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CERTAIN CLASSES OF RULED SURFACES IN 3-DIMENSIONAL ISOTROPIC SPACE

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Abstract In this paper, we study the ruled sufaces in the 3-dimensional isotropic space $\mathbb{I}^3$ whose the rulings are the lines associated to the Frenet vectors of the base curve. We obtain those ruled surfaces in $\mathbb{I}^3$ with zero relative curvature (analogue of the Gaussian curvature) and isotropic mean curvature.

1 Introduction

The ruled surfaces form an extensive class of surfaces in classical geometry and this fact gives rise to observe the ruled surfaces in different ambient spaces of arbitrary dimension. For example, see [2, 6, 7]. As well-known, the ruled surfaces are generated by a pair of the curves, so-called *base curve* and *director curve*. Explicitly, a ruled surface M^2 in a 3-dimensional Euclidean space $\mathbb{E}^3$ has locally the form ([9])

$$r(s,t) = \alpha(s) + t\beta(s), \tag{1.1}$$

where α and β are the base and director curves for a coordinate pair (s,t). The lines $t \longrightarrow \alpha(s_0) + t\beta(s_0)$ are called *rulings* of S. In particular; if we select the director curve to be a Frenet vector of α in (1.1), then a special class of ruled surfaces occurs. We call those *tangent developable, principal normal surface, binormal surface* of α, [1], [11]-[13]. Similarly such a surface is said to be *rectifying developable*, if the director curve is a Darboux vector of α, that is, $\tau V_1 + \kappa V_3$, where κ, τ are curvature and torsion, V_1, V_3 the tangent and binormal vectors.

On the other hand, the isotropic geometry naturally appears when properties of functions shall be geometrically visualized and interpreted via their graph surfaces [18]. As applications of isotropic geometry, the Image Processing, architectural design and microeconomics appear, [8, 19, 20].

Differential geometry of isotropic spaces have been introduced by K. Strubecker [22], H. Sachs [21], D. Palman [17] and others. Especially the reader can find a well bibliography for isotropic planes and isotropic 3-spaces in [21].

In this paper, we present several results relating to the zero curvature ruled surfaces in a 3-dimensional isotropic space whose the rulings are the Frenet vectors of the base curve.

2 Preliminaries

Isotropic space based on the following group G_6 of affine transformations (so-called *isotropic congruence transformations* or *i-motions*) is a Cayley-Klein space: (see [3]-[5], [14]-[16])

$$\begin{aligned} \mathbf{x}' &= a + \mathbf{x}\cos\phi - \mathbf{y}\sin\phi, \\ \mathbf{y}' &= b + \mathbf{x}\sin\phi - \mathbf{y}\cos\phi, \\ \mathbf{z}' &= c + d\mathbf{x} + e\mathbf{y} + \mathbf{z}. \end{aligned}$$

Consider the points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. The projection in $\mathbf{z}-$direction onto $\mathbb{R}^2$, $(x_1, x_2, x_3) \longmapsto (x_1, x_2, 0)$, is called the *top view*. In the sequel, many of metric properties in isotropic geometry (invariants under G_6) are Euclidean invariants in the top view such as the isotropic distance, so-called i-distance. *I-distance* of two points $\mathbf{x}$ and $\mathbf{y}$ is defined as the Euclidean distance of their top views, i.e.,

$$\|\mathbf{x}-\mathbf{y}\|_i = \sqrt{\sum_{j=1}^{2}(y_j - x_j)^2}.$$

Two points having the same top view are called *parallel points*. The i-metric is degenerate along the lines in $\mathbf{z}-$direction, and such lines are called *isotropic* lines. The plane containing an isotropic line is called an *isotropic plane.*

Let $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{I}^3$ be an admissible curve (i.e. without isotropic tangents) parametrized by arc-length $s \in I$. In coordinate form, one can be written as

$$\gamma(s) = (x(s), y(s), z(s)),$$

where x, y and z are smooth functions of one variable. Denote the first derivative with respect to s by a prime, etc. Then the curvature and torsion functions of γ are respectively defined by

$$\kappa(s) = x'(s)\,y''(s) - x''(s)\,y'(s) \tag{2.1}$$

and

$$\tau(s) = \frac{1}{\kappa(s)} \det(\gamma'(s), \gamma''(s), \gamma'''(s)). \tag{2.2}$$

Morever, the associated trihedron of γ is given by

$$\begin{aligned} V_1(s) &= (x'(s), y'(s), z'(s)), \\ V_2(s) &= \tfrac{1}{\kappa}(x''(s), y''(s), z''(s)), \\ V_3(s) &= (0,0,1). \end{aligned} \tag{2.3}$$

In the sequel, the Frenet's formulas of such vectors are

$$V_1' = \kappa V_2,\; V_2' = -\kappa V_1 + \tau V_2,\; V_3' = 0.$$

Let M^2 be a surface immersed in $\mathbb{I}^3$ which has no isotropic tangent planes. Such a surface M^2 is said to be *admissible* and can be parametrized by

$$X : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{I}^3 : \;\; (u_1, u_2) \longmapsto (X_1(u_1,u_2), X_2(u_1,u_2), X_3(u_1,u_2)),$$

where X_1, X_2 and X_3 are smooth and real-valued functions on a domain $D \subseteq \mathbb{R}^2$. Denote g the metric on M^2 induced from $\mathbb{I}^3$. The components of the first fundamental form of M^2 can be calculated via the induced metric g as follows:

$$E = g(X_{u_1}, X_{u_1}),\; F = g(X_{u_1}, X_{u_2}),\; G = g(X_{u_2}, X_{u_2}),$$

where $X_{u_i} = \frac{\partial X}{\partial u_i}$, $i, j \in \{1,2\}$. The unit normal vector field of M^2 is completely isotropic. Also, the components of the second fundamental form are

$$L = \frac{\det(X_{u_1u_1}, X_{u_1}, X_{u_2})}{\sqrt{g_{11}g_{22} - g_{12}^2}},\; M = \frac{\det(X_{u_1u_2}, X_{u_1}, X_{u_2})}{\sqrt{g_{11}g_{22} - g_{12}^2}},\; N = \frac{\det(X_{u_2u_2}, X_{u_1}, X_{u_2})}{\sqrt{g_{11}g_{22} - g_{12}^2}}, \tag{2.4}$$

where $X_{u_iu_j} = \frac{\partial^2 X}{\partial u_i \partial u_j}$, $i, j \in \{1,2\}$. Thus the *relative curvature* (so-called the *isotropic curvature* or *isotropic Gaussian curvature*) and the *isotropic mean curvature* are respectively defined by

$$K = \frac{LN - M^2}{EG - F^2} \tag{2.5}$$

and

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}. \tag{2.6}$$

3 Ruled Surfaces in Isotropic 3-Space

In this section, we consider the ruled surfaces in $\mathbb{I}^3$ whose rulings are the Frenet vectors of the base curve. For this, let $\alpha = \alpha(s)$ be an arc-length curve in $\mathbb{I}^3$ and $\{V_1, V_2, V_3\}$ the Frenet trihedron of α. Thus we have three types of the ruled surfaces:

$$r(s,t) = \alpha(s) + tV_1(s), \tag{3.1}$$

$$r(s,t) = \alpha(s) + tV_2(s), \tag{3.2}$$

$$r(s,t) = \alpha(s) + tV_3(s). \tag{3.3}$$

The ruled surfaces given by (3.1) and (3.2) are respectively called the *tangent developable* and the *principal normal surface* of α. We remark that the ruled surfaces given by (3.3) are not admissible. That is why we consider the following ruled surface instead of these given by

$$r(s,t) = t\alpha(s) + V_3(s), \tag{3.4}$$

which is indeed a generalized cone with the vertex at the end point of V_3.

Let M^2 be a tangent developable surface of the arc-length curve $\alpha = \alpha(s)$ in $\mathbb{I}^3$. Then we write

$$r(s,t) = \alpha(s) + tV_1(s) \tag{3.5}$$

The components of the first and the second fundamental form of M are

$$E = 1 + (t\kappa)^2, \ F = G = 1 \tag{3.6}$$

and

$$L = -t\kappa\tau, \ M = N = 0. \tag{3.7}$$

From (3.6) and (3.7), we get

$$K = 0 \text{ and } H = -\frac{\tau}{2t\kappa}. \tag{3.8}$$

Therefore, we obtain below results.

Theorem 3.1. *Let M^2 be a ruled surface given by (3.1) in $\mathbb{I}^3$. Then the following items hold:*

(i) M^2 is isotropic flat, $K = 0$,

(ii) M^2 is isotropic minimal if and only if the base curve is contained in a non-isotropic plane.

(iii) M^2 cannot have nonzero constant isotropic mean curvature.

Proof. The first item is clear. From (3.8), we have $H = 0$ if and only if $\tau = 0$. This implies the second item of the theorem. For the last item, since t is an independent variable, H is never a nonzero constant for all values (s,t). □

Suppose that M^2 is a principal surface of an arc-length curve $\alpha = \alpha(s)$ in $\mathbb{I}^3$. Thus we have

$$r(s,t) = \alpha(s) + tV_2(s). \tag{3.9}$$

The components of the fundamental forms are

$$E = (1 - t\kappa)^2 + (t\tau)^2, \ F = 0, \ G = 1 \tag{3.10}$$

and

$$L = \frac{1}{W}\left[t\tau' - t^2(\frac{\tau}{\kappa})'\kappa^2\right], \ M = \frac{\tau}{W}, \ N = 0, \tag{3.11}$$

where $W^2 = EG - F^2 = (1 - t\kappa)^2 + (t\tau)^2$. By (3.10) and (3.11), we have

$$K = \frac{\tau^2}{W^4} \text{ and } H = -\frac{t}{2W^3}\left[t(\frac{\tau}{\kappa})'\kappa^2 - \tau'\right]. \tag{3.12}$$

Theorem 3.2. *For a principal normal surface M^2 given by (3.2) in $\mathbb{I}^3$, we have:*

(i) M^2 is isotropic flat if and only if its base curve is contained in a non-isotropic plane.

(ii) M^2 is isotropic minimal if and only if its base curve is a constant curvature curve.

Proof. (3.12) immediately follows the first item of the theorem. For the second item of the theorem, we have from (3.12)

$$t(\frac{\tau}{\kappa})'\kappa^2 - \tau' = 0. \tag{3.13}$$

If we differentiate (3.13) with respect to t, then

$$(\frac{\tau}{\kappa})' = 0, \quad \frac{\tau}{\kappa} = const. \tag{3.14}$$

If we take into consideration (3.14) in (3.13), then we find $\tau = const.$ From (3.14), we obtain $\kappa = const.$ and $\tau = const.$ This completes the proof. □

Let M^2 be a generalized cone in $\mathbb{I}^3$ parametrized by

$$r(s,t) = t\alpha(s) + V_3(s). \tag{3.15}$$

Denote

$$\alpha(s) = \sum_{i=1}^{3} \lambda_i(s)V_i(s), \ \lambda_i(s) \in C^\infty. \tag{3.16}$$

Then the components of the first and the second fundamental form of M^2 are

$$E = t^2, \quad F = t\lambda_1, \quad G = \sum_{i=1}^{3} \lambda_i^2 \tag{3.17}$$

and

$$L = -\frac{\lambda_3 t^2 \kappa}{W}, \quad M = N = 0, \tag{3.18}$$

where $W^2 = t^2(\lambda_2^2 + \lambda_3^2)$. It follows from (3.17) and (3.18) that

$$K = 0 \text{ and } H = -\frac{1}{2W^3}\left[\lambda_3 t^2 \kappa \sum_{i=1}^{3} \lambda_i^2\right]. \tag{3.19}$$

Theorem 3.3. *Let M^2 be a ruled surface given by (3.15) in $\mathbb{I}^3$. Then*

(i) M^2 is isotropic flat.

(ii) M^2 is isotropic minimal if and only if the base curve is an osculating curve.

Proof. (3.19) immediately implies the first item of the theorem. Also, if $H = 0$ then we get from (3.19) that $\lambda_3 = 0$. By considering it into (3.16) , we write

$$\alpha(s) = \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s)$$

which means that the base curve $\alpha(s)$ is contained in the osculating plane of α. This proves the theorem. □

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