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# On the crosscap of the annihilating-ideal graph of a commutative ring

K. Selvakumar and P. Subbulakshmi

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**Abstract** Let $R$ be a non-domain commutative ring with identity and $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. We call an ideal $I_1$ of $R$, an *annihilating-ideal* if there exists a non-zero ideal $I_2$ of $R$ such that $I_1I_2 = (0)$. The *annihilating-ideal graph* of $R$ is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices $I_1$ and $I_2$ are adjacent if and only if $I_1I_2 = (0)$. In this paper, we characterize all commutative Artinian non-local rings $R$ for which $\mathbb{AG}(R)$ is planar and the crosscap of $\mathbb{AG}(R)$ is one.

## 1 Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups, see [3, 4, 5, 12, 20, 31, 23, 24, 25]. In ring theory, the structure of a ring $R$ is closely tied to ideal's behavior more than elements and so it is deserving to define a graph with vertex set as ideals instead of elements. Recently M. Behboodi and Z. Rakeei [13, 14] have introduced and investigated the annihilating-ideal graph of a commutative ring. For a non-domain commutative ring $R$, let $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. We call an ideal $I_1$ of $R$, an *annihilating-ideal* if there exists a non-zero ideal $I_2$ of $R$ such that $I_1I_2 = (0)$. The *annihilating-ideal graph* of $R$ is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices $I_1$ and $I_2$ are adjacent if and only if $I_1I_2 = (0)$. Several properties of $\mathbb{AG}(R)$ were studied by the authors in [1, 2, 13, 14, 26, 27].

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set $V$ and edge set $E$. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use $K_n$ to denote the complete graph with $n$ vertices. An $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes $m$ and $n$ is denoted by $K_{m,n}$. The girth of $G$ is the length of a shortest cycle in $G$ and is denoted by $gr(G)$. If G has no cycles, we define the girth of $G$ to be infinite. The *corona* of two graphs $G_1$ and $G_2$ is the graph $G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. A graph $G$ is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph $G$ is *planar* if and only if it contains no subdivision of $K_5$ or $K_{3,3}$(see [16, p.153]). A *minor* of $G$ is a graph obtained from $G$ by contracting edges in $G$ or deleting edges and isolated vertices in $G$. A classical theorem due to K. Wagner [30] states that a graph $G$ is planar if and only if $G$ does not have $K_5$ or $K_{3,3}$ as a minor. It is well known that if $G'$ is a minor of $G$, then $\gamma(G') \leq \gamma(G)$.

The main objective of topological graph theory is to embed a graph into a surface. By a surface, we mean a connected two-dimensional real manifold, i.e., a connected topological space such that each point has a neighborhood homeomorphic to an open disk. It is well known that any compact surface is either homeomorphic to a sphere, or to a connected sum of $g$ tori, or to a connected sum of $k$ projective planes (see [21, Theorem 5.1]). We denote $S_g$ for the surface formed by a connected sum of $g$ tori, and $N_k$ for the one formed by a connected sum of $k$ projective planes. The number $g$ is called the genus of the surface $S_g$ and $k$ is called the crosscap of $N_k$. When considering the orientability, the surfaces $S_g$ and sphere are among the orientable class and the surfaces $N_k$ are among the non-orientable one. In this paper, we mainly focus on the non-orientable cases.

A simple graph which can be embedded in $S_g$ but not in $S_{g-1}$ is called a graph of genus $g$. Similarly, if it can be embedded in $N_k$ but not in $N_{k-1}$, then we call it a graph of crosscap $k$. The notations $\gamma(G)$ and $\overline{\gamma}(G)$ are denoted for the genus and crosscap of a graph $G$, respectively. It is easy to see that $\gamma(H) \leq \gamma(G)$ and $\overline{\gamma}(H) \leq \overline{\gamma}(G)$ for all subgraph $H$ of $G$. For details on the notion of embedding of graphs in surface, one can refer to A. T. White [32].

## 2 Planarity of $\mathbb{AG}(R)$

The main goal of this section is to determine all commutative Artinian non-local rings $R$ for which $\mathbb{AG}(R)$ is planar.

**Theorem 2.1.** [26] Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a commutative ring with identity where each $F_i$ is a field and $n \geq 2$. Then $\mathbb{AG}(R)$ is planar if and only if $n = 2$ or $n = 3$.

**Theorem 2.2.** Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n \geq 2$. Let $n_i$ be the nilpotency of $\mathfrak{m}_i$. Then $\mathbb{AG}(R)$ is planar if and only if $n = 2$ and one of the following condition holds:

(i) $n_1 = 2, n_2 = 3$ and $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$ and $\mathfrak{m}_2, \mathfrak{m}_2^2$ are the only non-trivial ideals in $R_2$;

(ii) $n_1 = 3$, $n_2 = 2$ and $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in $R_1$ and $\mathfrak{m}_2$ is the only non-trivial ideal in $R_2$;

(iii) $n_1 = n_2 = 2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideal in $R_1$ and $R_2$ respectively.

*Proof.* Suppose that $n = 2$. Then $R = R_1 \times R_2$ and hence the proof follows from Fig. 2.1.

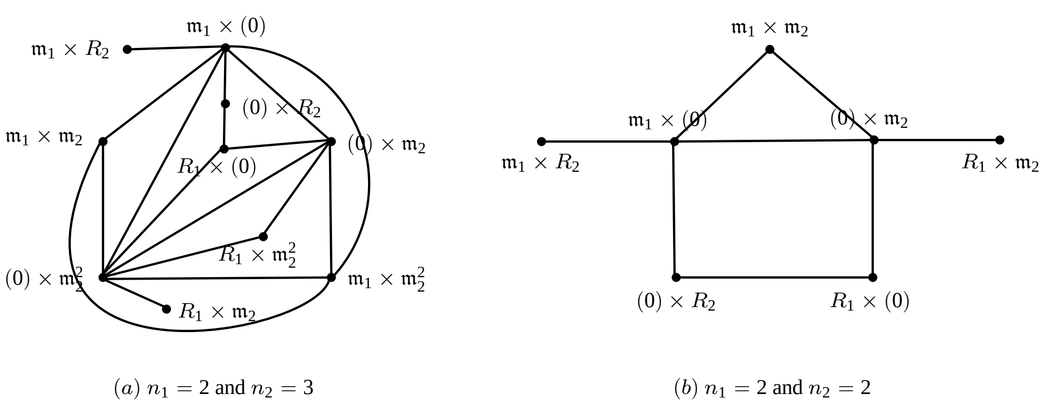



Fig 2.1: $\mathbb{AG}(R_1 \times R_2)$

Conversely, assume that $\mathbb{AG}(R)$ is planar. Suppose that $n > 2$. Consider the non-trivial ideals $x_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0) \times \cdots \times (0)$, $x_2 = (0) \times \mathfrak{m}_2^{n_2-1} \times (0) \times \cdots \times (0)$, $x_3 = (0) \times (0) \times \mathfrak{m}_3^{n_3-1} \times \cdots \times (0)$, $y_1 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0) \times \cdots \times (0)$, $y_2 = (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times \cdots \times (0)$, $y_3 = \mathfrak{m}_1 \times (0) \times \mathfrak{m}_3 \times \cdots \times (0)$ in $R$. Then $x_i y_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $n = 2$.

Suppose that $n_1 > 2$ and $n_2 > 2$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1^{n_1-1} \times (0)$, $u_2 = (0) \times \mathfrak{m}_2^{n_2-1}$, $u_3 = \mathfrak{m}_1^{n_1-1} \times \mathfrak{m}_2^{n_2-1}$, $u_4 = \mathfrak{m}_1 \times (0)$, $u_5 = (0) \times \mathfrak{m}_2$ in $R$. Then $u_i u_j = (0)$ for every $i \neq j$ and so $K_5$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $n_1 = 2$ or $n_2 = 2$. Without loss of generality, we assume that $n_1 = 2$.

Suppose that $n_2 > 3$. Consider the non-trivial ideals $a_1 = (0) \times \mathfrak{m}_2^{n_2-1}$, $a_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-1}$, $a_3 = \mathfrak{m}_1 \times (0)$, $b_1 = (0) \times \mathfrak{m}_2$, $b_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $b_3 = (0) \times \mathfrak{m}_2^{n_2-2}$ in $R$. Then $a_i b_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $n_2 \leq 3$.

Suppose that $n_2 = 3$. Let $I_2$ be any non-trivial ideal in $R_2$ with $I_2 \neq \mathfrak{m}_2, \mathfrak{m}_2^2$. Consider the non-trivial ideals $x_1 = (0) \times \mathfrak{m}_2^2$, $x_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $x_3 = \mathfrak{m}_1 \times (0)$, $y_1 = (0) \times \mathfrak{m}_2$, $y_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $y_3 = (0) \times I_2$ in $R$. Then $x_i y_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_2, \mathfrak{m}_2^2$ are the only non-trivial ideals in $R_2$.

Let $I_1$ be any non-trivial ideal in $R_1$ with $I_1 \neq \mathfrak{m}_1$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $u_2 = (0) \times \mathfrak{m}_2^2$, $u_3 = \mathfrak{m}_1 \times (0)$, $u_4 = I_1 \times (0)$, $u_5 = (0) \times \mathfrak{m}_2$ in $R$. Then $u_i u_j = (0)$ for every $i \neq j$ and so $K_5$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$.

Suppose that $n_2 = 2$. Let $I_2$ be any non-trivial ideal in $R_2$ with $I_2 \neq \mathfrak{m}_1$. Consider the non-trivial ideals $x_1 = (0) \times \mathfrak{m}_2$, $x_2 = (0) \times I_2$, $x_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $x_4 = \mathfrak{m}_1 \times I_2$, $x_5 = \mathfrak{m}_1 \times (0)$ in $R$. Then $x_i x_j = (0)$ for every $i \neq j$ and so $K_5$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_2$ is the only non-trivial ideal in $R_2$. Similarly one can prove that $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$. □

**Lemma 2.3.** Let $(R, \mathfrak{m})$ be a local ring. If $\dim(\mathfrak{m}/\mathfrak{m}^2) = 1$ and for some positive integer $t$, $\mathfrak{m}^t = (0)$, then the set of all non-trivial ideals of $R$ is the set $\{\mathfrak{m}^i : 1 \leq i < t\}$.

*Proof.* Since $\dim(\mathfrak{m}/\mathfrak{m}^2) = 1$, by Nakayama's lemma, $\mathfrak{m} = Rx$ for some $x \in R$. Now, let $I$ be a non-trivial ideal of $R$. Since $\mathfrak{m}^t = (0)$, there exists a natural number $i \leq t$ such that $I \subseteq \mathfrak{m}^i$ and $I \nsubseteq \mathfrak{m}^{i+1}$. Let $a \in I \backslash \mathfrak{m}^{i+1}$. We have $a = bx^i$ for some $b \in R$. If $b \in \mathfrak{m}$, then $a \in \mathfrak{m}^{i+1}$, a contradiction. Thus $b$ is an unit. Hence $x^i \in I$. This implies that $I = \langle x^i \rangle = \mathfrak{m}^i$, as desired. Thus, the set of all non-trivial ideals of $R$ is the set $\{\mathfrak{m}^i : 1 \leq i < t\}$. □

The next Proposition has a crucial role in this paper.

**Proposition 2.4.** If $(R, \mathfrak{m})$ is a local ring and there is an ideal $I$ of $R$ such that $I \neq \mathfrak{m}^i$ for every $i$, then $R$ has at least three distinct non-trivial ideals $J$, $K$ and $L$ such that $J$, $K$, $L \neq \mathfrak{m}^i$ for every $i$.

*Proof.* Assume that $R$ has an ideal $I$ such that $I \neq \mathfrak{m}^i$ for every $i$. Then by Lemma 2.3, $\dim(\mathfrak{m}/\mathfrak{m}^2) = n \geq 2$. Therefore, by Nakayama's Lemma, we can find $x_1, x_2, \ldots, x_n \notin \mathfrak{m}^2$ such that $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$. Thus, $Rx_1$, $Rx_2$ and $R(x_1 + x_2)$ are the distinct non-trivial ideals with desired properties. □

**Theorem 2.5.** Let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $F_j$ is a field, $n, m \geq 1$. Let $n_i$ be the nilpotency of $\mathfrak{m}_i$. Then $\mathbb{AG}(R)$ is planar if and only if one of the following condition holds:

(i) $R = R_1 \times F_1 \times F_2$, $n_1 = 2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$;

(ii) $R = R_1 \times F_1$ and one of the following holds:

  (a) $n_1 = 2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$;

  (b) $n_1 = 3$ and $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in $R_1$;

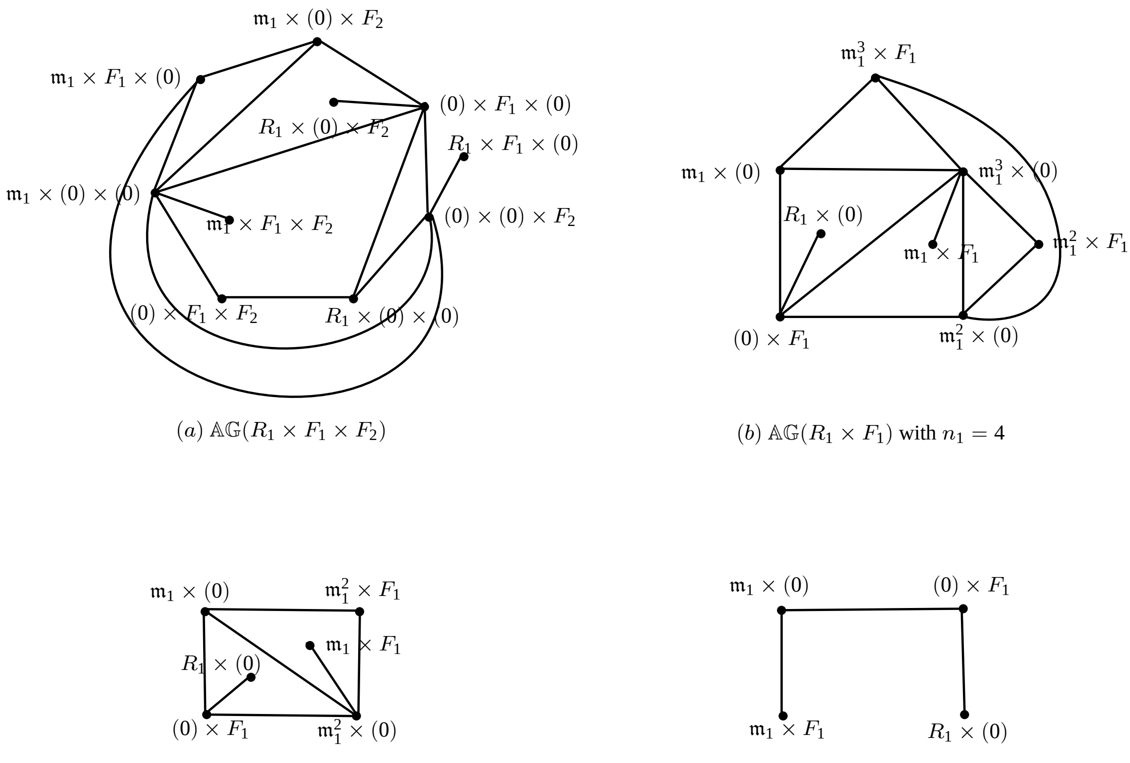
  (c) $n_1 = 4$ and $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in $R_1$.

*Proof.* Suppose $R = R_1 \times F_1 \times F_2$, $n_1 = 2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$. Then $\mathbb{AG}(R)$ is isomorphic to the graph given in Fig 2.2$(a)$. Hence $\mathbb{AG}(R)$ is planar.

Suppose $R = R_1 \times F_1$, $n_1 = 2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$. Then $\mathbb{AG}(R)$ is isomorphic to the graph given in Fig 2.2$(d)$. Hence $\mathbb{AG}(R)$ is planar.

Suppose $n_1 = 3$ and $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in $R_1$. Then $\mathbb{AG}(R)$ is isomorphic to the graph given in Fig 2.2$(c)$. Hence $\mathbb{AG}(R)$ is planar.

Suppose $n_1 = 4$ and $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in $R_1$. Then $\mathbb{AG}(R)$ is isomorphic to the graph given in Fig 2.2$(b)$. Hence $\mathbb{AG}(R)$ is planar.




$(a)$ $\mathbb{AG}(R_1 \times F_1 \times F_2)$

$(b)$ $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 4$

$(c)$ $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 3$

$(d)$ $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 2$

Fig 2.2

Conversely, assume that $\mathbb{AG}(R)$ is planar. Suppose $n > 1$. Consider the non-trivial ideals $x_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0) \times \cdots \times (0)$, $x_2 = (0) \times \mathfrak{m}_2^{n_2-1} \times (0) \times \cdots \times (0)$, $x_3 = \mathfrak{m}_1^{n_1-1} \times \mathfrak{m}_2^{n_2-1} \times (0) \times \cdots \times (0)$, $y_1 = (0) \times (0) \times \cdots \times F_1 \times (0) \times \cdots \times (0)$, $y_2 = \mathfrak{m}_1 \times (0) \times \cdots \times F_1 \times (0) \times \cdots \times (0)$, $y_3 = (0) \times \mathfrak{m}_2 \times \cdots \times F_1 \times (0) \times \cdots \times (0)$ in $R$. Then $x_i y_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $n = 1$.

Suppose that $m > 2$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times (0) \times (0) \times \cdots \times (0)$, $u_2 = \mathfrak{m}_1 \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $u_3 = (0) \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $v_1 = (0) \times (0) \times F_2 \times (0) \times \cdots \times (0)$, $v_2 = (0) \times (0) \times (0) \times F_3 \times \cdots \times (0)$, $v_3 = (0) \times (0) \times F_2 \times F_3 \times \cdots \times (0)$ in $R$. Then $u_i v_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $m \leq 2$.

Suppose that $m = 2$. Let $n_1 > 2$. Consider the non-trivial ideals $a_1 = R_1 \times (0) \times (0)$, $a_2 = \mathfrak{m}_1 \times (0) \times (0)$, $a_3 = \mathfrak{m}_1^2 \times (0) \times (0)$, $b_1 = (0) \times F_1 \times (0)$, $b_2 = (0) \times (0) \times F_2$, $b_3 = (0) \times F_1 \times F_2$ in $R$. Then $a_i b_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $n_1 = 2$.

Suppose that $I$ is any non-trivial ideal in $R_1$ with $I \neq \mathfrak{m}_1$. Consider the non-trivial ideals $d_1 = R_1 \times (0) \times (0)$, $d_2 = \mathfrak{m}_1 \times (0) \times (0)$, $d_3 = I \times (0) \times (0)$, $e_1 = (0) \times F_1 \times (0)$, $e_2 = (0) \times (0) \times F_2$, $e_3 = (0) \times F_1 \times F_2$ in $R$. Then $d_i e_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$.

Suppose that $m = 1$. Let $n_1 > 4$. Consider the non-trivial ideals $x_1 = \mathfrak{m}_1^{n_1-1} \times (0)$, $x_2 = \mathfrak{m}_1^{n_1-2} \times (0)$, $x_3 = \mathfrak{m}_1^{n_1-3} \times (0)$, $y_1 = (0) \times F_1$, $y_2 = \mathfrak{m}_1^{n_1-1} \times F_1$, $y_3 = \mathfrak{m}_1^{n_1-2} \times F_1$ in $R$. Then $x_i y_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $n_1 \leq 4$.

Assume that $n_1 = 2$. Suppose there is an ideal $I$ of $R_1$ such that $I \neq \mathfrak{m}_1^i$ for $i = 1, 2$. Then by Proposition 2.4, $R_1$ has at least three distinct non-trivial ideals $I_1$, $I_2$ and $I_3$ such that $I_1$, $I_2$, $I_3 \neq \mathfrak{m}_1$. Consider the non-trivial ideals $d_1 = \mathfrak{m}_1 \times (0)$, $d_2 = I_1 \times (0)$, $d_3 = I_2 \times (0)$, $e_1 = (0) \times F_1$, $e_2 = \mathfrak{m}_1 \times F_1$, $e_3 = I_1 \times F_1$ in $R$. Then $d_i e_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$.

Assume that $n_1 = 3$. Suppose there is an ideal $I$ of $R_1$ such that $I \neq \mathfrak{m}_1^i$ for all $i = 1, 2, 3$. Then by Proposition 2.4, $R_1$ has at least three distinct non-trivial ideals $I_1$, $I_2$ and $I_3$ such that $I_1$, $I_2$, $I_3 \neq \mathfrak{m}_1^i$ for all $i = 1, 2$. Consider the non-trivial ideals $u_1 = \mathfrak{m}_1 \times (0)$, $u_2 = I_1 \times (0)$, $u_3 = I_2 \times (0)$, $v_1 = (0) \times F_1$, $v_2 = \mathfrak{m}_1^2 \times F_1$, $v_3 = \mathfrak{m}_1^2 \times (0)$ in $R$. Then $u_i v_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_1, \mathfrak{m}_1^2$ are the only non-trivial ideals in $R_1$.

Assume that $n_1 = 4$. Let $I$ be any non-trivial ideal in $R_1$ with $I \neq \mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$. Consider the non-trivial ideals $q_1 = \mathfrak{m}_1^2 \times (0)$, $q_2 = \mathfrak{m}_1 \times (0)$, $q_3 = I \times (0)$, $w_1 = (0) \times F_1$, $w_2 = \mathfrak{m}_1^3 \times F_1$, $w_3 = \mathfrak{m}_1^3 \times (0)$ in $R$. Then $q_i w_j = (0)$ for every $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$, a contradiction. Hence $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in $R_1$. $\square$

## 3 Crosscap of $\mathbb{AG}(R)$

The main goal of this section is to determine all commutative Artinian non-local rings $R$ for which $\mathbb{AG}(R)$ has crosscap one. The following two results about the crosscap formulae of a complete graph and a complete bipartite graph are very useful in the subsequent sections.

**Lemma 3.1.** Let $m$, $n$ be integers and for a real number $x$, $\lceil x \rceil$ is the least integer that is greater than or equal to $x$. Then

$$(i)\ \overline{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil & \text{if } n \geq 3 \text{ and } n \neq 7 \\ 3 & if\ n = 7 \end{cases}$$

In particular, $\overline{\gamma}(K_n) = 1$ if $n = 5, 6$.

$(ii)$ $\overline{\gamma}(K_{m,n}) = \lceil \frac{1}{2}(m-2)(n-2) \rceil$, where $n, m > 1$. In particular, $\overline{\gamma}(K_{3,n}) = 1$ if $n = 3, 4$.

**Theorem 3.2.** Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a commutative ring with identity where each $F_i$ is a field and $n > 1$. Then $\overline{\gamma}(\mathbb{AG}(R)) = 1$ if and only if $n = 4$.

*Proof.* Assume that $\overline{\gamma}(\mathbb{AG}(R)) = 1$. Suppose that $n > 4$. Consider the non-trivial ideals $x_1 = F_1 \times (0) \times (0) \times (0) \times (0) \times \cdots \times (0)$, $x_2 = (0) \times F_2 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $x_3 = F_1 \times F_2 \times (0) \times (0) \times (0) \times \cdots \times (0)$, $y_1 = (0) \times (0) \times F_3 \times (0) \times (0) \times \cdots \times (0)$, $y_2 = (0) \times (0) \times (0) \times F_4 \times (0) \times \cdots \times (0)$, $y_3 = (0) \times (0) \times (0) \times (0) \times F_5 \times \cdots \times (0)$, $y_4 = (0) \times (0) \times F_3 \times F_4 \times (0) \times \cdots \times (0)$, $y_5 = (0) \times (0) \times F_3 \times (0) \times F_5 \times \cdots \times (0)$ in $R$. Then $x_i y_j = (0)$ for every $i, j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Theorem 2.1, $n = 4$.

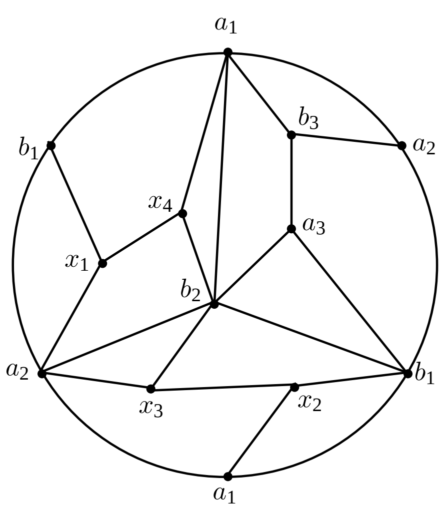



Fig 3.1: Projective embedding of $\mathbb{AG}(F_1 \times F_2 \times F_3 \times F_4)$

Conversely, suppose that $n = 4$. Consider all the non-trivial ideals $a_1 = F_1 \times (0) \times (0) \times (0)$, $a_2 = (0) \times F_2 \times (0) \times (0)$, $a_3 = F_1 \times F_2 \times (0) \times (0)$, $b_1 = (0) \times (0) \times F_3 \times (0)$, $b_2 = (0) \times (0) \times (0) \times F_4$, $b_3 = (0) \times (0) \times F_3 \times F_4$, $x_1 = F_1 \times (0) \times (0) \times F_4$, $x_2 = (0) \times F_2 \times (0) \times F_4$, $x_3 = F_1 \times (0) \times F_3 \times (0)$,

$x_4 = (0)\times F_2\times F_3\times(0)$, $x_5 = (0)\times F_2\times F_3\times F_4$, $x_6 = F_1\times F_2\times(0)\times F_4$, $x_7 = F_1\times F_2\times F_3\times(0)$, $x_8 = F_1\times(0)\times F_3\times F_4$ in $R$. Then $a_ib_j = (0)$ for every $i,j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(R)$. Therefore by Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) \geq 1$. The embedding given in Fig 3.1 explicitly shows that $\overline{\gamma}(\mathbb{AG}(R)) = 1$. □

**Theorem 3.3.** Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n > 1$. Let $n_i$ be the nilpotency of $\mathfrak{m}_i$. If $\mathbb{AG}(R)$ is non-planar, then $\overline{\gamma}(\mathbb{AG}(R)) > 1$.

*Proof.* Assume that $\mathbb{AG}(R)$ is non-planar. Suppose that $n > 2$. Consider the non-trivial ideals $x_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0) \times \cdots \times (0)$, $x_2 = (0) \times \mathfrak{m}_2^{n_2-1} \times (0) \times \cdots \times (0)$, $x_3 = \mathfrak{m}_1^{n_1-1} \times \mathfrak{m}_2^{n_2-1} \times \cdots \times (0)$, $y_1 = (0) \times (0) \times \mathfrak{m}_3 \times (0) \cdots \times (0)$, $y_2 = \mathfrak{m}_1 \times (0) \times \mathfrak{m}_3 \times (0) \times \cdots \times (0)$, $y_3 = (0)\times\mathfrak{m}_2\times\mathfrak{m}_3\times(0)\times\cdots\times(0)$, $y_4 = \mathfrak{m}_1\times\mathfrak{m}_2\times\mathfrak{m}_3\times(0)\times\cdots\times(0)$, $y_5 = (0)\times(0)\times R_3\times(0)\times\cdots\times(0)$ in $R$. Then $x_iy_j = (0)$ for every $i,j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$. Hence $n = 2$.

Suppose that $n_1 > 2$ and $n_2 > 2$. Consider the non-trivial ideals $a_1 = \mathfrak{m}_1^{n_1-1} \times (0)$, $a_2 = (0)\times\mathfrak{m}_2^{n_2-1}$, $a_3 = \mathfrak{m}_1^{n_1-1}\times\mathfrak{m}_2^{n_2-1}$, $b_1 = \mathfrak{m}_1\times(0)$, $b_2 = (0)\times\mathfrak{m}_2$, $b_3 = \mathfrak{m}_1\times\mathfrak{m}_2$, $b_4 = \mathfrak{m}_1^{n_1-1}\times\mathfrak{m}_2$, $b_5 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-1}$ in $R$. Then $a_ib_j = (0)$ for all $i,j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$. Hence $n_1 = 2$ or $n_2 = 2$. Without loss of generality, we assume that $n_1 = 2$.

Suppose that $n_2 > 3$. Consider the set $\Omega = \{e_1, \ldots, e_9\}$, where $e_1 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-2}$, $e_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^{n_2-1}$, $e_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $e_4 = \mathfrak{m}_1 \times (0)$, $e_5 = (0) \times \mathfrak{m}_2^{n_2-1}$, $e_6 = (0) \times \mathfrak{m}_2$, $e_7 = (0) \times \mathfrak{m}_2^{n_2-2}$, $e_8 = R_1 \times (0)$, $e_9 = R_1 \times \mathfrak{m}_2^{n_2-1}$ are the non-trivial ideals in $R$. Then the subgraph induced by $\Omega$ in $\mathbb{AG}(R)$ contains a subgraph isomorphic to the graph given in Fig 3.2. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$. Hence $n_2 \leq 3$.

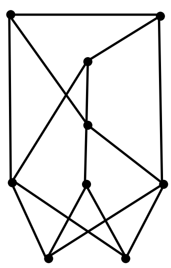


Fig 3.2: Forbidden subgraph for the projective plane

**Case 1.** Suppose that $n_2 = 3$. Let $J_1$ be a non-trivial ideal in $R_2$ such that $J_1 \neq \mathfrak{m}_2, \mathfrak{m}_2^2$. Consider the set $\Omega' = \{f_1, \ldots, f_9\}$ where $f_1 = \mathfrak{m}_1 \times J_1$, $f_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $f_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $f_4 = \mathfrak{m}_1 \times (0)$, $f_5 = (0) \times \mathfrak{m}_2^2$, $f_6 = (0) \times \mathfrak{m}_2$, $f_7 = (0) \times J_1$, $f_8 = R_1 \times (0)$, $f_9 = R_1 \times \mathfrak{m}_2^2$ are the non-trivial ideals in $R$. Then the subgraph induced by $\Omega'$ in $\mathbb{AG}(R)$ contains a subgraph isomorphic to the graph given in Fig 3.2. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$.

Let $I_1$ be a non-trivial ideal in $R_1$ such that $I_1 \neq \mathfrak{m}_1$. Consider the non-trivial ideals $x_1 = (0) \times \mathfrak{m}_2^2$, $x_2 = \mathfrak{m}_1 \times (0)$, $x_3 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $y_1 = (0) \times \mathfrak{m}_2$, $y_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $y_3 = I_1 \times (0)$, $y_4 = I_1 \times \mathfrak{m}_2$, $y_5 = I_1 \times \mathfrak{m}_2^2$ in $R$. Then $x_iy_j = (0)$ for every $i,j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$.

Suppose $\mathfrak{m}_1$ and $\mathfrak{m}_2$, $\mathfrak{m}_2^2$ are the only non-trivial ideals in $R_1$ and $R_2$ respectively. Then by Theorem 2.2, $\overline{\gamma}(\mathbb{AG}(R)) = 0$, a contradiction.

**Case 2.** Assume that $n_2 = 2$. Suppose there is an ideal $I$ of $R_2$ such that $I \neq \mathfrak{m}_2^i$ for all $i = 1, 2$. Then by Proposition 2.4, $R_2$ has at least three distinct non-trivial ideals $J_1$, $J_2$ and $J_3$ such that $J_1$, $J_2$, $J_3 \neq \mathfrak{m}_2$. Consider the non-trivial ideals $a_1 = \mathfrak{m}_1 \times (0)$, $a_2 = (0) \times \mathfrak{m}_2$, $a_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $a_4 = (0) \times J_1$, $a_5 = \mathfrak{m}_1 \times J_1$, $a_6 = (0) \times J_2$, $a_7 = \mathfrak{m}_1 \times J_2$ in $R$. Then $a_ia_j = (0)$ for every $i \neq j$ and so $K_7$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$.

If $R_2$ has at most two non-trivial ideals different from $\mathfrak{m}_2$, then by Proposition 2.4 and Lemma 2.3, $\mathfrak{m}_2$ is the only non-trivial ideal in $R_2$ and if $R_1$ has at most two non-trivial ideals different from $\mathfrak{m}_1$, then by Proposition 2.4 and Lemma 2.3, $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$. Hence by Theorem 2.5, $\overline{\gamma}(\mathbb{AG}(R)) = 0$, a contradiction. □

**Theorem 3.4.** Let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times F_2 \times \cdots \times F_m$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and each $F_j$ is a field and $n, m \geq 1$. Let $n_i$ be the nilpotency of $\mathfrak{m}_i$. Then $\overline{\gamma}(\mathbb{AG}(R)) = 1$ if and only if $R = R_1 \times F_1$ and one of the following condition holds:

$(i)$ $n_1 = 3$ and $R_1$ has exactly 5 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, I_1, I_2, I_3$ with $I_i\mathfrak{m}_1 \neq (0)$ for every $i = 1, 2, 3$ and $I_iI_j = (0)$ for some $i \neq j$.

$(ii)$ $n_1 = 5$ and $R_1$ has exactly 4 distinct non-trivial ideals, say $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4$.

*Proof.* Assume that $\overline{\gamma}(\mathbb{AG}(R)) = 1$. Suppose that $n > 1$. Consider the set $\Omega = \{x_1, \ldots, x_4, y_1, \ldots, y_4\}$, where $x_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$, $x_2 = (0) \times \mathfrak{m}_2^{n_2-1} \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$, $x_3 = (0) \times (0) \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$, $x_4 = \mathfrak{m}_1^{n_1-1} \times \mathfrak{m}_2^{n_2-1} \times \cdots \times (0) \times F_1 \times (0) \times \cdots \times (0)$, $y_1 = \mathfrak{m}_1 \times (0) \times \cdots \times (0) \times \cdots \times (0)$, $y_2 = (0) \times \mathfrak{m}_2 \times \cdots \times (0) \times \cdots \times (0)$, $y_3 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \cdots \times (0) \times \cdots \times (0)$, $y_4 = [R_1 \times (0) \times \cdots \times (0), (0) \times R_2 \times \cdots \times (0)]$. Then the subgraph induced by $\Omega$ in $\mathbb{AG}(R)$ contains a subgraph isomorphic to the graph given in Fig 3.3. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n = 1$

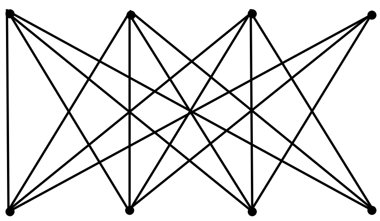


Fig 3.3: Forbidden subgraph for the projective plane

Suppose that $m > 2$. Consider the non-trivial ideals $a_1 = (0) \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $a_2 = \mathfrak{m}_1^{n_1-1} \times F_1 \times (0) \times (0) \times \cdots \times (0)$, $a_3 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0) \times \cdots \times (0)$, $b_1 = (0) \times (0) \times F_2 \times (0) \times \cdots \times (0)$, $b_2 = (0) \times (0) \times (0) \times F_3 \times \cdots \times (0)$, $b_3 = (0) \times (0) \times F_2 \times F_3 \times \cdots \times (0)$, $b_4 = \mathfrak{m}_1 \times (0) \times F_2 \times (0) \times \cdots \times (0)$, $b_5 = \mathfrak{m}_1 \times (0) \times (0) \times F_3 \times \cdots \times (0)$ in $R$. Then $a_ib_j = (0)$ for every $i, j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $m \leq 2$.

**Case 1.** Assume that $m = 2$.

Suppose that $n_1 > 2$. Consider the set $\Omega' = \{e_1, \ldots, e_9\}$, where $e_1 = \mathfrak{m}_1^{n_1-1} \times (0) \times F_2$, $e_2 = \mathfrak{m}_1^{n_1-1} \times F_1 \times (0)$, $e_3 = \mathfrak{m}_1 \times (0) \times F_2$, $e_4 = \mathfrak{m}_1^{n_1-1} \times (0) \times (0)$, $e_5 = (0) \times F_1 \times (0)$, $e_6 = (0) \times F_1 \times F_2$, $e_7 = (0) \times (0) \times F_2$, $e_8 = R_1 \times (0) \times (0)$, $e_9 = \mathfrak{m}_1 \times (0) \times (0)$ are the non-trivial ideals in $R$. Then the subgraph induced by $\Omega'$ in $\mathbb{AG}(R)$ contains a subgraph isomorphic to the graph given in Fig 3.2. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n_1 = 2$.

Let $I_1$ be any non-trivial ideal in $R_1$ with $I_1 \neq \mathfrak{m}_1$. Consider the set $\Omega_1 = \{f_1, \ldots, f_9\}$, where $f_1 = I_1 \times (0) \times F_2$, $f_2 = I_1 \times F_1 \times (0)$, $f_3 = \mathfrak{m}_1 \times (0) \times F_2$, $f_4 = I_1 \times (0) \times (0)$, $f_5 = (0) \times F_1 \times (0)$, $f_6 = (0) \times F_1 \times F_2$, $f_7 = (0) \times (0) \times F_2$, $f_8 = R_1 \times (0) \times (0)$, $f_9 = \mathfrak{m}_1 \times (0) \times (0)$ are the non-trivial ideals in $R$. Then the subgraph induced by $\Omega_1$ in $\mathbb{AG}(R)$ contains a subgraph isomorphic to the graph given in Fig 3.2. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$ and so by Theorem 2.5, $\overline{\gamma}(\mathbb{AG}(R)) = 0$, a contradiction.

**Case 2.** Assume that $m = 1$.

Let $n_1 > 5$. Consider the set $\Omega_2 = \{x_1, \ldots, x_4, y_1, \ldots, y_4\}$, where $x_1 = (0) \times F_1$, $x_2 = \mathfrak{m}_1^{n_1-1} \times F_1$, $x_3 = \mathfrak{m}_1^{n_1-1} \times (0)$, $x_4 = \mathfrak{m}_1^{n_1-2} \times F_1$, $y_1 = \mathfrak{m}_1^4 \times (0)$, $y_2 = \mathfrak{m}_1^3 \times (0)$, $y_3 = \mathfrak{m}_1^2 \times (0)$, $y_4 = \mathfrak{m}_1 \times (0)$ are the non-trivial ideals in $R$. Then the subgraph induced by $\Omega_2$ in $\mathbb{AG}(R)$ contain a subgraph isomorphic to the graph given in Fig 3.3. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $n_1 \leq 5$.

**Subcase 2.1.** $n_1 = 2$.

Suppose there is an ideal $I$ of $R_1$ such that $I \neq \mathfrak{m}_1^i$ for all $i = 1, 2$. Then by Proposition 2.4, $R_1$ has at least three distinct non-trivial ideals $I_1$, $I_2$ and $I_3$ such that $I_1$, $I_2$, $I_3 \neq \mathfrak{m}_1^i$ for all $i = 1, 2$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1 \times F_1$, $u_3 = I_1 \times F_1$, $u_4 = I_2 \times F_1$, $v_1 = \mathfrak{m}_1 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$ in $R$. Then $u_i v_j = (0)$ for every $i, j$ and so $K_{4,4}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Proposition 2.4 and Lemma 2.3, $\mathfrak{m}_1$ is the only non-trivial ideal in $R_1$ and so by Theorem 2.5, $\overline{\gamma}(\mathbb{AG}(R)) = 0$, a contradiction.

**Subcase 2.2.** $n_1 = 3$.

Suppose there is an ideal $I$ of $R_1$ such that $I \neq \mathfrak{m}_1^i$ for all $i = 1, 2, 3$. Then by Proposition 2.4, $R_1$ has at least three distinct non-trivial ideals $I_1$, $I_2$ and $I_3$ such that $I_1$, $I_2$, $I_3 \neq \mathfrak{m}_1^i$ for all $i = 1, 2$. Suppose $R_1$ has at least 4 non-trivial ideals different from $\mathfrak{m}_1^i$ for all $i = 1, 2$. Let $I_1$, $I_2$, $I_3$, $I_4$ be the distinct non-trivial ideals in $R_1$ such that $I_i \neq \mathfrak{m}_1, \mathfrak{m}_1^2$ for every $i$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^2 \times F_1$, $u_3 = \mathfrak{m}_1^2 \times (0)$, $v_1 = \mathfrak{m}_1 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$, $v_5 = I_4 \times (0)$ in $R$. Then $u_i v_j = (0)$ for every $i, j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $R_1$ has exactly 3 non-trivial ideals different from $\mathfrak{m}_1^i$ for $i = 1, 2$. Hence $\mathfrak{m}_1, \mathfrak{m}_1^2, I_1, I_2, I_3$ are the only non-trivial ideals in $R_1$.

Suppose $I_i \mathfrak{m}_1 = (0)$ for some $i$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^2 \times F_1$, $u_3 = \mathfrak{m}_1^2 \times (0)$, $u_4 = I_i \times F_1$, $v_1 = \mathfrak{m}_1 \times (0)$, $v_2 = I_1 \times (0)$, $v_3 = I_2 \times (0)$, $v_4 = I_3 \times (0)$ in $R$. Then $u_i v_j = (0)$ for every $i, j$ and so $K_{4,4}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_i \mathfrak{m}_1 \neq (0)$ for every $i$.

Suppose $I_i$ is adjacent to $I_j$ for every $j \neq i$. Let us assume that $I_1 I_2 = (0)$, $I_2 I_3 = (0)$ and $I_1 I_3 = (0)$. Consider the set $\Omega = \{u_1 \ldots u_9\}$, where $u_1 = \mathfrak{m}_1^2 \times (0)$, $u_2 = I_1 \times (0)$, $u_3 = I_3 \times F_1$, $u_4 = (0) \times F_1$, $u_5 = I_2 \times F_1$, $u_6 = I_1 \times F_1$, $u_7 = \mathfrak{m}_1^2 \times F_1$, $u_8 = I_2 \times (0)$, $u_9 = I_3 \times (0)$ are the non-trivial ideals in $R$. Then the subgraph induced by $\Omega$ in $\mathbb{AG}(R)$ contain a subgraph isomorphic to the graph given in Fig 3.4. and so by Theorem 6.5.1 [15, p.197], $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $I_1 I_2 \neq (0)$ or $I_2 I_3 \neq (0)$ or $I_1 I_3 \neq (0)$.

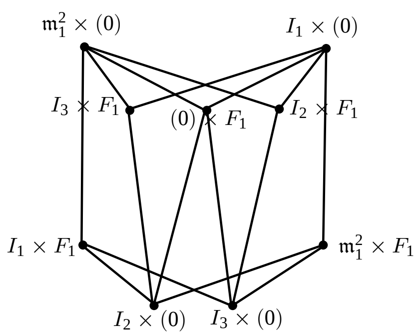



Fig 3.4: Forbidden subgraph for the projective plane

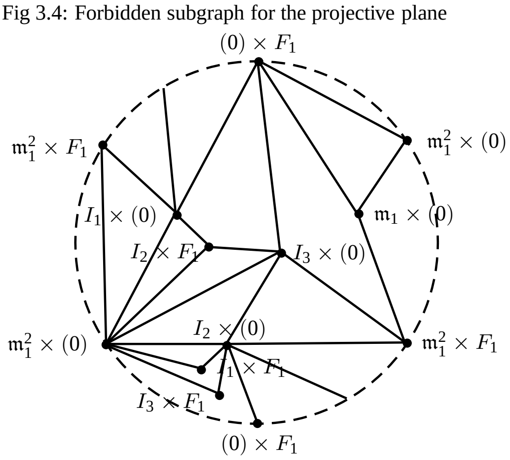



Fig 3.5: Projective embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 3$, $I_i \mathfrak{m}_1 \neq (0)\ \forall\ i$, $I_1 I_2 = (0)$, $I_2 I_3 = (0)$ and $I_1 I_3 \neq (0)$

**Subcase 2.3.** $n_1 = 4$.
Suppose $R_1$ has at least 3 non-trivial ideals different from $\mathfrak{m}_1^i$ for every $i$. Let $J_1$, $J_2$, $J_3$ be the distinct non-trivial ideals in $R_1$ such that $J_i \neq \mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ for every $i$. Consider the non-trivial ideals $u_1 = (0) \times F_1$, $u_2 = \mathfrak{m}_1^3 \times F_1$, $u_3 = \mathfrak{m}_1^3 \times (0)$, $v_1 = \mathfrak{m}_1^2 \times (0)$, $v_2 = \mathfrak{m}_1 \times (0)$, $v_3 = J_1 \times (0)$, $v_4 = J_2 \times (0)$, $v_5 = J_3 \times (0)$ in $R$. Then $u_i v_j = (0)$ for every $i, j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence by Proposition 2.4 and Lemma 2.3, $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$ are the only non-trivial ideals in $R_1$ and so by Theorem 2.5, $\overline{\gamma}(\mathbb{AG}(R)) = 0$, a contradiction.
**Subcase 2.4.** $n_1 = 5$.
Suppose $R_1$ contains at least two distinct non-trivial ideals $I_1$, $I_2$ such that $I_i \neq \mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, $\mathfrak{m}_1^4$ for $i = 1, 2$. Consider the non-trivial ideals $c_1 = (0) \times F_1$, $c_2 = \mathfrak{m}_1^4 \times F_1$, $c_3 = \mathfrak{m}_1^4 \times (0)$, $d_1 = \mathfrak{m}_1^3 \times (0)$, $d_2 = \mathfrak{m}_1^2 \times (0)$, $d_3 = \mathfrak{m}_1 \times (0)$, $d_4 = I_1 \times (0)$, $d_5 = I_2 \times (0)$ in $R$. Then $c_i d_j = (0)$ for every $i, j$ and so $K_{3,5}$ is a subgraph of $\mathbb{AG}(R)$. By Lemma 3.1, $\overline{\gamma}(\mathbb{AG}(R)) > 1$, a contradiction. Hence $R_1$ contains at most one non-trivial ideal $I$ such that $I \neq \mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, $\mathfrak{m}_1^4$. By Proposition 2.4, $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, $\mathfrak{m}_1^4$ are the only non-trivial ideals in $R_1$.

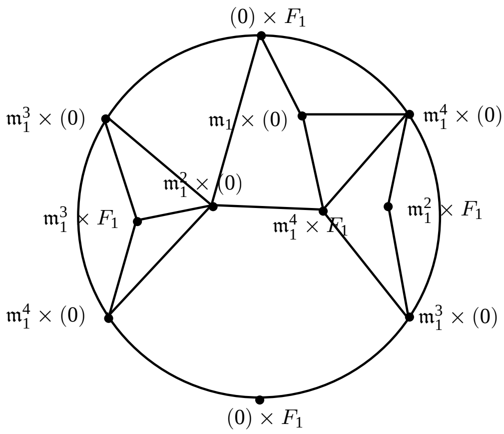



Fig 3.6: Projective embedding of $\mathbb{AG}(R_1 \times F_1)$ with $n_1 = 5$
and $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3, \mathfrak{m}_1^4$ are the only non-trivial ideals in $R_1$

Converse follows from Fig. 3.5 and Fig. 3.6. □

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## Author information

K. Selvakumar and P. Subbulakshmi, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, 627 012, India..
E-mail: `selva_158@yahoo.co.in`

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