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ARITHMEIC PROPERTIES OF PARTITION 5 AND 7 TUPLES WITH ODD PARTS DISTINCT

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AbstractLet $pod_{-k}(n)$ denote the number of partition $k-$tuples of any positive integer n where the odd parts in each partition are distinct. The arithmetic properties of $pod_{-k}(n)$ for the particular cases $k = 1, 2, 3$ and 4 have been studied by different authors in recent times. In this paper, we study the arithmetic properties of the partition functions $pod_{-5}(n)$ and $pod_{-7}(n)$ and establish infinite family of congruences by using Ramanujan's theta-functions. We also prove some other congruences for $pod_{-5}(n)$ and $pod_{-7}(n)$.

1 Introduction

Let $pod_{-k}(n)$ denote the number of partition k-tuples of any positive integer n where the odd parts in each partition are distinct. The generating function of $pod_{-k}(n)$ [7, p.1, (1.1)] is given by

$$\sum_{n=0}^{\infty} pod_{-k}(n)q^n = \frac{(-q;q^2)_\infty^k}{(q^2;q^2)_\infty^k} = \frac{1}{\psi(-q)^k}, \tag{1.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

and

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} \tag{1.2}$$

is a special case of the Ramanujan's general theta-function $f(a,b)$ which is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.3}$$

Other two special cases of $f(a,b)$ and of importance in this paper are

$$\phi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2} \tag{1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty. \tag{1.5}$$

The arithmetic properties of $pod_{-k}(n)$ have drawn much attention in recent years. Hirschorn and Sellers [5] studied the partition function $pod_{-1}(n)$ (often denoted by $pod(n)$) and proved some infinite family of congruences including the following congruence: For $\alpha \geq 1$,

$$pod\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod{3}. \tag{1.6}$$

They also proved some internal congruences of the following type

$$pod(81n + 17) \equiv pod(9n + 2) \pmod{27}. \tag{1.7}$$

The congruences modulo 5 and 7 for $pod(n)$ are proved by Radu and Sellers [6] by employing the method of modular form. Wang [9] also found infinite family of congruences modulo 5 for $pod(n)$. Chen and Lin [3] investigated the arithmetic properties of $pod_{-2}(n)$ and proved infinite family of congruences modulo 3 and 5. Wang [9] proved infinite family of congruences modulo 7, 9 and 11 for the partition function $pod_{-3}(n)$. More recently, Wang [7] established infinite family of congruences modulo 9 for $pod_{-4}(n)$ and also proved some internal congruences.

In sequel to above works, in this paper we study arithmetic properties the partition functions $pod_{-5}(n)$ and $pod_{-7}(n)$ and prove some congruences by employing Ramanujan's theta-function identities.

In section 3, we prove congruences modulo 2, 3, and 5 for $pod_{-5}(n)$. For example, in Theorem 3.5 we prove that, for $\alpha \geq 1$ and any odd prime p,

$$pod_{-5}\left(3p^{2\alpha}n + \frac{3(8i+p)p^{2\alpha-1} + 5}{8}\right) \equiv 0 \pmod{3}, \tag{1.8}$$

where $i = 1, 2, \cdots, p-1$. We also prove some other congruences for $pod_{-5}(n)$ modulo 2, 3, and 5.

In section 4, we investigate the partition function $pod_{-7}(n)$ and prove congruences modulo 3 and 7. For example, in Theorem 4.4 we prove, for $\alpha \geq 1$ and any odd prime p, we have

$$pod_{-7}\left(9p^{2\alpha}n + \frac{9(8i+p)p^{2\alpha-1} + 7}{8}\right) \equiv 0 \pmod{3},$$

where $i = 1, 2, \cdots, p-1$.

Section 2 is devoted to record some preliminary results for ready references in this paper.

2 Preliminary Results

Lemma 2.1. *[1, p. 286, Lemma 3.11]We have*

$$\psi(-q) = \frac{(q;q)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty}, \tag{2.1}$$

$$\chi(q) := (-q;q^2)_\infty = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty}. \tag{2.2}$$

$$f(q) = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty}. \tag{2.3}$$

Lemma 2.2. *[2, p. 49, Corollary (i)& (ii)]We have*

$$\phi(q) = \phi(q^9) + 2qf(q^3, q^{15}), \tag{2.4}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{2.5}$$

Lemma 2.3. *For any prime p and positive integer m, we have*

$$(q^{pm};q^{pm})_\infty \equiv (q^m;q^m)_\infty^p \pmod{p}, \tag{2.6}$$

$$\phi(q^p) \equiv \phi(q)^p \pmod{p}. \tag{2.7}$$

Proof. (2.6) follows from binomial theorem. (2.7) follows from (2.6) and the product representations of $\phi(q)$ in (1.4). □

Lemma 2.4. *[2, p. 51 , Example(v)]We have*

$$f(q,q^5)=\psi(-q^3)\chi(q).$$

Lemma 2.5. *[2, p. 350 , Eqn. (2.3)]We have*

$$f(q,q^2)=\frac{\phi(-q^3)}{\chi(-q)}.$$

Lemma 2.6. *[2, p. 46, Entry 30(ii) & (iii)]We have*

$$f(a,b)+f(-a,-b)=2f\left(a^3b,ab^3\right),$$

$$f(a,b)-f(-a,-b)=2af\left(\frac{b}{a},a^5b^3\right).$$

Lemma 2.7. *[2, p. 49]For any odd prime p, we have*

$$\phi(q)=\phi(q^{p^2})+\sum_{r=0}^{p-1}q^{r^2}f(q^{p(p-2r)},q^{p(p+2r)}).$$

Lemma 2.8. *[4, Theorem 2.1]For any odd prime p, we have*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f\left(q^{\frac{p^2+(2k+1)p}{2}},q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Furthermore, $\dfrac{k^2+k}{2}\not\equiv\dfrac{p^2-1}{8}\ (mod\ p)$ *for* $0\leq k\leq\frac{p-3}{2}$.

3 Congruences for $pod_{-5}(n)$

Theorem 3.1. *For any positive integer n, we have*

$$(i)\ \sum_{n=0}^{\infty}pod_{-5}(3n)q^n\equiv\frac{\phi(q)^3}{(q;q)_\infty(q^4;q^4)_\infty}\pmod 3,$$

$$(ii)\ \sum_{n=0}^{\infty}pod_{-5}(3n+1)q^n\equiv 2\psi(-q)\pmod 3,$$

$$(iii)\ pod_{-5}(3n+2)\equiv 0\pmod 3.$$

Proof. Setting $k=5$ in (1.1), we find that

$$\sum_{n=0}^{\infty}pod_{-5}(n)q^n=\frac{(-q;q^2)_\infty^5}{(q^2;q^2)_\infty^5}=\frac{1}{\psi(-q)^5},\tag{3.1}$$

Employing (2.1) and (1.4) in (3.1), we find that

$$\sum_{n=0}^{\infty}pod_{-5}(n)q^n=\frac{\phi(q)}{(q;q)_\infty^3(q^4;q^4)_\infty^3}\tag{3.2}$$

Employing (2.4) and (2.6) in (3.2), we obtain

$$\sum_{n=0}^{\infty}pod_{-5}(n)q^n\equiv\frac{\phi(q^9)}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}+2q\frac{f(q^3,q^{15})}{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}\pmod 3.\tag{3.3}$$

Extracting the terms involving q^{3n} from (3.3), then replacing q^3 by q, and using (2.7), we arrive at (i).

Extracting the terms involving q^{3n+1} from (3.3), then dividing by q and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} pod_{-5}(3n+1)q^n \equiv \frac{2f(q,q^5)}{(q;q)_\infty (q^4;q^4)_\infty} \pmod 3. \tag{3.4}$$

Employing Lemma 2.4 in (3.4) and simplifying using (2.1) and (2.2), we obtain

$$\sum_{n=0}^{\infty} pod_{-5}(3n+1)q^n \equiv 2\frac{(q^3;q^3)_\infty (q^{12};q^{12})_\infty (q^2;q^2)_\infty^2}{(q;q)_\infty^2 (q^4;q^4)_\infty^2 (q^6;q^6)_\infty} \pmod 3. \tag{3.5}$$

Simplifying (3.5) by employing (2.6) with $p=3$ and using (2.1), we complete the proof of (ii).

Since right hand side of (3.3) contains no terms involving q^{3n+2}, extracting the terms involving q^{3n+2} from (3.3), dividing by q^2 and replacing q^3 by q, we readily arrive at (iii). □

Theorem 3.2. *For any positive integer n, we have*

$$pod_{-5}(16n+j) \equiv 0 \pmod 2,$$

where j =1, 3, 5, and 6.

Proof. Using (2.1) in (3.1) and simplifying using (2.6) with $p=2$, we obtain

$$\sum_{n=0}^{\infty} pod_{-5}(n)q^n \equiv \frac{1}{(q;q)_\infty^{15}} = \frac{(q;q)_\infty}{(q^{16};q^{16})_\infty} \pmod 2. \tag{3.6}$$

Employing (1.5) in (3.6), we find that

$$\sum_{n=0}^{\infty} pod_{-5}(n)q^n \equiv \frac{1}{(q^{16};q^{16})_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \pmod 2. \tag{3.7}$$

Extracting the terms involving q^{16n+j} for j =1, 3, 5, and 6 from (3.7) and employing the fact that there exist no positive integer n such that $n(3n+1)/2$ is congruent to 1, 3, 5, or 6 modulo 16, we arrive at the desired result. □

Theorem 3.3. *We have*

(i) $\displaystyle\sum_{n=0}^{\infty} pod_{-5}(9n+1)q^n \equiv 2f(-q,q^2) \pmod 3,$

(ii) $\displaystyle pod_{-5}(9n+1) \equiv pod_{-5}\left(9p^2n + \frac{3p^2+5}{8}\right) \pmod 3,$

where p is a prime such that $p \equiv \pm 1 \pmod 8$.

Proof. Replacing q by $-q$ in (2.5) and employing in Theorem 3.1(ii), we obtain

$$\sum_{n=0}^{\infty} pod_{-5}(3n+1)q^n \equiv 2\left(f(-q^3,q^6) - q\psi(-q^9)\right) \pmod 3. \tag{3.8}$$

Extracting the terms involving q^{3n} from (3.8) and replacing q^3 by q, we arrive at (i).

To prove (ii), we set

$$\sum_{n=0}^{\infty} c(n)q^n = \phi(q). \tag{3.9}$$

Employing (2.4) in (3.9) , we obtain

$$\sum_{n=0}^{\infty} c(n)q^n = \phi(q^9) + 2qf(q^3,q^{15}). \tag{3.10}$$

Extracting the terms involving q^{3n+1}, dividing by q then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} c(3n+1)q^n = 2f(q,q^5). \tag{3.11}$$

Setting $a = q$ and $b = q^5$ in Lemma 2.6, we obtain

$$f(q,q^5) + f(-q,-q^5) = 2f(q^8,q^{16}) \tag{3.12}$$

and

$$f(q,q^5) - f(-q,-q^5) = 2qf(q^4,q^{20}). \tag{3.13}$$

Adding (3.12) and (3.13) and simplifying, we obtain

$$f(q,q^5) = f(q^8,q^{16}) + qf(q^4,q^{20}). \tag{3.14}$$

Employing (3.14) in (3.11), then extracting the terms involving q^{8n} and replacing q^8 by $-q$, we obtain

$$\sum_{n=0}^{\infty} c(24n+1)(-1)^n q^n = 2f(-q,q^2). \tag{3.15}$$

Employing Theorem 3.3(i) in (3.15) and equating the terms involving q^n, we obtain

$$c(24n+1) \equiv (-1)^n pod_{-5}(9n+1) \pmod 3. \tag{3.16}$$

Employing Lemma 2.7 in (3.9), then extracting the terms involving q^{p^2n} and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty} c(p^2n)q^n = \phi(q). \tag{3.17}$$

From (3.9) and (3.17), we deduce that

$$c(n) = c(p^2n). \tag{3.18}$$

Replacing n by $24n+1$ in (3.18), we obtain

$$c(24n+1) = c(24p^2n+p^2) = c\left(24\left(p^2n + \frac{p^2-1}{24}\right)+1\right). \tag{3.19}$$

Employing (3.16) in (3.19) and simplifying, we find that

$$pod_{-5}(9n+1) \equiv (-1)^{(p^2+1)n+(p^2-1)/24} pod_{-5}\left(9p^2n + \frac{3p^2+5}{8}\right) \pmod 3. \tag{3.20}$$

Noting $(p^2+1)n + (p^2-1)/24$ is even for any prime p and $p \equiv \pm 1 \pmod 8$, (ii) follows from (3.20). □

Theorem 3.4. *For $\alpha \geq 1$ and any odd prime p, we have*

$$\sum_{n=0}^{\infty} (-1)^{p^{2\alpha}n+(p^{2\alpha}-1)/8} pod_{-5}\left(3p^{2\alpha}n + 3\left(\frac{p^{2\alpha}-1}{8}\right)+1\right) q^n \equiv 2\psi(q) \pmod 3.$$

Proof. We will prove the result by using the method of mathematical induction on α.

Replacing q by $-q$ in Theorem 3.1(ii) and employing Lemma 2.8, we obtain

$$\sum_{n=0}^{\infty} pod_{-5}(3n+1)(-1)^n q^n \equiv 2\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \pmod 3. \tag{3.21}$$

Extracting the terms involving $q^{p^2n+(p^2-1)/8}$ from (3.21), then dividing by $q^{(p^2-1)/8}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^2n+(p^2-1)/8}pod_{-5}\left(3p^2n+3\left(\frac{p^2-1}{8}\right)+1\right)q^n \equiv 2\psi(q) \pmod 3. \tag{3.22}$$

So the result is true for $\alpha=1$.

Assume that the result is true for $\alpha=k$, so

$$\sum_{n=0}^{\infty}(-1)^{p^{2k}n+(p^{2k}-1)/8}pod_{-5}\left(3p^{2k}n+3\left(\frac{p^{2k}-1}{8}\right)+1\right)q^n \equiv 2\psi(q) \pmod 3. \tag{3.23}$$

Employing Lemma 2.8 in (3.23), we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^{2k}n+(p^{2k}-1)/8}pod_{-5}\left(3p^{2k}n+3\left(\frac{p^{2k}-1}{8}\right)+1\right)q^n$$
$$\equiv 2\sum_{k=0}^{\frac{p-3}{2}}q^{\frac{k^2+k}{2}}f\left(q^{\frac{p^2+(2k+1)p}{2}},q^{\frac{p^2-(2k+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \pmod 3. \tag{3.24}$$

Extracting the terms involving $q^{p^2n+(p^2-1)/8}$ from (3.24), then dividing by $q^{(p^2-1)/8}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^{2k}\left(p^2n+\frac{p^2-1}{8}\right)+\frac{p^{2k}-1}{8}}pod_{-5}\left(3p^{2k}\left(p^2n+\frac{p^2-1}{8}\right)+3\left(\frac{p^{2k}-1}{8}\right)+1\right)q^n$$
$$=\sum_{n=0}^{\infty}(-1)^{p^{2(k+1)n}+\frac{p^{2(k+1)}-1}{8}}pod_{-5}\left(3p^{2(k+1)n}+3\left(\frac{p^{2(k+1)}-1}{8}\right)+1\right)q^n$$
$$\equiv 2\psi(q) \pmod 3. \tag{3.25}$$

Thus, the theorem is true for $\alpha=k+1$ whenever it is true for $\alpha=k$. As the result is also true for $\alpha=1$. Hence, by principle of mathematical induction the result is true for any $\alpha\geq 1$. □

Theorem 3.5. *For $\alpha\geq 1$ and any odd prime p, we have*

$$pod_{-5}\left(3p^{2\alpha}n+\frac{3(8i+p)p^{2\alpha-1}+5}{8}\right)\equiv 0 \pmod 3,$$

where $i=1,2,\cdots,p-1$.

Proof. Extracting the terms involving $q^{pn+\frac{p^2-1}{8}}$ from (3.24), then dividing by $q^{\frac{p^2-1}{8}}$ and replacing q^n by q, we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}}pod_{-5}\left(3p^{2\alpha+1}n+3\left(\frac{p^{2\alpha+2}-1}{8}\right)+1\right)q^n \equiv 2\psi(q^p) \pmod 3, \tag{3.26}$$

where we replaced k by α. The right hand side of (3.26) contains no terms involving q^{pn+i} for $i=1,2,\cdots,p-1$, so extracting the terms involving q^{pn+i} from (3.26) and simplifying, we arrive at the desired result. □

Theorem 3.6. *For any positive integer n, we have*

$$pod_{-5}(5n+j)\equiv 0 \pmod 5,$$

where j =1, 2, 3, and 4.

Proof. Employing (2.1) in (3.1) and simplifying using (2.6) with $p=5$, we obtain

$$\sum_{n=0}^{\infty}pod_{-5}(n)q^n=\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^5(q^4;q^4)_\infty^5}\equiv\frac{(q^{10};q^{10})_\infty}{(q^5;q^5)_\infty(q^{20};q^{20})_\infty} \pmod 5. \tag{3.27}$$

Since right hand side of (3.27) contains no term involving q^{5n+j} for $j=1,2,3,$ and 4, extracting the terms involving q^{5n+j}, we complete the proof. □

4 Congruences for $pod_{-7}(n)$

Theorem 4.1. *We have*

$$(i)\ \sum_{n=0}^{\infty} pod_{-7}(3n)q^n \equiv \frac{\phi^2(q^3)}{(q;q)_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty} \pmod 3.$$

$$(ii)\ \sum_{n=0}^{\infty} pod_{-7}(3n+1)q^n \equiv \frac{\phi^3(q)f(q,q^5)}{(q;q)_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty} \pmod 3.$$

$$(iii)\ \sum_{n=0}^{\infty} pod_{-7}(3n+2)q^n \equiv \frac{f^2(q,q^5)}{(q;q)_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty} \pmod 3.$$

Proof. Setting $k = 7$ in (1.1) and simplifying by employing (2.1) and (1.4), we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(n)q^n = \frac{\phi^2(q)}{(q;q)_\infty^3(q^2;q^2)_\infty^3(q^4;q^4)_\infty^3}. \tag{4.1}$$

Employing (2.6) with $p = 3$ in (4.1), we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(n)q^n \equiv \frac{\phi^2(q)}{(q^3;q^3)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty} \pmod 3. \tag{4.2}$$

Employing (2.4) in (4.2), we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(n)q^n \equiv \frac{\phi^2(q^9) + 4q\phi(q^9)f(q^3,q^{15}) + 4q^2f^2(q^3,q^{15})}{(q^3;q^3)_\infty(q^6;q^6)_\infty(q^{12};q^{12})_\infty} \pmod 3. \tag{4.3}$$

Extracting the terms involving q^{3n} from (4.3) and replacing q^3 by q, we arrive at (i). Extracting the terms involving q^{3n+1} from (4.3), dividing by q and replacing q^3 by q, we arrive at (ii). Extracting the terms involving q^{3n+2} from (4.3), dividing by q^2 and replacing q^3 by q we complete the proof of (iii). □

Theorem 4.2. *We have*

$$(i)\ \sum_{n=0}^{\infty} pod_{-7}(9n+2)q^n \equiv \psi(-q) \pmod 3,$$

$$(ii)\ pod_{-7}(9n+5) \equiv 0 \pmod 3,$$

$$(iii)\ pod_{-7}(9n+8) \equiv 0 \pmod 3.$$

Proof. Employing Lemma 2.4 in Theorem 4.1(iii) and simplifying using (2.1), (2.2) and (2.6), we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(3n+2)q^n \equiv \frac{(q^3;q^3)_\infty^2(q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^3(q;q)_\infty^3} \equiv \frac{(q^3;q^3)_\infty(q^{12};q^{12})_\infty}{(q^6;q^6)_\infty} = \psi(-q^3) \pmod 3. \tag{4.4}$$

Extracting the terms involving q^{3n} from (4.4) and replacing q^3 by q, we arrive at (i).

Since right hand side of (4.4) contains no terms involving q^{3n+1} and q^{3n+2}, extracting the terms involving q^{3n+1} and q^{3n+2} from (4.4), we complete the proof of (ii) and (iii), respectively. □

Theorem 4.3. *For any odd prime p and $\alpha \geq 1$, we have*

$$\sum_{n=0}^{\infty}(-1)^{p^{2\alpha}n+\frac{p^{2\alpha}-1}{8}} pod_{-7}\left(9p^{2\alpha}n + 9\left(\frac{p^{2\alpha}-1}{8}\right)+2\right)q^n \equiv \psi(q) \pmod 3.$$

Proof. We will prove the result by induction on α.

Replacing q by $-q$ in Theorem 4.2(i) and employing Lemma 2.8, we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(9n+2)(-1)^n q^n \equiv \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, ^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \pmod 3. \tag{4.5}$$

Extracting the terms involving $q^{p^2n+(p^2-1)/8}$ from (4.5), then dividing by $q^{(p^2-1)/8}$ and replacing q^{p^2} by q, we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^2n+(p^2-1)/8} pod_{-7}\left(9p^2n+9\left(\frac{p^2-1}{8}\right)+2\right)q^n \equiv \psi(q) \pmod 3. \tag{4.6}$$

So the result is true for $\alpha = 1$.

Assume that the result is true for $\alpha = k$, so

$$\sum_{n=0}^{\infty}(-1)^{p^{2k}n+(p^{2k}-1)/8} pod_{-7}\left(9p^{2k}n+9\left(\frac{p^{2k}-1}{8}\right)+2\right)q^n \equiv \psi(q) \pmod 3. \tag{4.7}$$

Employing Lemma 2.8 in (4.7), we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^{2k}n+(p^{2k}-1)/8} pod_{-7}\left(9p^{2k}n+9\left(\frac{p^{2k}-1}{8}\right)+2\right)q^n$$

$$\equiv \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, ^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}) \pmod 3. \tag{4.8}$$

Extracting the terms involving $q^{p^2n+(p^2-1)/8}$ from (4.8), then dividing by $q^{(p^2-1)/8}$ and replacing q^{p^2} by q, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^{p^{2k}\left(p^2n+\frac{p^2-1}{8}\right)+\frac{p^{2k}-1}{8}} pod_{-7}\left(9p^{2k}\left(p^2n+\frac{p^2-1}{8}\right)+9\left(\frac{p^{2k}-1}{8}\right)+2\right)q^n \\
&\qquad = \sum_{n=0}^{\infty}(-1)^{p^{2(k+1)n}+\frac{p^{2(k+1)}-1}{8}} pod_{-7}\left(9p^{2(k+1)n}+9\left(\frac{p^{2(k+1)}-1}{8}\right)+2\right)q^n \\
&\equiv \psi(q) \pmod 3.
\end{aligned} \tag{4.9}$$

Thus, the theorem is true for $\alpha = k+1$ whenever it is true for $\alpha = k$. As the result is also true for $\alpha = 1$. Hence, by principle of mathematical induction the result is true for any $\alpha \geq 1$. □

Theorem 4.4. *For $\alpha \geq 1$ and any odd prime p, we have*

$$pod_{-7}\left(9p^{2\alpha}n + \frac{9(8i+p)p^{2\alpha-1}+7}{8}\right) \equiv 0 \pmod 3,$$

where $i = 1, 2, \cdots, p-1$.

Proof. Extracting the terms involving $q^{pn+\frac{p^2-1}{8}}$ from (4.8), then dividing by $q^{\frac{p^2-1}{8}}$ and replacing q^n by q, we obtain

$$\sum_{n=0}^{\infty}(-1)^{p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{8}} pod_{-7}\left(9p^{2\alpha+1}n+9\left(\frac{p^{2\alpha+2}-1}{8}\right)+2\right)q^n \equiv \psi(q^p) \pmod 3, \tag{4.10}$$

where we replaced k by α. Since right hand side of (4.10) contains no terms involving q^{pn+i} for $i = 1, 2, \cdots, p-1$, extracting the terms involving q^{pn+i} from (4.10) and simplifying, we arrive at the desired result. □

Theorem 4.5. *We have*

(*i*) $\sum_{n=0}^{\infty} pod_{-7}(3n+1)q^n \equiv f(-q,q^2) \pmod 3,$

(*ii*) $pod_{-5}(9n+1) \equiv 2pod_{-7}(3n+1) \pmod 3,$

(*iii*) $pod_{-7}(3n+1) \equiv pod_{-7}\left(3p^2n + \frac{p^2+7}{8}\right) \pmod 3,$

where p *is a prime with* $p \equiv \pm 1 \pmod 8$.

Proof. Simplifying Theorem 4.1(ii) with the help of (2.7), (1.4), and Lemma 2.4, we arrive at

$$\sum_{n=0}^{\infty} pod_{-7}(3n+1)q^n \equiv \phi^2(q)\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty} = \frac{\phi^3(q)}{\phi(q)}\frac{(q^2;q^2)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty} \pmod 3. \quad (4.11)$$

Employing (2.7), (1.4) and (2.2) in (4.11) and simplifying , we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(3n+1)q^n \equiv \frac{\phi(q^3)}{\chi(q)} \pmod 3. \quad (4.12)$$

Replacing q by $-q$ in Lemma 2.5 and employing in (4.12), we complete the proof of (i).

(ii) follows easily from Theorem 3.3(i) and Theorem 4.5(i).

To prove (iii), combining (3.15) and Theorem 4.5(i), we obtain

$$c(24n+1) \equiv (-1)^n pod_{-7}(3n+1) \pmod 3. \quad (4.13)$$

Employing (4.13) in (3.18) and simplifying, we obtain

$$pod_{-7}(3n+1) \equiv (-1)^{(p^2+1)n+\frac{p^2-1}{24}} pod_{-7}\left(3p^2n+\frac{p^2+7}{8}\right) \pmod 3. \quad (4.14)$$

Noting $(p^2+1)n + (p^2-1)/24$ is even for any prime p and $p \equiv \pm 1 \pmod 8$, we complete the proof of (iii). □

Theorem 4.6. *For any positive integer* n*, we have*

$$pod_{-7}(7n+j) \equiv 0 \pmod 7,$$

where j *=1, 2, 3, 4, 5, and 6.*

Proof. Employing Lemma 2.1 in (1.1) with $k=7$ and simplifying using (2.6), we obtain

$$\sum_{n=0}^{\infty} pod_{-7}(n)q^n = \frac{(q^2;q^2)_\infty^7}{(q;q)_\infty^7(q^4;q^4)_\infty^7} \equiv \frac{(q^{14};q^{14})_\infty}{(q^7;q^7)_\infty(q^{28};q^{28})_\infty} \pmod 7. \quad (4.15)$$

The right hand side of (4.15) contains no term involving q^{7n+j} for j =1, 2, 3, 4, 5, and 6, so extracting the terms involving q^{7n+j}, we complete the proof. □

References

[1] N. D. Baruah, J. Bora and N. Saikia: Some new proofs of modular relations for the Göllnitz-Gordon functions. *Ramanujan J.* **15**, 281-301, (2008)

[2] B. C. Berndt: *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[3] W. Y. C. Chen and B. L. S. Lin: Congruences for bipartitions with odd parts distinct. *Ramanujan J.* **25**(2), 277-293 (2011)

[4] S. P. Cui and N. S. S. Gu: Arithmetic properties of l-regular partitions. *Adv. Appl. Math.* **51**, 507-523 (2013)

[5] M. D. Hirschhorn and J. A. Sellers: Arithmetic properties of partitions with odd parts distinct. *Ramanujan J.* **22**, 273-284 (2010)

[6] S. Radu and J. A. Sellers: Congruence properties modulo 5 and 7 for the pod function. *Int. J. Number Theory*. **07**, 2249-2259 (2011)

[7] L. Wang: Arithmetic properties of partition quadruples with odd parts distinct. *Bull. Aust. Math. Soc.* **142**, 441-448 (2015)

[8] L. Wang: New congruences for partitions where the odd parts are distinct.*J. Integer Seq.* **18**, article 15.4.2 (2015)

[9] L. Wang: Arithmetic properties of partition triples with odd parts distinct. *Int. J. Number Theory.* **11**, 1791 (2015)

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