Palestine Journal of Mathematics

Vol. 7(1)(2018) , 107–114

FEKETE-SZEGO INEQUALITY WITH SIGMOID FUNCTION FOR CERTAIN SUBCLASSES OF MULTIVALENT BESSEL FUNCTIONS

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 30C45, 30C50; Secondary 33B99, 33C10.

Keywords and phrases: Analytic functions, $p-$valent functions, Starlike function, Convex function, Sigmoid function.

Abstract The coeffcient bounds $|b_{p+2} - \mu b_{p+1}^2|$ and $|b_{p+3}|$ for certain subclasses of p-valent Bessel functions were established. Furthemore, the Fekete-Szego type inequalities were obtained for these classes. The work was concluded by applying the modified sigmoid function.

1 Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad p \in \aleph \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : |z| < 1\}$.
Similarly, let $A_{p,k}(j)$ denote the class of functions of the form

$$g(z) = (\frac{z}{2})^p + \sum_{k=j+p}^{\infty} b_k (\frac{z}{2})^k \qquad p, j \in N = \{1, 2, 3...\} \tag{1.2}$$

which are also analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. The class is known as p valent Bessel function and was established in [7].
For the class defined in (1.1) the normalization conditions

$$\frac{f(z)}{z^{p-1}}\|_{z=0} = 0 \;\; and \;\; \frac{f'(z)}{z^{p-1}}\|_{z=0} = p$$

are classical, for the class of function defined in (1.2) the normalization

$$\frac{g(z)}{(\frac{z}{2})^{p-1}}\|_{z=0} = 0 \;\; and \;\; \frac{g'(z)}{(\frac{z}{2})^{p-1}}\|_{z=0} = \frac{p}{2}$$

holds.

Definition 1.1. Let f and g be analytic in U, then the function f is subordinate to g, if there exist a Schwarz function $w(z)$ analytic in U such that $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ and $f(z) = g(w(z))$ for all $z \in U$. This is denoted by $f \prec g$. It is also known that if g is univalent in U then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.2. Let $\varphi(z)$ be an analytic function with positive real part in U such that $\varphi(0) = 1$ and $\varphi'(0) > 0$ and maps U unto a region starlike with respect to 1 and symmetric with respect to the real axis.
A function $f(z) \in A_p$ is said to be in the class $S^*_{b,p}(\varphi)$

if

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right)\prec\varphi(z)\qquad(p\in N,\ z\in\ U)\tag{1.3}$$

A function $f(z)\in A_p$ is said to be in the class $C_{b,p}(\varphi)$ if it satisfies

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi(z)\qquad(p\in N\ \ z\in U)\tag{1.4}$$

The classes $S^*_{b,p}(\varphi)$ and $C_{b.p}(\varphi)$ were studied in [2]. For $b=1$ we have the classes $S^*_p(\varphi)$ and $C_p(\varphi)$ (see [1]) and for $p=b=1$ the classes reduced to the classes $S^*(\varphi)$ and $C(\varphi)$ which were earlier introduced and investigated in [10].These classes become the classes of starlike and convex functions of order $\alpha(0\le\alpha<1)$ respectively when

$$\varphi(z)=\frac{1+(1-2\alpha)z}{1-z}\qquad(0\le\alpha<1)$$

Similarly, for the class of functions defined in (1.3) and (1.4), we define equivalent classes for the class of function defined in (1.2)
A function $g(z)\in A_{p,k}(j)$ is in $S^*_{b,k,p}(\varphi)$ if it satisfies

$$\frac{1}{b}+\left(\frac{1}{p}\frac{zg'(z)}{g(z)}-1\right)\prec\varphi(z)\qquad(p\in N,\ z\in\ U)\tag{1.5}$$

and in $C_{b,k,p}(\varphi)$ if it satisfies

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zg''(z)}{g'(z)}\right)\prec\varphi(z)\qquad(p\in N\ \ z\in U)\tag{1.6}$$

Fekete and Szego in 1933 gave the sharp bound for the functional $|a_3-\mu a_2^2|$ for $f(z)\in S$ when μ is real. The determination of the sharp bounds for the functional $|a_3-\mu a_2^2|$ is known as the Fekete-Szego problem. And this has been investigated by several authors for different subclasses of S. [3], [5], [14]
In this paper sharp bounds for the Fekete-Szego coefficient functional are obtained for the classes of functions defined in (1.5) and (1.6).
Let Ω be the class of analytic functions of the form

$$w(z)=w_1z+w_2z^2+w_3z^3+\cdots=\sum_{k=1}^{\infty}w_kz^k\tag{1.7}$$

in the open unit disk U satisfying $|w(z)|<1$.
To prove our result we shall make use of the following lemmas

Lemma 1.3. *If $w\in\Omega$. then*

$$|w_2-tw_1^2|\le\left\{\begin{array}{cc}-t & \text{if } t\le-1\\ 1 & \text{if } -1\le t\le 1\\ t & \text{if } t>1\end{array}\right\}$$

when $t<-1$ or $t>1$, the equality holds if and only if $w(z)=z$ or one of its rotation. If $-1<t<1$, then equality holds only if $w(z)=z^2$ or one of its rotations. Equality holds for $t=-1$ if and only if

$$w(z)=z\frac{\lambda+z}{1+\lambda z}\qquad(0\le\lambda\le1)$$

or one of its rotations, while for $t=1$, the equality holds if and only if

$$w(z)=-z\frac{\lambda+z}{1+\lambda z}\qquad(0\le\lambda\le1)$$

or one of its rotation
Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$.

$$|w_2 - tw_1^2| + (1+t)|w_1|^2 \leq 1 \qquad (-1 < t \leq 0)$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \leq 1 \qquad (0 < t < 1)$$

[2], [11],[12] for details.

Lemma 1.4. *If* $w \in \Omega$*, then for any complex number* t

$$|w_2 - tw_1^2| \leq \ Max\ (1,\ |t|)$$

The result is sharp for the functions $w(z) = z$ *or* $w(z) = z^2$

[2], [10] and [12]

Lemma 1.5. *If* $w \in \Omega$*, then for any real numbers* q_1 *and* q_2*, then the following sharp estimate holds*

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^2| \leq H(q_1, q_2)$$

$$H(q_1,q_2) = \begin{cases} 1 & \textit{for } (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| & \textit{for } (q_1,q_2) \in \bigcup_{k=3}^{7} D_k \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3|q_1|+1+q_2}\right)^{\frac{1}{2}} & \textit{for } (q_1,q_2) \in D_8 \cup D_9. \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_1-1)}\right)^{\frac{1}{2}} & \textit{for } (q_1,q_2) \in D_{10} \cup D_{11} \sim \{\pm 2,\} \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \textit{for } (q_1,q_2) \in D_{12} \end{cases}$$

The extremal functions up to rotations are of the form $w(z) = z^3,\ w(z) = z$

$$w(z) = w_0(z) = \frac{(z\,[(1-\lambda)\epsilon_2 + \lambda\epsilon_1] - \epsilon_1\epsilon_2 z)}{1 - [(1-\lambda)\epsilon_1 + \lambda\epsilon_2]z}$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \qquad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z}$$

$$|\epsilon_1| = |\epsilon_2| = 1, \quad \epsilon_1 = t_0 - e^{-\frac{i\theta_0}{2}}(a \pm b), \quad t_2 = -e^{-\frac{\theta_0}{2}}(ia \pm b)$$

$$a = t_0 \cos\frac{\theta_0}{2}, \qquad b = \sqrt{i - t_0^2 \sin^2\frac{\theta_2}{2}}, \qquad \lambda = \frac{b \pm a}{2b}$$

$$t_0 = \left(\frac{2q_2(q_1^2+2) - 3q_2}{3(q_2-1)(q_1^2-4q_2)}\right)^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}}$$

$$t_2 = \left(\frac{|q_2|-1}{3(|q_1|-q_2)}\right)^{\frac{1}{2}},$$

$$\cos\frac{\theta_2}{2} = \frac{q_1}{2}\left(\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right)$$

The sets $D_k,\ \ k = 1, 2, \cdots, 12$ *are defined as follows*

$$D_1 = \left\{(q_1,q_2) : |q_1| \leq \tfrac{1}{2}, |q_2| \leq 1\right\}$$

$$D_2 = \left\{(q_1,q_2) : \tfrac{1}{2} \leq |q_1| \leq 2,\ \tfrac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \leq q_2 \leq 1\right\}$$

$$D_3 = \left\{(q_1,q_2) : |q_1| \leq \tfrac{1}{2},\ q_2 \leq -1\right\}$$

$$D_4 = \left\{(q_1,q_2) : |q_1| \geq \tfrac{1}{2},\ q_2 \leq -\tfrac{2}{3}(|q_1|+1)\right\}$$

$D_5 = \{(q_1, q_2) : |q_1| \le 2,\ q_2 \ge 1\}$

$D_6 = \left\{(q_1, q_2) : 2 \le |q_1| \le 4,\ q_2 \ge \frac{1}{12}(q_1^2 + 8)\right\}$

$D_7 = \left\{(q_1, q_2) : |q_1| \ge 4,\ q_2 \ge \frac{2}{3}(|q_1| - 1)\right\}$

$D_8 = \left\{(q_1, q_2) : \frac{1}{2} \le |q_1| \le 2,\ -\frac{2}{3}(|q_1| + 1) \le q_2 \le \frac{4}{27}(|q_1| + 1)^3 - (|q_1| + 1)\right\}$

$D_9 = \left\{(q_1, q_2) : |q - 1| \ge 2, -\frac{2}{3}(|q_1| + 1) \le q_2 \le \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\}$

$D_{10} = \left\{(q_1, q_2) : 2 \le |q_1| \le 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{1}{12}(q_1^2 + 8)\right\}$

$D_{11} = \left\{(q_1, q_2) : |q_1| \ge 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\}$

$D_{12} = \left\{(q_1, q_2) : |q_1| \ge 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2}{3}(|q_1| - 1)\right\}$

See [2], [13] for details.

Definition 1.6. Sigmoid Function

A sigmoid function which is an example of a special function is a mathematical function having an $\mathcal{S}$ shape. It is a function of the form

$$f(z) = \frac{1}{1 + e^{-z}} = \frac{1}{2} + \frac{1}{4}z - \frac{1}{48}z^3 + \frac{1}{480}z^5 - \frac{17}{8060}z^7 + \ldots$$

The modified sigmoid function, established in [8] is defined as

$$2f(z) = \frac{2}{1 + e^{-z}} = G(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \cdots$$

It was proved in [8] that $G(z)$ belongs to the class of Caratheodory function i.e $G(z) \in P \ \Rightarrow\ G(0) = 1$ and Re $G'(z) > 0$.

2 Coefficient Bound

Theorem 2.1. *Let $g(z)$ be given by (1.2) and $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ If $g(z) \in S^*_{b,k,p}(\varphi)$ then for any real number μ*

$$|b_{p+2} - \mu b_{p+1}^2| \le \begin{cases} 2bpB_1\left(\frac{B_2}{B_1} + (1 - 2\mu)bpB_1\right) & \text{if } \mu \le \sigma_1, \\ 2bpB_1 & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ 2bpB_1\left((2\mu - 1)bpB_1 - \frac{B_2}{B_1}\right) & \text{if } \mu \ge \sigma_2. \end{cases} \quad (2.1)$$

$$\sigma_1 = \frac{1}{2bpB_1}\left(\frac{B_2}{B_1} - 1 + bpB1\right), \quad \sigma_2 = \frac{1}{2bpB_1}\left(1 + \frac{B_2}{B_1} + bpB_1\right), \quad \sigma_3 = \frac{1}{2bpB_1}\left(\frac{B_2}{B_1} + bpB_1\right)$$

Furthermore, if $\sigma_1 \le \mu \le \sigma_3$ then

$$|b_{p+2} - \mu b_{p+1}^2| + \frac{1}{2bpB_1}(1 - \frac{B_2}{B_1} + bpB_1(2\mu - 1)|b_{p+1}|^2 \le 2pbB_1$$

If $\sigma_3 \le \mu \le \sigma_2$ then

$$|b_{p+2} - \mu b_{p+1}^2| + \frac{1}{2bpB_1}(1 + \frac{B_2}{B_1} - bpB_1(2\mu - 1)|b_{p+1}|^2 \le 2bpB_1$$

For any complex μ

$$|b_{p+2}-\mu b_{p+1}^2|\leq 2bpB_1\ Max\left\{1:\left|\frac{B_2}{B_1}+bpB_1(1-2\mu)\right|\right\}$$

Furthermore,

$$|b_{p+3}|\leq\frac{8pB_1}{3}H(q_1,q_2)$$

where $H(q_1,q_2)$ is as defined in Lemma 1.5 and

$$q_1=\frac{4bB_2+3pB_1^2}{2B_1}\ \ and\ \ q_2=\frac{2bB_3+3bpB_1(B_2+bpB_1^2)}{2B_1}$$

These results are sharp

Proof. If $g(z)\in S_{b,k,p}^*(\varphi)$ then there exists an analytic function $w(z)=w_1z+w_2z^2+w_3z^3+\cdots\in\Omega$ such that

$$1+\frac{1}{b}\left(\frac{zg'(z)}{pg(z)}-1\right)=\varphi(w(z))$$

$$1+\frac{1}{b}\left(\frac{zg'(z)}{pg(z)}-1\right)=1+\left(\frac{b_{p+1}(p+1)}{2bp}-\frac{b_{p+1}}{2b}\right)z+\left(\frac{b_{p+1}^2-b_{p+2}}{4b}-\frac{b_{p+1}^2(p+1)}{4bp}+\frac{b_{p+2}(p+2)}{4bp}\right)z^2$$

$$+\left(\frac{b_{p+1}b_{p+2}}{4b}-\frac{b_{p+3}b_{p+1}^3}{8b}+\frac{[b_{p+1}^3-b_{p+1}b_{p+2}](p+1)}{8bp}+\frac{b_{p+1}b_{p+2}(p+2)}{8bp}+\frac{b_{p+3}(p+3)}{8bp}\right)z^3+\cdots \tag{2.2}$$

$$\varphi(w(z))=1+B_1(w_1z+w_2z^2+w_3z^3+\cdots)+B_2(w_1z+w_2z^2+w_3z^3+\cdots)^2+B_3(w_1z+w_2z^2+w_3z^3+\cdots)^3$$

$$=1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+(B_1w_3+2B_2w_1w_2+B_3w_1^3)z^3+\cdots \tag{2.3}$$

Equating (2.2) and (2.3) we have

$$b_{p+1}=2bpB_1w_1 \tag{2.4}$$

$$b_{p+2}=2b\left\{pB_1w_2+p(B_2+bpB_1^2)w_1^2\right\} \tag{2.5}$$

and

$$b_{p+3}=\frac{8pB_1}{3}\left\{w_3+\frac{4bB_2+3pB_1^2}{2B_1}w_1w_2+\frac{2bB_3+3bpB_1(B_2+bpB_1^2)}{2B_1}w_1^3\right\} \tag{2.6}$$

By (2.4) and (2.5) we have

$$b_{p+2}-\mu b_{p+1}^2=2bpB_1\left\{w_2-vw_1^2\right\} \tag{2.7}$$

where

$$v=bpB_1\,(2\mu-1)-\frac{B_2}{B_1} \tag{2.8}$$

If $v\leq-1$, then $bpB_1(2\mu-1)-\frac{B_2}{B_1}\leq-1$, which implies

$$\mu\leq\frac{1}{2bpB_1}\left(\frac{B_2}{B_1}-1+bpB_1\right)=\sigma_1$$

By application of lemma 1.3 we have

$$|b_{p+2}-\mu b_{p+1}^2|\leq 2bpB_1\left(\frac{B_2}{B_1}+(1-2\mu)pB_1\right)$$

Next if $v\geq 1$ then we have, $bpB_1(2\mu-1)-\frac{B_2}{B_1}\geq 1$ which implies

$$\mu\geq\frac{1}{2bpB_1}\left(1+\frac{B_2}{B_1}+bpB_1\right)=\sigma_2$$

Furthermore,

$$|b_{p+2} - \mu b_{p+1}^2| \leq 2bpB_1 \left(pB_1(2\mu - 1) - \frac{B_2}{B_1} \right)$$

Suppose $-1 \leq v \leq 1$ then

$$-1 \leq bpB_1(1 - 2\mu) - \frac{B_2}{B_1} \leq 1$$

which shows that

$$|b_{p+2} - \mu b_{p+1}^2| \leq 2bpB_1$$

This is the second part of (2.1)
The sharpness of the results is a direct consequence of Lemma 1.3. Furthermore, when $\sigma_1 < \mu < \sigma_2$ the result can be improved as follows
if $-1 < v \leq 0$, then

$$-1 < bpB_1(2\mu - 1) - \frac{B_2}{B_1} \leq 0$$

which implies that $\sigma_1 < \mu \leq \sigma_3$ where

$$\sigma_3 = \frac{1}{2bpB_1} \left(\frac{B_2}{B_1} + bpB_1 \right)$$

By Lemma 1.3, (2.7) and (2.8), we have

$$\frac{1}{2bpB_1} \left| b_{p+2} - \mu b_{p+1}^2 \right| + \left(1 - \frac{B_2}{B_1} + bpB_1(2\mu - 1) \right) |w_1^2| \leq 1 \tag{2.9}$$

From (2.4) and (2.9) we have

$$\frac{1}{2bpB_1} \left| b_{p+2} - \mu b_{p+1}^2 \right| + \frac{\left(1 - \frac{B_2}{B_1} + bpB_1(2\mu - 1) \right)}{4p^2 B_1^2} |b_{p+1}|^2 \leq 1$$

$$|b_{p+2} - \mu b_{p+1}^2| + \frac{1}{2bpB_1} \left(1 - \frac{B_2}{B_1} + bpB_1(2\mu - 1) \right) |b_{p+1}|^2 \leq 2bpB_1$$

Further if $0 \leq v < 1$, then $\sigma_3 \leq \mu \leq \sigma_2$ by Lemma 1.3

$$\frac{1}{2bpB_1} \left| b_{p+2} - \mu b_{p+1}^2 \right| + \left(1 + \frac{B_2}{B_1} - bpB_1(2\mu - 1) \right) |w_1^2| \leq 1$$

which becomes

$$|b_{p+2} - \mu b_{p+1}^2| + \frac{1}{2bpB_1} \left(1 + \frac{B_2}{B_1} - bpB_1(2\mu - 1) \right) |b_{p+1}|^2 \leq 2bpB_1$$

By using Lemma 1.4 we can write

$$|b_{p+2} - \mu b_{p+1}^2| \leq 2bpB_1 \ Max \left\{ 1 : \left| \frac{B_2}{B_1} + bpB_1(1 - 2\mu) \right| \right\}$$

for any complex number μ
By Lemma 1.5 , (2.6) becomes

$$b_{p+3} = \frac{8pB_1}{3} \left\{ w_3 + q_1 w_1 w_2 + q_2 w_1^3 \right\}$$

which can further be written as

$$|b_{p+3}| \leq \frac{8pB_1}{3} H(q_1, q_2)$$

where

$$q_1 = \frac{4bB_2 + 3pB_1^2}{2B_1} \quad and \quad q_2 = \frac{2bB_3 + 3bpB_1B_2 + b^2p^2B_1^3}{2B_1}$$

Theorem 2.2. *Let $g(z)$ be given by (1.2) and $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ if $g(z) \in C_{b,k,p}(\varphi)$ then for any real number μ*

$$|b_{p+2} - \mu b_{p+1}^2| \leq \begin{cases} \frac{2bp^2 B_1}{(p+2)} \left(\frac{B_2}{B_1} - \frac{PB_1}{\eta^2}(\lambda + 2\mu p(p+2)) \right) & \text{if } \mu \leq \sigma_1, \\ \frac{2bp^2 B_1}{(p+2)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{2bp^2 B_1}{(p+2)} \left(\frac{pB_1}{\eta^2}(\lambda + 2\mu p(p+2)) - \frac{B_2}{B_1} \right) & \text{if } \mu \geq \sigma_2. \end{cases} \quad (2.10)$$

.

where

$$\sigma_1 = \frac{\eta^2}{2bp^2 B_1 (p+2)} \left(\frac{B_2}{B_1} - 1 - \frac{pB_1 \lambda}{\eta^2} \right)$$

and

$$\sigma_2 = \frac{\eta^2}{2bp^2 B_1 (p+2)} \left(1 + \frac{B_2}{B_1} - \frac{pB_1 \lambda}{\eta^2} \right), \quad \sigma_3 = \frac{\eta^2}{2p^2 B_1 (p+2)} \left(\frac{B_2}{B_1} - \frac{pB_1 \lambda}{\eta^2} \right)$$

The inequality (2.10) is sharp.
Further, the result is improved as follows
if $\sigma_1 < \mu \leq \sigma_3$ then

$$|b_{p+2} - \mu b_{p+1}^2| + \frac{\eta^2}{2p^2 B_1 (p+2)} \left(1 - \frac{B_2}{B_1} + \frac{pB_1}{R^2}(\lambda + 2\mu bp(p+2)) \right) |b_{p+1}|^2 \leq \frac{2bp^2 B_1}{(p+2)}$$

and also if $\sigma_3 \leq \mu < \sigma_2$,then

$$|b_{p+2} - \mu b_{p+1}^2| + \frac{\eta^2}{2bp^2 B_1 (p+2)} \left(1 + \frac{B_2}{B_1} + \frac{pB_1}{\eta^2}(\lambda + 2\mu bp(p+2)) \right) |b_{p+1}|^2 \leq \frac{2bp^2 B_1}{(p+2)}$$

For any complex number μ

$$|b_{p+2} - \mu b_{p+1}^2| \leq \frac{2bp^2 B_1}{(p+2)} \, Max \left\{ 1, \left| \frac{pB_1}{\eta^2}(\lambda + 2\mu bp(p+2)) - \frac{B_2}{B_1} \right| \right\}$$

Furthermore,

$$|b_{p+3}| \leq \frac{8bp^2 B_1}{Y} H(q_1, q_2)$$

where $H(q_1, q_2)$ is as defined in Lemma 1.5

$$q_1 = \frac{4(p+2)\eta^2 (B_2 - pB_1^2)}{2B_1 (p+2)\eta^3}, \qquad q_2 = \frac{2(p+2)\eta^3 B_3 + 2\kappa(p+2)B_1^3 b^2 p^2 + bp^2 \lambda k - pB_1 B_2 \eta^2}{2B_1 (p+2)\eta^3}$$

The results are sharp
where $\lambda = p^3 + 4p - 3$, $\eta = p(1-p) + 3$, $\kappa = [(p+1)^2 (p^2 + p + 1)]$, $k = p^3 - 3p^2 - 8p - 8$ and $Y = 2p^2 + 7p + 3$
Proof. If $g(z) \in C_{b,k,p}(\varphi)$ then there exist an analytic function $w_1 z + w_2 z^2 + w_3 z^3 + \cdots \in \Omega$ such that

$$1 - \frac{1}{b} + \frac{1}{bp} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} = \varphi(w(z)) \quad (2.11)$$

The proof follows from the proof of Theorem 2.1

Corollary 2.3. *Let $g(z)$ be given by (1.2) and G(z)=$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $g(z) \in S^*_{b,k,p}(\varphi)$ then for any real number μ*

$$|b_{p+2} - \mu b_{p+1}^2| \leq \begin{cases} \frac{b^2 p^2}{2} (1 - 2\mu) & \text{if } \mu \leq \sigma_1, \\ bp & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{b^2 p^2}{2} (2\mu - 1) & \text{if } \mu \geq \sigma_2. \end{cases} \quad (2.12)$$

Corollary 2.4. *Let $g(z)$ be given by (1.2) and G(z)=$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ If $g(z) \in C_{b,k,p}(\varphi)$ then for any real number μ*

$$|b_{p+2} - \mu b_{p+1}^2| \leq \begin{cases} \frac{bp^2}{(p+2)}\left(-\frac{p}{2\eta^2}(\lambda + 2\mu p(p+2))\right) & \textit{if } \mu \leq \sigma_1, \\ \frac{bp^2}{(p+2)} & \textit{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{bp^2}{(p+2)}\left(\frac{p}{2\eta^2}(\lambda + 2\mu p(p+2))\right) & \textit{if } \mu \geq \sigma_2. \end{cases} \quad (2.13)$$

3 Conclusion

In this work, certain subclasses of p-valent Bessel functions of starlike and convex functions were defined and Fekete-Szego functionals were established for these classes.

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Received: May 27, 2016.

Accepted: January 12, 2017.