Palestine Journal of Mathematics

Vol. 7(1)(2018) , 9–13

A NEW TRI-DIAGONAL MATRIX INVARIANCE PROPERTY

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11C20.

Keywords and phrases: Tri-diagonal matrix, anti-diagonals ratio invariance.

Abstract We state and prove an invariance property, with respect to matrix power, for those $n-1$ immediate off-diagonal ratios of a tri-diagonal n-square matrix. Illustrative examples are given.

1 Introduction and Background

Let

$$\mathbf{N} = \mathbf{N}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

be a general 2×2 matrix. In a previous publication [2] it was established that (unless otherwise indeterminate) the ratio of the two anti-diagonal terms in $\mathbf{N}^k$ is the quantity B/C, being invariant with respect to integer power $k \geq 1$ (four proofs were given). The result is broadened in this paper to accommodate a more general case of which that just described is merely a particular instance.

If $A = A(x)$, $B = B(x)$, $C = C(x)$ are drawn from $\mathbb{Z}[x]$ (with $D = 0$), then the matrix $\mathbf{N}(x) = \mathbf{N}(-B(x), A(x), -C(x), 0)$ can be regarded as characterising a particular class of polynomial families; they are those general functional coefficients of a quadratic equation

$$0 = A(x)T^2(x) + B(x)T(x) + C(x) \tag{1.2}$$

satisfied by the (ordinary) generating function $T(x)$ of an integer sequence which, in any instance, defines an infinite polynomial family $\alpha_0(x), \alpha_1(x), \alpha_2(x), \ldots$, within the class according to (where T denotes transposition)

$$\alpha_n(x) = \alpha_n(A(x), B(x), C(x)) = (1, 0)\mathbf{N}^n(-B(x), A(x), -C(x), 0)(1, 0)^T \tag{1.3}$$

for $n \geq 0$. The class of polynomial families delivered by (1.3) is a large one—commensurate with those sequences governed by (1.2), of which there are many in discrete mathematics. This work, and that of [2], have their origins in conditions found for cross-family member equality [4], and for anti-diagonals product invariance across powers of 2×2 matrix sets [5].

Closed form expressions for the entries of an exponentiated arbitrary dimension square matrix have been developed in the not too distant past [1, 7], but in no great quantity nor in any meaningful way that delivers our result here (see also, as an aside, the paper [8] in which the authors consider terms of exponentiated 3×3 and $n \times n$ matrices (albeit of specialised forms) pertaining to work on polynomial root extraction). Anti-diagonals ratio invariance in the 2×2 case introduced above is seen as an immediate consequence of a 2004 result by McLaughlin [6] (who, unknowingly it seems, reproduced that by Pla from 1996 [9], although the key parameters in terms of which the formulations were made differ). Very recently, McLaughlin's result was established by Larcombe [3] based on the use of so called Catalan polynomials which underpinned the work on polynomial families in [4] and [5]. The 2×2 anti-diagonals ratio invariance property is a surprisingly little known one and, supported by some illustrative examples, we offer its natural extension in that spirit.

2 Result, Proof and Examples

2.1 Result and Proof

We state the result formally, and offer a direct proof.

Theorem 2.1. *Suppose* $\mathbf{M} = \mathbf{M}(a_1, \ldots, a_n, u_1, \ldots, u_{n-1}, l_1, \ldots, l_{n-1}) = \mathbf{M}(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1})$ *is an* $n \times n$ *tri-diagonal matrix*

$$\mathbf{M} = \begin{pmatrix} a_1 & u_1 & & & & & \\ l_1 & a_2 & u_2 & & & & \\ & l_2 & a_3 & u_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & l_{n-2} & a_{n-1} & u_{n-1} \\ & & & & & l_{n-1} & a_n \end{pmatrix},$$

with finite anti-diagonal ratios $u_1/l_1, u_2/l_2, \ldots, u_{n-1}/l_{n-1} \neq 0$. *Then the immediate off-diagonal terms of* $\mathbf{M}^k$ *form corresponding anti-diagonal ratios that, when finite, remain invariant as the power* $k > 1$ *to which* $\mathbf{M}$ *is raised increases.*

Proof. Let $\mathbf{D}(\mathbf{d}_n) = \mathbf{D}(d_1, \ldots, d_n)$ be the n-square diagonal matrix

$$\mathbf{D}(\mathbf{d}_n) = \begin{pmatrix} d_1 & & & & & \\ & d_2 & & & & \\ & & d_3 & & & \\ & & & \ddots & & \\ & & & & d_{n-1} & \\ & & & & & d_n \end{pmatrix}, \tag{P.1}$$

where $d_1, \ldots, d_n \neq 0$. Then, writing

$$\mathbf{X}(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1}, \mathbf{d}_n) = \mathbf{D}(\mathbf{d}_n)\mathbf{M}(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1})\mathbf{D}^{-1}(\mathbf{d}_n), \tag{P.2}$$

we see that (tri-diagonal)

$$\begin{aligned} &\mathbf{X}(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1}, \mathbf{d}_n) \\ &= \begin{pmatrix} a_1 & u_1\frac{d_1}{d_2} & & & & & \\ l_1\frac{d_2}{d_1} & a_2 & u_2\frac{d_2}{d_3} & & & & \\ & l_2\frac{d_3}{d_2} & a_3 & u_3\frac{d_3}{d_4} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & l_{n-2}\frac{d_{n-1}}{d_{n-2}} & a_{n-1} & u_{n-1}\frac{d_{n-1}}{d_n} \\ & & & & & l_{n-1}\frac{d_n}{d_{n-1}} & a_n \end{pmatrix} \end{aligned} \tag{P.3}$$

is symmetric if, on setting $d_1 = 1$,

$$\begin{aligned} d_2 &= \pm\sqrt{u_1/l_1}, \\ d_3 &= \pm\sqrt{u_1 u_2/(l_1 l_2)}, \\ &\vdots \\ d_n &= \pm\sqrt{u_1 u_2 \cdots u_{n-1}/(l_1 l_2 \cdots l_{n-1})}. \end{aligned} \tag{P.4}$$

Given these conditions on $d_1, \ldots, d_n$ then the matrix

$$\mathbf{X}^k(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1}) = \mathbf{X}^k(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1}, \mathbf{d}_n(\mathbf{u}_{n-1}, \mathbf{l}_{n-1})) \tag{P.5}$$

will also be symmetric (though not tri-diagonal for $k > 1$). Noting that it takes the form

$$\mathbf{X}^k(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1}) = \begin{pmatrix} \theta_1^{(k)} & \sigma_1^{(k)} & \cdots & & & & \\ \sigma_1^{(k)} & \theta_2^{(k)} & \sigma_2^{(k)} & \cdots & & & \\ \vdots & \sigma_2^{(k)} & \theta_3^{(k)} & \sigma_3^{(k)} & \cdots & & \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \cdots & \sigma_{n-2}^{(k)} & \theta_{n-1}^{(k)} & \sigma_{n-1}^{(k)} \\ & & & & \cdots & \sigma_{n-1}^{(k)} & \theta_n^{(k)} \end{pmatrix}, \tag{P.6}$$

we have, using (P.1) and (P.6),

$$\begin{aligned} &\mathbf{D}^{-1}(\mathbf{d}_n(\mathbf{u}_{n-1}, \mathbf{l}_{n-1}))\mathbf{X}^k(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1})\mathbf{D}(\mathbf{d}_n(\mathbf{u}_{n-1}, \mathbf{l}_{n-1})) \\ &= \begin{pmatrix} \theta_1^{(k)} & \sigma_1^{(k)}\frac{d_2}{d_1} & \cdots & & & & \\ \sigma_1^{(k)}\frac{d_1}{d_2} & \theta_2^{(k)} & \sigma_2^{(k)}\frac{d_3}{d_2} & \cdots & & & \\ \vdots & \sigma_2^{(k)}\frac{d_2}{d_3} & \theta_3^{(k)} & \sigma_3^{(k)}\frac{d_4}{d_3} & \cdots & & \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \cdots & \sigma_{n-2}^{(k)}\frac{d_{n-2}}{d_{n-1}} & \theta_{n-1}^{(k)} & \sigma_{n-1}^{(k)}\frac{d_n}{d_{n-1}} \\ & & & & \cdots & \sigma_{n-1}^{(k)}\frac{d_{n-1}}{d_n} & \theta_n^{(k)} \end{pmatrix} \\ &= \mathbf{M}^k(\mathbf{a}_n, \mathbf{u}_{n-1}, \mathbf{l}_{n-1}) \end{aligned} \tag{P.7}$$

by (P.2). The $n-1$ immediate off-diagonal ratios of $\mathbf{M}^k$ are $(d_2/d_1)^2, (d_3/d_2)^2, \ldots, (d_n/d_{n-1})^2$, which by (P.4) reduce to $u_1/l_1, u_2/l_2, \ldots, u_{n-1}/l_{n-1}$. Being independent of matrix power k, their invariance with respect to it is established. □

2.2 Examples

We finish with some instances of Theorem 2.1 for the benefit of the reader.

Example 1. We begin with a 3×3 example where

$$\mathbf{M} = \begin{pmatrix} 2 & -2\sqrt{2} & 0 \\ -2 & -4 & -10 \\ 0 & -1 & 8 \end{pmatrix}, \tag{2.1}$$

for which the two anti-diagonal ratios are $\sqrt{2}$ and 10. These are seen to be repeated in the matrix powers

$$\begin{aligned} \mathbf{M}^2 &= \begin{pmatrix} 4(1+\sqrt{2}) & 4\sqrt{2} & 20\sqrt{2} \\ 4 & 2(13+2\sqrt{2}) & -40 \\ 2 & -4 & 74 \end{pmatrix}, \\ \mathbf{M}^3 &= \begin{pmatrix} 8 & -4(4+11\sqrt{2}) & 120\sqrt{2} \\ -4(11+2\sqrt{2}) & -8(8+3\sqrt{2}) & -20(29+2\sqrt{2}) \\ 12 & -2(29+2\sqrt{2}) & 632 \end{pmatrix}, \\ \mathbf{M}^4 &= \begin{pmatrix} 8(6+11\sqrt{2}) & 8(8+5\sqrt{2}) & 40(4+35\sqrt{2}) \\ 8(5+4\sqrt{2}) & 4(217+56\sqrt{2}) & -80(50+\sqrt{2}) \\ 4(35+2\sqrt{2}) & -8(50+\sqrt{2}) & 4(1409+10\sqrt{2}) \end{pmatrix}, \ldots \end{aligned} \tag{2.2}$$

Example 2. We next offer a 4×4 matrix

$$\mathbf{M} = \begin{pmatrix} 8 & -1 & 0 & 0 \\ 2 & -2 & 6 & 0 \\ 0 & -8 & -2 & -6 \\ 0 & 0 & -3 & 0 \end{pmatrix}, \tag{2.3}$$

which has the three anti-diagonal ratios $-1/2$, $-3/4$ and 2. These ratios are repeated in the matrix powers

$$\mathbf{M}^2 = \begin{pmatrix} 62 & -6 & -6 & 0 \\ 12 & -46 & -24 & -36 \\ -16 & 32 & -26 & 12 \\ 0 & 24 & 6 & 18 \end{pmatrix},$$

$$\mathbf{M}^3 = \begin{pmatrix} 484 & -2 & -24 & 36 \\ 4 & 272 & -120 & 144 \\ -64 & 160 & 208 & 156 \\ 48 & -96 & 78 & -36 \end{pmatrix},$$

$$\mathbf{M}^4 = \begin{pmatrix} 3868 & -288 & -72 & 144 \\ 576 & 412 & 1440 & 720 \\ -192 & -1920 & 76 & -1248 \\ 192 & -480 & -624 & -468 \end{pmatrix}, \ldots \tag{2.4}$$

Example 3. In our final example $\mathbf{M}$ is the 5×5 matrix

$$\mathbf{M} = \begin{pmatrix} -2 & 2(-1+i) & 0 & 0 & 0 \\ 3(1-i) & -i & 2i & 0 & 0 \\ 0 & -(1+2i) & 1 & -2 & 0 \\ 0 & 0 & 2 & i & i \\ 0 & 0 & 0 & -2(1+i) & -1 \end{pmatrix} \tag{2.5}$$

(with some complex entries, for variation), possessing four anti-diagonal ratios $-2/3$, $-2(2+i)/5$, -1 and $-(1+i)/4$ that are repeated in the matrix powers

$$\mathbf{M}^2 = \begin{pmatrix} 4(1+3i) & 2(3-i) & -4(1+i) & 0 & 0 \\ 3(-3+i) & 3+10i & 2(1+i) & -4i & 0 \\ -3(3+i) & -(3+i) & 1-2i & -2(1+i) & -2i \\ 0 & -2(1+2i) & 2(1+i) & -(3+2i) & -(1+i) \\ 0 & 0 & -4(1+i) & 4 & 3-2i \end{pmatrix} \tag{2.6}$$

and

$$\mathbf{M}^3 = \begin{pmatrix} 4(1-12i) & -2(19+5i) & 8i & 8(1+i) & 0 \\ 3(19+5i) & 3(8-11i) & -18 & -4i & 4 \\ 6(1+2i) & 9(2-i) & -(1+12i) & 2(-2+3i) & 2 \\ -6(3+i) & -2(1+2i) & 2(2-3i) & -(2+3i) & 3-2i \\ 0 & 4(-1+3i) & 4(1-i) & 2(-1+5i) & 3(-1+2i) \end{pmatrix}, \tag{2.7}$$

and so on (see Remark 2.3, however).

Remark 2.2. For $n = 2$ Theorem 2.1, with $a_1 = A$, $a_2 = D$, $u_1 = B$, $l_1 = C$, reproduces Theorem 1.1 of [2] for the 2×2 matrix $\mathbf{M}(\mathbf{a}_2, \mathbf{u}_1, \mathbf{l}_1) = \mathbf{M}(a_1, a_2, u_1, l_1) = \mathbf{N}(A, B, C, D)$ (1.1); our general proof for n-square tri-diagonal $\mathbf{M}$ collapses to Proof III therein for which, with $d_1 = 1$, $d_2 = x = x(B, C) = \pm\sqrt{B/C}$, the variables $\theta_1^{(k)}, \sigma_1^{(k)}, \theta_2^{(k)}$ here correspond to r_k, s_k, t_k in that proof.

Remark 2.3. In Example 3 one of the anti-diagonal ratios of $\mathbf{M}^4$ is indeterminate (of form $0/0$). Such an occurrence is a rarity, and it reverts to the correct (finite) ratio in subsequent powers as far as we have computed.

3 Summary

In this paper we have extended a previous result which arose as part of work involving polynomial families, each one associated with an integer sequence through the functional coefficients of its governing quadratic (ordinary) generating function equation. Both the theorem in [2], and the generalised version here, would appear to have gone unnoticed in the literature—this is something of a surprise, and will hopefully interest some readers.

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Received: January 11, 2017

Accepted: April 7, 2017