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On heredity group rings of groups acting on trees

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Abstract. Following the notation of [1, page 111], if G is a group and R is a nonzero ring of unit element 1, let $R(G)$ denote the group ring. The map $\omega : R(G) \to R$ given by $\sum\limits_{g\in G} r_g g \to \sum\limits_{g\in G} r_g$ is called the augmentation homomorphism. It is a homomorphism of rings. The kernel of this homomorphism is denoted by $\omega_R(G)$ and is called the augmentation ideal of $R(G)$. The group ring $R(G)$ is called a right hereditary ring if G is finitely generated and $\omega_R(G)$ is a right $R(G)$ projective. The group G is called R^{-1} group if all finite subgroups of G have order invertible in R. Equivalently, if H is a finite subgroup of G of order n, then the element $n.1$ has inverse in R, where 1 is the unit element of R. In this paper we prove that if R is a nonzero ring of unit element and G is a group acting on a tree X without inversions such that the group ring $R(G_v)$ of the stabilize G_v for each vertex v of X is a right hereditary ring and, $G_v \neq G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then the group ring $R(G)$ is a right hereditary ring. Furthermore, we prove that if G is a group and R is a nonzero ring of unit element such that the group ring $R(G)$ is a right hereditary ring, and H is a subgroup of G then

(1) If H is of finite index in G, then the group ring $R(H)$ is a right hereditary ring.

(2) If H is finite and normal, then the group ring $R(G/H)$ is a right hereditary ring, where G/H is the quotient group of G over H.

We have applications to tree product of the groups and HNN extension groups.

1 Introduction

For the structures of group rings we refer the readers to [8] and for the augmentation ideals, and right hereditary ring to [1, page 111]. In [2, Theorem 2.12, page 118], W. Dicks proved that if G is a group and R is a nonzero ring of unit element then the group ring $R(G)$ is a right hereditary ring if and only if there exists a tree X on which G acts without inversions in such a way that the orders of the stabilizers of the vertices are finite and R^{-1} are groups. In this paper we use this result to show that the hereditary group rings of groups acting on trees without inversions are inherited from the group rings of the stabilizers of the vertices in the sense that if R is a nonzero ring of unit element and G is a group acting on a tree X without inversions such that the group ring $R(G_v)$ of the stabilize G_v for each vertex v of X is a right hereditary ring and, $G_v \neq G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then the group ring $R(G)$ is a right hereditary ring. Now we begin a general background of groups acting on trees without inversions introduced in [1], [2], [6] or in [9] as follows. A graph X consists of two disjoint sets $V(X)$ (the set of vertices of X) and $E(X)$ (the set of edges of X) with $V(X)$ non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$ and $\eta : E(X) \to E(X)$ is an involution satisfying the conditions $\partial_0\eta = \partial_1$ and $\partial_1\eta = \partial_0$. For simplicity, if $e \in E(X)$ then we write $\partial_0(e) = 0(e)$, $\partial_1(e) = t(e)$ and $\eta(e) = \overline{e}$. This implies that $o(\overline{e}) = t(e)$, $t(\overline{e}) = o(e)$ and $\overline{\overline{e}} = e$ We say that a group G acts on a graph X without inversions if there is a group homomorphism $\phi : G \to Aut(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$, then we write $g(x)$ for $(\phi(g))(x)$. Thus, if $g \in G$, $y \in E(X)$ and then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, $g(\overline{y}) = \overline{g(y)}$. If the group G acts on the graph X and $x \in X$ (x is a vertex or edge), then

1. the stabilizer of x, denoted G_x is defined to be the set $G_x = \{g \in G : g(x) = x\}$. It is

clear that $G_x \leq G$, and if $x \in E(X)$ and $u \in \{o(x), t(x)\}$ then $G_{\overline{x}} = G_x$ and $G_x \leq G_u$.

2. the orbit of x denoted $G(x)$ is defined to be the set $G(x) = \{g(x) : g \in G\}$ It is clear that G acts on the graph X without inversions if and only if $G(\overline{e}) \neq G(e)$ for any $e \in E(X)$.

3. the set of the orbits G/X of the action of G on X is defined as $G/X = \{G(x) : x \in X\}$. G/X forms a graph called the quotient graph for the action of G on X, where $V(G/X) = \{G(v) : v \in V(X)\}$, $E(G/X) = \{G(e) : e \in E(X)\}$ and if $e \in E(X)$, then $o(G(e)) = G(o(e))$, $t(G(e)) = G(t(e))$ and $\overline{G(e)} = G(\overline{e})$.

Definition 1.1. Let G be a group acting on a tree X without inversions, and let T and Y be two subtrees of X such that $T \subseteq Y$, and each edge of Y has at least one end in T. Assume that T and Y are satisfying the following:

(i) T contains exactly one vertex from each vertex orbit.

(ii) Y contains exactly one edge y (say) from edge orbit if $G(y) = G(\overline{y})$. The pair $(T;Y)$ is called a fundamental domain for the action of G on X. For the existence of fundamental domains, we refer the readers to [4].

For the rest of this section, G is a group acting on a tree X without inversions $(T;Y)$ is the fundamental domain for the action of G on X. We have the following notation:

(1) For any vertex $v \in V(X)$, there exist a unique vertex denoted v^* of T and an element g (not necessarily unique) of G such that $g(v^*) = v$. That is, $G(v^*) = G(v)$, Moreover, if $v \in V(T)$, then $v^* = v$.

(2) For each edge $y \in E(Y)$there exists an element denoted $[y]$ of G satisfying the following:

(a) if $o(y) \in V(T)$, then $[y]((t(y))^*) = t(y)$, and $[y] = 1$ in case $y \in E(T)$.

(b) if $t(y) \in V(T)$, then $[y](o(y)) = (o(y))^*$, and $[y] = [\overline{y}]^{-1}$.

(3) For each edge $y \in E(Y)$,let $+y$ be the edge $+y = y$ if $o(y) \in V(T)$, and $+y = [y](y)$ if $t(y) \in V(T)$. It is clear that $o(+y) = (o(y))^*$, and $G_{+y} \leq G_{(o(y))^*}$ and if $y \in E(T)$, then $G_{+y} = G_y$.

2 The Main Results

We start the following lemma needed for the main results of this section For the proof, we refer the readers to [2, Theorem 2.12, page. 118].

Lemma 2.1. *Let R be a nonzero commutative ring with unit element and let G be a group. Then the augmented ideal $\omega_R(G)$ is a right $R(G)$ projective if and only if there exists a tree X such that G acts on X without inversions, the stabilize G_v for each vertex v of X is finite and is a R^{-1} group. We have the following applications of Lemma 2.1.*

Proposition 2.2. *Let R be a nonzero ring of unit element. Let H be a subgroup of finite index of a group G such that the group ring $R(G)$ is a right hereditary ring. Then the group ring $R(H)$ is a right hereditary ring.*

Proof. Since G is finitely generated and H is of finite index in G, by the Reidemeister-Schreier subgroup theorem [5, Corollary 2.8, page 93], H is finitely generated. Let the augmented ideal $\omega_R(G)$ be a right $R(G)$ projective. By Lemma 2.1, exists a tree X such that G acts on X without inversions, the stabilize G_v for each vertex v of X is finite and is a R^{-1} group. Then H acts on X by restriction. It is clear that the vertex stabilize H_v of the vertex v of X satisfies $H_v = H \cap G_v$. Since G_v is finite, H_v is finite. Since finite subgroups of H are finite subgroups of G, and G is a R^{-1} group, therefore H is a R^{-1} group. This shows that the group ring $R(H)$ is a right hereditary ring. This completes the proof. □

Proposition 2.3. *If H is a finite and normal subgroup of the group G such that the group ring $R(G)$ is a right hereditary ring, then $R(G/H)$ is a right hereditary ring.*

Proof. Since G is finitely generated, it is clear that is finitely generated. Let the augmented ideal $\omega_R(G)$ be a right $R(G)$ projective. By Lemma 2.1, exists a tree X such that G acts on X without inversions, the stabilize G_v for each vertex v of X is finite and is a R^{-1} group. Let $X^H = \{x \in X : H \leq G_x\}$. The set $X^H \neq \phi$, because H is finite, and every finite subgroup of groups acting on trees without inversions is contained in a vertex stabilizer (see [1], Th. 8.1, p.

27). Now we show that X^H is a subtree of X. Let X^H consist of more than one vertex. Let e be an edge of X^H. Then $H \leq G_e$. Since $G_e = G_{o(e)} \cap G_{t(e)}$, therefore $H \leq G_{o(e)}$, and $H \leq G_{t(e)}$. Then $o(e)$ and $t(e)$ are in X^H. Since $G_{\overline{e}} = G_e$, therefore $\overline{e}$ is in X^H. This implies that X^H is a subgraph of X. Let u and v be two vertices of X^H, and let $h \in H$. Then there exists a unique reduced path $e_1, e_2, ..., e_n$ in X joining u and v. Then it is clear that $h(e_1), h(e_2), ..., h(e_n)$ is a unique reduced path in X joining the vertices $h(u)$ and $h(v)$. Since $H \leq G_u$, and $H \leq G_v$, therefore $h(e_1), h(e_2), ..., h(e_n)$ is a unique reduced path in X joining u and v. This implies that $h(e_i) = e_i$ for $i = 1, ..., n$. Then $H \leq G_{e_i}$, for $i = 1, ..., n$, and $e_1, e_2, ..., e_n$ is a unique reduced path in X^H joining u and v. This implies that X^H is a subtree of X. Let $g \in G$, and $x \in X^H$. Then $g^{-1}Hg = H$, because H is a normal subgroup of G and, $H \leq G_x$. This implies that $H = g^{-1}Hg \leq g^{-1}G_xg = G_{g(x)}$. So $g(x) \in X^H$, and the rule $gH(x) = g(x)$ defines an action of G/H on X^H. If G/H acts on $X^H$with inversions, then there exist an element $g \in G$, and edge $e \in E(X^H)$ such that $gH(e) = g(e) = \overline{e}$. This is a contradiction because the action of G on X is without inversions. It is clear that the stabilizer of $x \in X^H$ under the action of G/H on X^H is $(G/H\ )_x = G_x/H$, where G_x is the stabilizer of x under the action of G on X. Since stabilizer of each $x \in X$ under the action of G on X is finite, therefore stabilizer of each $x \in X^H$ under the action of G/H on X^H is finite. Any finite subgroup of G/H is of the form K/H, where K is a finite subgroup of G. The assumption that G is a R^{-1} group shows K has order invertible in R. Therefore G/H is a R^{-1} group. This shows that the group ring $R(G/H)$ is a right hereditary ring. This completes the proof. □

Before we prove the main result of this paper, we introduce the following concept taken from [1, page 78]. Let H be a subgroup of a group G and let H act on a set X. Define $\equiv$ to be the relation on $G \times X$ defined as $(f, u) \equiv (g, v)$, if there exists $h \in H$ such that $f = gh$ and $u = h^{-1}(v)$. It is easy to show that $\equiv$ is an equivalence relation on $G \times X$. The equivalence class containing $(f; u)$ is denoted by $f \otimes_H u$ Thus $f \otimes_H u = \{(fh; h^{-1}(u)) : h \in H\}$. Define $G \otimes_H X$ to be the set $G \otimes_H X = \{g \otimes_H x : g \in G, x \in X\}$.

The main result of this section is the following theorem.

Theorem 2.4. *Let R be a nonzero ring of unit element. If G is a group acting on a tree X without inversions such that the group ring $R(G_v)$ of the stabilize G_v for each vertex v of X is a right hereditary ring $G_v \neq G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the ac tion of G on X is finite, then the group ring $R(G)$ is a right hereditary ring.*

Proof. Since for each vertex v of X, $G_v \neq G$ and the group ring $R(G_v)$ of the stabilize G_v is a right hereditary ring, G_v is finitely generated and the augmented ideal $\omega_R(G_v)$ is a right $R(G_v)$ projective. The case that the quotient graph G/X for the action of G on X is finite, by Lemma 4.4 of [6], G is finitely generated. The case the augmented ideal $\omega_R(G_v)$ is a right $R(G_v)$ projective, Lemma 2.1 shows that there exists a tree denoted X_v (X_v could consist of the single vertex $\{v\}$) such that G_v acts on X_v without inversions, the stabilize $(G_v)_u$ for each vertex u of X_v is finite and is a R^{-1} group. Now we show that G is a R^{-1} group. Let H be a finite subgroup of G of order n. Then H acts on X without inversions. Since $H$is finite, by Corollary 4.9 of [1, page 18], H stabilizes a vertex v of X. Thus, H is a finite subgroup of G_z, z is a vertex of X. Since G_z is R^{-1} group, this implies that $n.1$ has an inverse in R. Consequently, G is R^{-1} group. Let $(T; Y)$ be a fundamental domain for the action of G on X. Since the quotient graph G/X for the action of G on X is finite, T and Y are finite. By Lemma 4.4 of [6], G is generated by the generators of G_v, $v \in V(T)$ and by the elements $[y]$, $y \in E(Y)$. By Theorem 3.4 of [7], there exists a tree denoted as $\widetilde{X} = \widehat{X} \cup \left(\underset{v \in V(T)}{\cup} (G \otimes_{G_v} X_v) \right)$, where $\widehat{X} = \{[g; e] : g \in G, e \in E(Y)\}$ and $[g; e] = (gG_{+e}, +e)$, $V(\widetilde{X}) = \underset{v \in V(T)}{\cup} (G \otimes_{G_v} V(X_v))$ and $E(\widetilde{X}) = \widehat{X} \cup \left(\underset{v \in V(T)}{\cup} (G \otimes_{G_v} E(X_v)) \right)$. The ends and the inverse of the edge $g \otimes_{G_v} e$ are defined as follows: $t(g \otimes_{G_v} e) = g \otimes_{G_v} t(e)$, $o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e)$ and $\overline{g \otimes_{G_v} e} = g \otimes_{G_v} \overline{e}$ where $t(e), o(e)$, and $\overline{e}$ are the ends and the inverse of the edge e in X_v. G acts on $\widetilde{X}$ as follows: if $f, g \in G$, $y \in E(Y)$, $v \in V(T)$, and $u \in V(X_v)$ then $f[g; y] = [fg; y]$, $f(g \otimes_{G_v} e) = fg \otimes_{G_v} e$, and $f(g \otimes_{G_v} u) = fg \otimes_{G_v} u$. If $g \in G$ and $e \in E(Y)$ such that $g(1 \otimes_{G_v} e) = \overline{1 \otimes_{G_v} e} = 1 \otimes_{G_v} \overline{e}$, then $g \in G_v$ and $e \in E(X_v)$, $g(e) = \overline{e}$. Hence, G_v acts on X_v with inversions. This is a contradiction. This implies that G acts on $\widetilde{X}$ without inversions. Now for $g \in G$ and $x \in X_v$, it is clear that the stabilizer $G_{g\otimes_{G_v}x}$ of the vertex $g \otimes_{G_v} x$ is $g(_{G_v})g^{-1}$where $(G_v)_x$ is the stabilizer of x under the action of G_v on X_v. Since $(G_v)_x$ is finite, therefore, $G_{g\otimes_{G_v}x}$ is finite. So the stabilizer of each vertex of $\widetilde{X}$ under the action of G on $\widetilde{X}$ is finite. Thus, G satisfies the conditions of Lemma 2.1. So the augmented ideal $\omega_R(R)$ is a right $R(G)$ projective. Since G is finitely generated, this shows that the group ring $R(G)$ is a right hereditary ring. This completes the proof. □

3 Application

Now we apply Theorem 2.4 to tree product of groups and HNN groups introduced in [3]. Tree product of groups and HNN groups are examples of groups acting on trees without inversions. If $A = \prod^*_{i\in I}(A_i; U_{ij} = U_{ji})$ is a tree product of the groups A_i, $i \in I$ with amalgamation subgroups U_{ij}, $i, j \in I$, then A acts on the tree X without inversions defined as follow: $V(X) = \{(gA_i, i) : g \in A, i \in I\}$ and $E(X) = \{(gU_{ij}, ij) : g \in A, i, j \in I\}$. If y is the edge $y = (gU_{ij}, ij)$, then $o(y) = (gA_i, i)$, $t(y) = (gA_j, j)$, and $\overline{y} = (gU_{ji}, ji)$. A acts on X as follows: let $f \in A$. Then $f((gA_i, i)) = (fgA_i, i)$ and $f(gU_{ij}, ij) = (fgU_{ij}, ij)$. If $v = (gA_i, i) \in V(X)$ and $f(gU_{ij}, ij) \in E(X)$, then the stabilizer of v is $A_v = gA_ig^{-1} \cong A_i$, a conjugate of A_j and the stabilizer of y is $A_y = gU_{ij}g^{-1} \cong U_{ij}$, a conjugate of U_{ij}. The orbit of v is $A(v) = \{(agA_i, i) : a \in A, i \in I\}$ and the orbit of y is $A(y) = \{(agU_{ij}, ij) : a \in A, i \in I\}$. Now, we turn to the definition of an HNN group. Let G be a group and let I be an index set. Let $\{A_i, i \in I\}$ and $\{B_i, i \in I\}$ be two families of subgroups of G. For each $i \in I$, let $\phi_i : A_i \to B_i$ be an onto isomorphism. The group G^* of the presentation $G^* =< gen(G), t_i \mid rel(G), t_iA_it_i^{-1} = B_i, i \in I >$ is called an HNN group of base G and of associated pairs (A_i, B_i) of isomorphic subgroups of G, $i \in I$, where $< gen(G) \mid rel(G) >$ stands for any presentation of G, and $t_iA_it_i^{-1} = B_i, i \in I$ stands for the relations $t_ia_it_i^{-1} = \phi_i(a_i), a_i \in A_i$. The HNN group G^* acts on the tree X without inversions defined as follow: $V(X) = \{gG : g \in G^*\}$, and $E(X) = \{(gB_i, t_i), (gA_i, t_i^{-1}),$ where $g \in G^*$ and $i \in I$. For the edges (gB_i, t_i) and $(gA_i, t_i^{-1}), i \in I$, define $o(gB_i, t_i) = o(gA_i, t_i^{-1}) = gG$, $t(gB_i, t_i) = gt_iG$, $t(gA_i, t_i^{-1}) = gt_i^{-1}G$, and $\overline{(gB_i, t_i)} = (gt_iA_i, t_i^{-1})$ and $\overline{(gA_i, t_i^{-1})} = (gt_i^{-1}B_i, t_i)$. Let $f \in G^*$. Then for the vertex gG and the edges (gB_i, t_i) and (gA_i, t_i^{-1}) of X, define $f(gG) = fgG$, $f(gB_i, t_i) = (fgB_i, t_i)$, and $f(gA_i, t_i^{-1}) = (fgA_i, t_i^{-1})$. The stabilizer of the vertex $v = gG$ is $G^*_v = gGg^{-1}$, a conjugate of G, the stabilizers of the edges (gB_i, t_i) and (gA_i, t_i^{-1}) are gB_ig^{-1}, a conjugate of B_i and gA_ig^{-1}, a conjugate of A_i are finite for all $i \in I$. The orbits of gG, (gB_i, t_i) and (gA_i, t_i^{-1}) are $\{fG : f \in G^*\}$ and $\{(fB_i, t_i) : f \in G^*\}$.

Proposition 3.1. *Let R be a nonzero ring of unit element and $A = \prod^*_{i\in I}(A_i; U_{ij} = U_{ji})$ be a tree product of the groups $A_i, i \in I$ with amalgamation subgroups U_{ji}, $i, j \in I$ such that the group ring $R(A_i)$ a right hereditary ring , I is finite, and U_{ij} is finite for all $i, j \in I$. Then the group ring $R(A)$ is a right hereditary ring.*

Corollary 3.2. *Let R be a nonzero ring of unit element and $A = *_C A_i$, $i \in I$ be the free product of the groups A_i, $i \in I$ with amalgamation subgroup C U_{ij}, $i, j \in I$ such that the group ring $R(A_i)$ a right hereditary ring, I is finite, and C is finite for all $i, j \in I$. Then the group ring $R(A)$ is a right hereditary ring.*

Proposition 3.3. *Let R be a nonzero ring of unit element and G^* be the HNN group $G^* =< gen(G), t_i \mid rel(G), t_iA_it_i^{-1} = B_i, i \in I >$ of G base and of associated pairs (A_i, B_i) of isomorphic subgroups of G such that the group ring $R(G)$ a right hereditary ring, A_i is finite, $i, j \in I$ and I is finite. Then the group ring $R(G^*)$ is a right hereditary ring.*

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