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SOME UNIFIED INTEGRALS OF MITTAG LEFFLER FUNCTIONS

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Abstract In this paper we establish integral formulas involving generalized Mittag Leffler functions which are expressed in terms of Wright hypergeometric function. Some useful special cases of our main results are also considered.

1 Introduction

The Mittag Leffler functions and its various generalizations has become a center of attraction of scientists in applied mathematics and mathematical physics through several years. This function has a recent growing interest by mathematicians in the last few years due to its close relation to the fractional calculus.
In 1903, Mittag-Leffler, G. M. [2] discovered his remarkable function $E_\alpha(z)$ as follows:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where $z \in C, \alpha \in \mathcal{C}, \Re(\alpha) > 0$, the generalization of Mittag-Leffler function define as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

where $\alpha, \beta \in \mathcal{C}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, z \in C$ with C being the set of complex numbers which is known as Wiman function [10].
In 1971, Prabhakar [4] defined the function $E^{\gamma}_{\alpha,\beta}(z)$ in the following form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1.3}$$

where $\alpha, \beta, \gamma \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, z \in C$ and $(\gamma)_n$ is the pochhammer's symbol [5]. The generalization of $(\gamma)_n$ is given as

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2) \ldots (\gamma+n-1), \; n \geq 1, \tag{1.4}$$

$$= \prod_{m=1}^{n} (\gamma + m - 1), \quad (\gamma)_0 = 1, \quad \gamma \neq 0,$$

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(n)}, \qquad (\gamma)_0 = 1.$$

For this factorial function, we get

$$(\gamma)_{mn} = m^{nm} \left(\frac{\gamma}{m}\right)_n \left(\frac{\gamma+1}{m}\right)_n \ldots \left(\frac{\gamma+m-1}{m}\right)_n, \tag{1.5}$$

where m is a positive integers and n is a non-negative integer.
In 2007, Shukla and Prajaputi [8] introduced the the function

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{1.6}$$

where $\alpha,\beta\in C$, $\Re(\alpha)>0, \Re(\beta>0, \Re(\gamma)>0$, and $q\in(0,1)\cup N$.
In 2009, Salim, [7]. introduced the function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ as follows

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n\, z^n}{\Gamma(\alpha n+\beta)(\delta)_n}, \tag{1.7}$$

where $\alpha,\ \beta,\ \gamma,\ \delta\in\mathcal{C},\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(\gamma)>0,\ \Re(\delta)>0.$

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [1] and Wright ([11], [12], [13]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function (see [6] and [9]).

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),...,(\alpha_p,A_p);\\(\beta_1,B_1),...,(\beta_q,B_q);\end{array}\ z\right]=\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j+A_jk)}{\prod_{j=1}^{q}\Gamma(\beta_j+B_jk)}\frac{z^k}{k!}, \tag{1.8}$$

where $A_1,...,A_P$ and $B_1,...,B_q$ are positive real numbers such that:

$$i)\ 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>0\ and\ 0<|z|<\infty; z\neq\ 0. \tag{1.9a}$$

$$ii)\ 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j=0\ and\ 0<|z|<A_1^{-A_1}\ ...A_p^{-A_p}B_1^{B_1}...B_q^{B_q}. \tag{1.9b}$$

A special case of (1.8) is

$${}_p\Psi_q\left[\begin{array}{c}(\alpha_1,1),...,(\alpha_p,1);\\(\beta_l,1),...,(\beta_q,1);\end{array}\ z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left[\begin{array}{c}\alpha_1,...,\alpha_p;\\\beta_l,...,\beta_q;\end{array}\ z\right], \tag{1.10}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see[[11], section 1.5])

$$\begin{aligned}{}_pF_q\left[\begin{array}{c}\alpha_1,...,\alpha_p;\\\beta_l,...,\beta_q;\end{array}\ z\right]&=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n...(\alpha_p)_n}{(\beta_1)_n...(\beta_q)_n}\frac{z^n}{n!}\\&={}_p\ F_q(\alpha_1,...,\alpha_p;\beta_1...\beta_q;z).\end{aligned} \tag{1.11}$$

2 Result Required:

For our present investigation, the following interesting and useful result due to Oberhettinger, will be required [3]:

$$\int_0^{\infty}x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\,dx=2\,\lambda\,a^{-\lambda}\,\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{2.1}$$

provided $0<\Re(\mu)<\Re(\lambda)$.

3 Main results

We establish integrals involving generalized Mittag-Leffler $E_{\alpha,\beta}^{\gamma,q}$, which are expressed in terms of Wright hypergeometric function.

$$i)\ \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\ \right)^{-\lambda} E_{\alpha,\beta}^{\gamma,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=\frac{a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)}{\Gamma(\frac{\gamma}{q})\,\Gamma(\frac{\gamma+1}{q})\,...\,\Gamma(\frac{\gamma+q-1}{q})}$$

$$\times_{q+2}\psi_3\left[\begin{array}{l}(\frac{\gamma}{q},1),(\frac{\gamma+1}{q},1),\ldots,(\frac{\gamma+q-1}{q},1),(\lambda+1,1),(\lambda-\mu,1);\ \frac{y\,q^q}{a}\\ (\lambda+\mu+1,1),(\lambda,1),(\beta,\alpha),\ \_\_\_\_,\ \_\_\_\_;\end{array}\right]. \tag{3.1}$$

$$ii)\ \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\ \right)^{-\lambda} E_{\alpha,\beta}^{\gamma,q}\left(\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=\frac{a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)}{\Gamma(\frac{\gamma}{q})\,\Gamma(\frac{\gamma+1}{q})\,...\,\Gamma(\frac{\gamma+q-1}{q})}$$

$$\times_{q+2}\psi_3\left[\begin{array}{l}(\frac{\gamma}{q},1),(\frac{\gamma+1}{q},1),...,(\frac{\gamma+q-1}{q},1),(\lambda+1,1),(2\mu,2);\ \frac{y\,q^q}{2}\\ (\lambda,1),(\lambda+\mu+1,2),(\beta,\alpha)\ ,\ \_\_\_\_,\_\_\_\_;\end{array}\right]. \tag{3.2}$$

Proof of (3.1): In order to prove the main integral (3.1), we consider the left hand side of (3.1) by I, expressing $E_{\alpha,\beta}^{\gamma,q}$ as a series with the help of definition (1.6), we have

$$(i)\ I=\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\ \right)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha\,n+\beta)\,n!}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)^n\,dx,$$

changing the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha\,n+\beta)}\,\frac{y^n}{n!}\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\ \right)^{-(\lambda+n)}\,dx. \tag{3.3}$$

Evaluating the above integral with the help of (2.1) and then using (1.4) and (1.5), after a little simplification, we get the our main result(3.1).
similarly, by the same technique one can establish the result (3.2).

4 Variation of (3.1):

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\ \right)^{-\lambda} E_{\alpha,\beta}^{\gamma,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=\frac{\lambda\,a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(\beta)\Gamma(\lambda+\mu+1)}$$

$$\times_{q+2}F_3\left[\begin{array}{l}\Delta(1;\frac{\gamma}{q}),\Delta(1;\frac{\gamma+1}{q}),\ \ldots,\Delta(1;\frac{\gamma+q-1}{q}),\Delta(1;\lambda+1),\Delta(1;\lambda-\mu);\ \frac{yq^q}{a}\\ \Delta(\alpha;\beta),\Delta(1;\lambda+\mu+1),\Delta(1;\lambda),\ \_\_\_\_,\_\_\_\_,\_\_\_\_;\end{array}\right]. \tag{4.1}$$

5 Special cases:

(**i**) Taking $q=1$ in (3.1)and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} E_{\alpha,\beta}^{\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=\frac{a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)}{\Gamma(\gamma)\,\Gamma(\gamma+1)}\;{}_3\psi_3\left[\begin{array}{c}(\gamma,1),(\lambda+1,1),(\lambda-\mu,1);\\(\lambda+\mu+1,1),(\lambda,1),(\beta,\alpha);\end{array}\;\frac{y}{a}\right] \tag{5.1}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} E_{\alpha,\beta}^{\gamma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=\frac{a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)}{\Gamma(\gamma)\,\Gamma(\gamma+1)}\;{}_3\psi_3\left[\begin{array}{c}(\gamma,1),(\lambda+1,1),(2\mu,2);\\(\lambda,1),(\lambda+\mu+1,2),(\beta,\alpha);\end{array}\;\frac{y}{2}\right] \tag{5.2}$$

(**ii**) Taking $q=1$ and $\gamma=1$ in (3.1) and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} E_{\alpha,\beta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\;{}_3\psi_3\left[\begin{array}{c}(1,1),(\lambda+1,1),(\lambda-\mu,1);\\(\lambda+\mu+1,1),(\lambda,1),(\beta,\alpha);\end{array}\;\frac{y}{a}\right] \tag{5.3}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} E_{\alpha,\beta}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)\;{}_3\psi_3\left[\begin{array}{c}(1,1),(\lambda+1,1),(2\mu,2);\\(\lambda,1),(\lambda+\mu+1,2),(\beta,\alpha);\end{array}\;\frac{y}{2}\right] \tag{5.4}$$

(**iii**) Taking $q=\gamma=\beta=1$ in (3.1) and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} E_{\alpha}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\;{}_3\psi_3\left[\begin{array}{c}(1,1),(\lambda+1,1),(\lambda-\mu,1);\\(\lambda+\mu+1,1),(\lambda,1),(1,\alpha);\end{array}\;\frac{y}{a}\right] \tag{5.5}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} E_{\alpha}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)\,dx$$

$$=a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)\;{}_3\psi_3\left[\begin{array}{c}(1,1),(\lambda+1,1),(2\mu,2);\\(\lambda,1),(\lambda+\mu+1,2),(1,\alpha);\end{array}\;\frac{y}{2}\right] \tag{5.6}$$

(**iv**) Taking $q=\gamma=\beta=1$ and $\alpha=0$ in (3.1) and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda}\left(1-\frac{y}{x+a+\sqrt{x^2+2ax}}\right)^{-1}dx$$

$$=a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\;{}_3\psi_2\left[\begin{array}{c}(1,1),(\lambda+1,1),(\lambda-\mu,1);\\(\lambda+\mu+1,1),(\lambda,1);\end{array}\;\frac{y}{a}\right] \tag{5.7}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda}\left(1-\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)^{-1}dx$$

$$=a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)\;{}_3\psi_2\left[\begin{array}{c}(1,1),(\lambda+1,1),(2\mu,2);\\(\lambda,1),(\lambda+\mu+1,2);\end{array}\;\frac{y}{2}\right] \tag{5.8}$$

(**v**) Taking $q = \gamma = \beta = \alpha = 1$ in (3.1) and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda}\exp(\frac{y}{x+a+\sqrt{x^2+2ax}})\,dx$$

$$= a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\,{}_2\psi_2\left[\begin{array}{cc}(\lambda+1,1),(\lambda-\mu,1); & \frac{y}{a}\\ (\lambda+\mu+1,1),(\lambda,1); & \end{array}\right] \tag{5.9}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda}\exp(\frac{xy}{x+a+\sqrt{x^2+2ax}})\,dx$$

$$= a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)\,{}_2\psi_2\left[\begin{array}{cc}(\lambda+1,1),(2\mu,2); & \frac{y}{2}\\ (\lambda,1),(\lambda+\mu+1,2); & \end{array}\right] \tag{5.10}$$

(**vi**) Taking $q = \gamma = \beta = 1$ and $\alpha = \frac{1}{2}$ in (3.1) and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} erfc\left(\mp\frac{y}{x+a+\sqrt{x^2+2ax}}\right)^{\frac{1}{2}}$$

$$\exp\left(\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)^2\right)\,dx = a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\,{}_3\psi_3\left[\begin{array}{cc}(1,1),(\lambda+1,1),(\lambda-\mu,1); & \frac{y}{a}\\ (\lambda+\mu+1,1),(\lambda,1),(1,\frac{1}{2}); & \end{array}\right] \tag{5.11}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda} erfc\left(\mp\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)^{\frac{1}{2}}$$

$$\exp\left(\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)^2\right)\,dx = a^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)\,{}_3\psi_3\left[\begin{array}{cc}(1,1)(\lambda+1,1),(2\mu,2); & \frac{y}{2}\\ (\lambda,1),(\lambda+\mu+1,2)(1,\frac{1}{2}); & \end{array}\right] \tag{5.12}$$

where erfc denote the complementary error function.

(**vii**) Taking $q = \gamma = \alpha = 1$ and $\beta = 2$ in (3.1) and (3.2), after a little simplification, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda+1}\left(\exp\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)-1\right)\,dx$$

$$= ya^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(2\mu)\,{}_3\psi_3\left[\begin{array}{cc}(1,1),(\lambda+1,1),(\lambda-\mu,1); & \frac{y}{a}\\ (\lambda+\mu+1,1),(\lambda,1),(2,1); & \end{array}\right] \tag{5.13}$$

$$\int_0^\infty x^{\mu-2}\left(x+a+\sqrt{x^2+2ax}\,\right)^{-\lambda+1}\left(\exp\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)-1\right)\,dx$$

$$= ya^{\mu-\lambda}\,2^{1-\mu}\,\Gamma(\lambda-\mu)\,{}_3\psi_3\left[\begin{array}{cc}(1,1)(\lambda+1,1),(2\mu,2); & \frac{y}{2}\\ (\lambda,1),(\lambda+\mu+1,2),(2,1); & \end{array}\right] \tag{5.14}$$

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