$$\mathrm{d}\eta = \Omega \quad \text{and} \quad \nabla\xi = \alpha \sum_{\mathrm{i}=1}^{2} \omega_{\mathrm{i}} \otimes \psi_{\mathrm{i}} + \beta\psi_1 \wedge \psi_2,$$

from $\mathrm{d}\eta = \Omega$ we obtain $\mathrm{d}\Omega = 0$ and using formulas (4.6) and (4.3) respectively we get $\beta = 1$ and $\alpha = 0$, then $\nabla\xi = \psi_1 \wedge \psi_2$.
(2): A Golden almost contact metric manifold is cosymplectic if and only if

$$\mathrm{d}\Omega = \mathrm{d}\eta = 0 \quad \text{and} \quad \nabla\xi = \alpha \sum_{\mathrm{i}=1}^{2} \omega_{\mathrm{i}} \otimes \psi_{\mathrm{i}} + \beta\psi_1 \wedge \psi_2,$$

from $\mathrm{d}\Omega = \mathrm{d}\eta = 0$ and using formulas (4.3) and (4.6) we get $\alpha = \beta = 0$, then $\nabla\xi = 0$.
(3): A Golden almost contact metric manifold is Kenmotsu if and only if

$$\mathrm{d}\eta = 0, \quad \mathrm{d}\Omega = 2\eta \wedge \Phi \quad \text{and} \quad \nabla\xi = \alpha \sum_{\mathrm{i}=1}^{2} \omega_{\mathrm{i}} \otimes \psi_{\mathrm{i}} + \beta\psi_1 \wedge \psi_2,$$

from $\mathrm{d}\eta = 0$ and $\mathrm{d}\Omega = 2\eta \wedge \Phi$ and using formulas (4.3) and (4.6) respectively we get $\beta = 0$ and $\alpha = 1$, then $\nabla\xi = \sum_{i=1}^{2} \omega_i \otimes \psi_i$.

□

Example 4.7. We denote the Cartesian coordinates in a 3-dimensional Euclidean space E^3 by (x, y, z) and define a Golden structure Φ on E^3 by

$$\Phi = \begin{pmatrix} \phi & \sqrt{5}f & 0 \\ 0 & \phi^* & 0 \\ \sqrt{5}h & \sqrt{5}fh & \phi^* \end{pmatrix},$$

where f and h are functions on E^3.
Using proposition (4.2), We can find the Golden basis as follows

$$\xi = \frac{\partial}{\partial x} + h\frac{\partial}{\partial z}, \quad \psi_1 = -f\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad \psi_2 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric difined by

$$g(\xi, \psi_1) = g(\xi, \psi_2) = g(\psi_1, \psi_2) = 0,$$

$$g(\xi, \xi) = g(\psi_1, \psi_1) = g(\psi_2, \psi_2) = 1$$

that is, the form of the metric becomes

$$g = \begin{pmatrix} 1 + h^2 & f(1 + h^2) & -h \\ f(1 + h^2) & 1 + f^2(1 + h^2) & -fh \\ -h & -fh & 1 \end{pmatrix},$$

and the corresponding 1-forms are

$$\eta = dx + f\,dy, \quad \omega_1 = dy, \quad \omega_2 = -h\,dx - fh\,dy + dz.$$

To define φ, let's use the formula $\varphi = \psi_2 \wedge \psi_1$, we get

$$\varphi = \begin{pmatrix} -fh & -f^2h & f \\ h & fh & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

where we can check that (φ, ξ, η, g) is a Golden almost contact metric structure on E^3.

On the other hand, using the Koszul formula for the Levi-Civita connection of a Riemannian metric,

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) - g([e_j, e_k], e_i),$$

where $\{e_i\}$ is an orthonormal basis, one can obtain the following:

$$\nabla_\xi \xi = \begin{pmatrix} -f(f_1 + hf_3) \\ f_1 + hf_3 \\ 0 \end{pmatrix}, \quad \nabla_{\psi_1} \xi = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\left(h_2 - f_3 - f_3h^2 - h_1 f - hf_1\right) \end{pmatrix},$$

$$\nabla_{\psi_2} \xi = \begin{pmatrix} -f\left(h_2 - f_3 - f_3h^2 - h_1 f - hf_1\right) \\ h_2 + f_3 - f_3h^2 - h_1 f - hf_1 \\ h_3 \end{pmatrix}.$$

where $f_i = \frac{\partial f}{\partial x_i}$ and $h_i = \frac{\partial h}{\partial x_i}$.

Now, according to the cases given in theorem (4.6), the structure (φ, ξ, η, g) is:

$$\begin{cases} (1) : Golden\ Sasakian \ \Leftrightarrow \quad f_1 = -2h,\ f_3 = 2,\ h_2 = h_1 f \ and\ h_3 = 0 \\ (2) : Golden\ cosymplectic \ \Leftrightarrow \quad f_1 = f_3 = h_3 = 0 \ and\ h_2 = h_1 f, \end{cases} \tag{4.8}$$

but it is never Kenmotsu. Henceforth, we can give non-trivial structures on E^3. For example:

$$\begin{cases} (1) : f = 2z,\ h = 0, \\ (2) : f = y,\ h = y^2 + 2x. \end{cases} \tag{4.9}$$

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Author information

Gherici BELDJILALI, Laboratory of Quantum Physics
and Mathematical Modeling (LPQ3M),
University of Mascara,, Algeria.
E-mail: ghercici.beldjilali@univ-mascara.dz

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