FUNCTIONS ON STATIONARY SETS

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Abstract A characterization of stationary sets is established using regressive functions, selection property, continuous functions on ordinals and real continuous functions.

1 Introduction

Whenever a number of pigeons, to be put into cages, is greater than the number of cages considered for this purpose; one cage, at least, has to contain more than one pigeon. This self-evident principle inspired many mathematicians over time and has been a basis for many generalizations indeed. The first extension of this principle to ordinals dates back to P. Alexandroff and P. Uryshon (1926), see [1]: A regressive function \( f \) on limit ordinals (i.e., \( f(\alpha) \neq \alpha \)) has a constant value on some uncountable set of ordinal numbers. Four years later, Ben Dushnik (1930), see [3], gave a more explicit generalization: Any regressive function on \( \omega_{\beta+1} \) into itself is constant on some set of size \( \aleph_{\beta+1} \). In 1950 P. Erdős extended Ben Dushnik result to any \( \omega_\nu \) of uncountable cofinality, see [4]. In the same year (1950) J. Novak, see [9], generalized Alexandroff-Uryshon’s theorem to closed (under the order topology) and unbounded subsets of \( \omega_1 \): Any regressive function on a club, closed and unbounded, \( C \) of \( \omega_1 \) into itself is constant on some uncountable subset of C. H. Bachman, see [2], strengthened Novak’s result by showing that, in fact, whenever a cardinal \( \kappa \) has no countable cofinal subset every regressive function on a club of \( \kappa \) into \( \kappa \) is constant on some subset of size \( \kappa \). It is with W. Neumer (1951), see [8], that the terminology of “stationary” set came into use (a set \( S \subseteq \kappa \) is stationary in \( \kappa \) whenever \( S \) meets all clubs in \( \kappa \)): If \( \kappa \) is an uncountable regular cardinal and \( f \) is a regressive function on a stationary set \( S \) in \( \kappa \), then \( f \) is constant on some cofinal subset of \( S \).

2 Stationary sets and regressive functions

Throughout this paper \( \kappa \) shall denote any regular uncountable cardinal, ordinals are assumed to be endowed with the order topology and by club we mean a set of \( \kappa \) that is closed under the order topology and unbounded in \( \kappa \). Now we say that a set \( S \subseteq \kappa \) is stationary in \( \kappa \) whenever it meets all clubs of \( \kappa \). Otherwise we say that \( S \) is non-stationary. The ideal \( \mathcal{NS}_\kappa \) of non-stationary sets of \( \kappa \) is indeed \( \lambda \)-complete for any \( \lambda < \kappa \), see Corollary 2.2 below. Next, the assumption of studying stationary sets in regular cardinals is not restrictive. For let \( \alpha \) be a limit ordinal and denote by \( cf(\alpha) \) the least cardinal \( \mu \geq \omega \) that \( \mu \) is unbounded in \( \alpha \). Now, putting \( \mu = cf(\alpha) > \omega \), then \( \mu \) may be regarded as a club in \( \alpha \) since one can choose \( \langle x_\nu \rangle_{\nu < \mu} \) a strictly increasing and continuous cofinal set in \( \alpha \). Now notice that \( S \cap \mu = T \cap \mu \) modulo \( \mathcal{NS}_\mu \) implies that there is a club \( D \subseteq \mu \) so that \( (S \cap \mu) \setminus (T \cap \mu) \cap D = \emptyset \), where \( \Delta \) is the symmetric difference. Thus \( (S \Delta T) \cap (D \cap \mu) = \emptyset \), where \( D \cap \mu \) is a club of \( \alpha \). Thus \( S = T \) modulo \( \mathcal{NS}_\alpha \). Now the mapping \( \theta \) defined by \( \theta(S) = \cap \{ \alpha : \mu, \mu \in \mathcal{NS}_\mu \} \cap (P(\alpha) / \mathcal{NS}_\alpha) \) and \( P(\alpha) / \mathcal{NS}_\alpha \) in one-to-one correspondence.

Proposition 2.1. Any intersection of less than \( \kappa \) clubs of \( \kappa \) is a club of \( \kappa \).

Proof. Let \( \lambda \) less than \( \kappa \), and let \( \langle C_\mu : \nu < \lambda \rangle \) a family of clubs in \( \kappa \). We shall use induction on \( \lambda \). For set \( D_\lambda := \cap \{ C_\nu : \nu < \lambda \} \), where \( \lambda_\alpha < \lambda_{\alpha+1} \) for all \( \alpha < cf(\lambda) \). By the
induction hypothesis \( \{D_{\lambda_n} : \alpha < \text{cf}(\lambda)\} \) is a decreasing sequence of clubs in \( \kappa \). Next, pick any \( \xi \in \kappa \) and construct an increasing sequence \( \{x_\alpha : \alpha < \text{cf}(\lambda)\} \) so that \( x_0 > \xi \) and \( x_\alpha \in D_{\lambda_\alpha} \). Thus \( \sup\{x_\alpha : \alpha < \text{cf}(\lambda)\} \in D_{\lambda_\alpha} \) for each \( \alpha < \text{cf}(\lambda) \) and therefore \( \sup\{x_\alpha : \alpha < \text{cf}(\lambda)\} \in \cap\{D_{\lambda_\alpha} : \alpha < \text{cf}(\lambda)\}(= \cap\{C_\nu : \nu < \lambda\}) \); moreover \( \xi < \sup\{x_\alpha : \alpha < \text{cf}(\lambda)\} \). This shows that \( \cap\{C_\nu : \nu < \lambda\} \) is a club in \( \kappa \). \( \Box \)

Corollary 2.2. The ideal \( N_\kappa \) is \( \lambda \)-complete for any \( \lambda < \kappa \).

Next, recall that a function \( f : \omega_1 \rightarrow \omega_1 \) is regressive whenever \( f(\alpha) < \alpha \) for every non-zero ordinal \( \alpha \) in \( \omega_1 \). To have an idea of what Fodor’s theorem on a stationary set of \( \kappa \) looks like, take \( f : \omega_1 \rightarrow \omega_1 \) a regressive function and suppose for all \( \xi \)'s in \( \omega_1 \) \( f^{-1}(\xi) = \{\nu : f(\nu) = \xi\} \) is a non-stationary set of \( \omega_1 \). So there is for each \( \xi \) a club \( C_\xi \) so that \( f^{-1}(\xi) \cap C_\xi = \emptyset \). Hence, \( \omega_1 = \cup f^{-1}(\xi) \). So, is there \( \alpha \in \omega_1 \) so that \( f(\alpha) \geq \alpha \)? This may be possible if e.g., \( f(\alpha) \neq \xi \), for all \( \xi < \alpha \). Now, if \( \alpha \in C_\xi \) for all \( \xi < \alpha \) this shall make \( f(\alpha) \neq \xi \) for all \( \xi < \alpha \) whenever \( \alpha \) is a limit ordinal. The question now is to find \( \alpha \) so that \( \alpha \in \cap\{C_\xi : \xi < \alpha\} \). This fix-point like situation is actually the key in proving Fodor’s theorem. To finish up the proof here construct a sequence \( (\beta_n)_{n<\omega} \) so that \( \beta_{n+1} \in \cap\{C_\nu : \nu < \beta_n\} \). Then \( \sup\{\beta_n : n < \omega\} = \beta \in \cap\{C_\nu : \nu < \beta\} \), and \( f(\beta) \geq \beta \); contradiction. Next, the set of \( \alpha \) so that \( \alpha \in \cap\{C_\nu : \nu < \alpha\} \) is actually a club as showed by the next proposition.

Proposition 2.3. Let \( (C_\alpha)_{\alpha < \kappa} \) be a family of clubs in \( \kappa \). The diagonal intersection of \( (C_\alpha)_{\alpha < \kappa} \) is a club of \( \kappa \) denoted by \( \Delta\{C_\alpha : \alpha < \kappa\} \) and defined by:

\[
\alpha \in \Delta\{C_\alpha : \nu < \kappa\} \iff \alpha \in \cap\{C_\nu : \nu < \alpha\}
\]

Proof. First, to see that \( \Delta\{C_\alpha : \alpha < \kappa\} \) is closed let \( (\xi_\nu : \nu < \lambda) \) be a strictly increasing sequence in \( \Delta\{C_\alpha : \alpha < \kappa\} \). Put \( \xi = \sup\{\xi_\nu : \nu < \lambda\} \). Now for each \( \alpha < \xi \), there is \( \nu_0 \) so that \( \alpha < \xi_\nu < \xi \) for \( \nu \geq \nu_0 \). Thus \( \xi_\nu \in C_\alpha \) for \( \nu \geq \nu_0 \). Hence \( \xi \in C_\alpha \) and therefore \( \xi \in \Delta\{C_\alpha : \alpha < \kappa\} \).

Second, to show that \( \Delta\{C_\alpha : \alpha < \kappa\} \) is cofinal in \( \kappa \), let \( \beta < \kappa \) and construct an increasing sequence \( (\beta_n : n < \omega) \) so that \( \beta_0 > \beta \), for some \( \beta_0 \in C_\alpha \) and \( \beta_{n+1} \in \cap\{C_\alpha : \alpha < \beta_n\} \), for each \( n < \omega \). Thus \( \beta = \sup\{\beta_n : n < \omega\} \) is in \( \Delta\{C_\alpha : \alpha < \kappa\} \) and is bigger than \( \beta \). \( \Box \)

Theorem 2.4 (G. Fodor). Let \( S \) be stationary in \( \kappa \). Then any regressive function on \( S \) is constant on some stationary subset \( S_0 \) of \( S \).

Proof. Towards a contradiction assume \( f^{-1}(\alpha) \) is non-stationary for all \( \alpha \)'s. For each \( \alpha < \kappa \), pick a club \( C_\alpha \) of \( \kappa \) so that \( C_\alpha \cap f^{-1}(\alpha) = \emptyset \). By Proposition 2.3 pick \( \xi \) limit in \( S \cap \Delta\{C_\alpha : \alpha < \kappa\} \). So \( f(\xi) \neq \alpha \) for all \( \alpha < \xi \) i.e., \( f(\xi) \geq \xi \) which is a contradiction. \( \Box \)

The following is a converse of Fodor’s theorem.

Proposition 2.5. For each non-stationary set in \( \kappa \), there is a regressive function \( f \) so that \( |f^{-1}(\xi)| < \kappa \) for all \( \xi < \kappa \).

Proof. Assume \( S \) is a non-stationary set in \( \kappa \) and pick \( C \) a club disjoint from \( S \); then write \( \kappa \setminus C = \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\} \) where \( (x_\alpha, x_{\alpha+1}) \) is a maximal open interval and \( (x_\alpha) \) is an increasing continuous enumeration of \( C \). Now since \( S \subseteq \cup\{(x_\alpha, x_{\alpha+1}) : \alpha < \kappa\} \), define \( f : S \rightarrow \kappa \) by \( f(\xi) = x_\alpha(\xi) \) with \( \alpha(\xi) \) is the unique \( \nu \) so that \( \xi \in (x_\alpha, x_{\nu+1}) \). \( f \) is regressive since \( S \) and \( C \) are disjoint sets, and \( |f^{-1}(\xi)| \leq |(x_\alpha(\xi), x_{\alpha(\xi)+1})| < \kappa \). \( \Box \)

3 Stationary sets and selection

Definition 3.1. We say that \( \Sigma(S) \) is a selector of \( S \) whenever \( S = \cup\{S_\alpha : \alpha < \kappa\} \) implies \( |\Sigma(S) \cap S_\alpha| = 1 \) for any \( \alpha < \kappa \).

Next, we give two consequences of Fodor’s theorem.
Theorem 3.2 (Selection). \( S \) is stationary in \( \kappa \) if any partition of \( S \) in non-stationary sets of \( \kappa \) has a stationary selector.

Proof. One direction is obvious. To see the other one, assume that \( \Sigma(S) = \{\min(S_\alpha) : \alpha < \kappa\} \) is non-stationary, where \( S = \bigcup (S_\alpha : \alpha < \kappa) \) such that \( S_\alpha \in N\Sigma \kappa \) for each \( \alpha < \kappa \). Pick then \( C \) a club of \( \kappa \) so that \( C \cap \Sigma(S) = \emptyset \). Next, define \( \psi : S \cap C \to \kappa \) by \( \psi(x) = \min(S_\alpha(x)) \) whenever \( x \in S_\alpha(x) \). \( \psi \) is a regressive function on \( S \cap C \); and thus \( \psi \upharpoonright S_0 \) is constant, where \( S_0 \) is stationary included in \( S \cap C \). Hence, \( x \in S_0 \) implies \( \psi(x) = \min(S_\alpha(x)) \). Thus \( S_0 \subseteq S_{\alpha(0)} \) and this contradicts the fact that \( S_{\alpha(0)} \) is non-stationary. \( \square \)

Proposition 3.3. Let \( S \) be a stationary set \( \kappa \) and \( f \) be a function from \( S \) into \( \kappa \). Then there is a stationary set \( S_0 \) so that at least one of the following three statements holds:

(a) \( f \upharpoonright S_0 \) is a constant function;
(b) \( f \upharpoonright S_0 \) is the identity;
(c) \( f \upharpoonright S_0 \) is strictly increasing and thus injective.

Moreover \( f(\alpha) \geq \alpha \) for all \( \alpha \in S_0 \) and \( f^\kappa S_0 \) is non stationary.

Proof. Split \( S \) into three sets \( S_1, S_2, S_3 \) defined as follows:

\[
S_1 = \{ \alpha \in S : f(\alpha) < \alpha \}, \quad S_2 = \{ \alpha \in S : f(\alpha) = \alpha \}, \quad \text{and} \quad S_3 = \{ \alpha \in S : f(\alpha) > \alpha \}.
\]

Now, if \( S_1 \) is stationary then by Fodor’s theorem \( f \) is constant on some stationary set of \( S_1 \). Hence we may assume \( S_1 = \emptyset \). If \( S_2 \) is stationary then \( f \upharpoonright S_2 \) is the identity and thus \( S_0 = S_2 \). So, we may assume \( S = S_3 \). Now, put

\[
A(\alpha_0) = \{ \nu \in S : f(\nu) \leq f(\alpha_0) \}, \quad A(\alpha_0) = \min S;
\]

\[
A(\alpha_1) = \{ \nu \in S : f(\nu) \leq f(\alpha_1) \}, \quad A(\alpha_1) = \min(S \setminus A(\alpha_0));
\]

\[
\vdots
\]

\[
A(\alpha_\xi) = \{ \nu \in S : f(\nu) \leq f(\alpha_\xi) \}, \quad A(\alpha_\xi) = \min(S \setminus \cup \{ A(\alpha_\nu) : \nu < \xi \}).
\]

Notice that \( \langle \alpha_\xi : \xi < \alpha \rangle \) is increasing, and \( S = \cup \{ A(\alpha_\xi) : \xi < \kappa \} \) and \( A(\alpha_\xi) \) are non stationary sets. Hence by selection property (Theorem 3.2) pick \( \Sigma(S) \) a selector of \( S \). \( \Sigma(S) \) is stationary, \( \Sigma(S) = \{ \min(A(\alpha_\xi)) : \xi < \kappa \} = \{ \alpha_\xi : \xi < \kappa \} \) and \( f \upharpoonright \Sigma(S) \) is increasing. \( \square \)

4 Stationary sets and continuous functions

We characterize stationary sets using continuous functions. This feature actually makes stationary sets a very important tool in set theory distinguishing between objects and therefore constructing, at will, incomparably many of them in many areas of mathematics see [6].

Theorem 4.1. For any subset \( S \) of \( \kappa \), the following statements are equivalent:

(a) \( S \) is stationary in \( \kappa \).

(b) Every continuous function \( f \) from \( S \) into \( \kappa \) is either constant on a final segment of \( S \) or \( \Fix(f) := \{ \nu \in S : f(\nu) = \nu \} \) is cofinal in \( \kappa \).

Proof. (a) implies (b). Suppose \( S \) stationary and let \( f : S \to \kappa \) be a continuous function.

Case 1. \( S_0 = \{ \alpha \in S : f(\alpha) < \alpha \} \) is stationary in \( \kappa \).

Find a stationary \( \Sigma \subseteq S_0 \), and \( \alpha \in \kappa \) so that \( f^\kappa \Sigma = \{ a \} \). Now since two clubs intersect in \( \kappa \), it follows that for each \( \nu < \alpha \), \( f(\nu) \) is bounded in \( \kappa \). Thus \( \sup \{ f^{-1}(\nu) : \nu < a \} = \delta < \kappa \). Hence for \( t \in \Sigma \cap [\delta + 1, \rightarrow) \), we have \( f(t) \geq a \). Next, assume that \( f^{-1}( [ a + 1, \rightarrow] ) \) is cofinal in \( \kappa \). It follows then that \( f^{-1}( [ a + 1, \rightarrow] ) \) is a club of \( S \) and thus there is \( t \in \Sigma \cap f^{-1}( [ a + 1, \rightarrow] ) \). Therefore \( f(t) \geq a + 1 \); this contradicts \( f^\kappa \Sigma = \{ a \} \). So, there is \( \delta_1 < \kappa \) so that \( \sup(f^{-1}( [ a + 1, \rightarrow] )) = \delta_1 \).
Hence, for $t \in S \cap [(\delta \lor \delta_1)+1,\rightarrow)$, $f(t) = a$ i.e., $f \upharpoonright S \cap [\gamma,\rightarrow)$ is constant for some $\gamma < \kappa$.

**Case 2.** $S_1 = \{\alpha \in S : f(\alpha) \geq \alpha\}$ is stationary in $\kappa$.

Construct, by induction, a sequence of size $\kappa$, $(x_\xi)_{\xi<\kappa}$ so that: $x_\xi \leq f(x_\xi) \leq x_{\xi+1}$ for each $\xi < \kappa$. Now, denote by $\text{cl}(X)$ and $\text{lim}(X)$ respectively the closure and the set of limit points of $X$ in the order topology on $\kappa$. Next, let $t \in S \cap \text{lim}(S) \cap \text{lim}(\text{cl}(x_\xi : \xi < \kappa))$. So, pick $(x_{\xi(n)})_n$ so that: $x_{\xi(n)} < f(x_{\xi(n)}) \leq x_{\xi(n)+1}$ and $\sup_n x_{\xi(n)} = t$. By continuity of $f$ we have $f(t) = t$, but this shows that $\text{Fix}(f) \neq \emptyset$, and hence is cofinal in $\kappa$.

(b) implies (a). Assume that $S$ is non-stationary. So pick $C$ a club set in $\kappa$ disjoint from $S$ and write $\kappa \setminus C = \{x_\alpha, x_{\alpha+1} : \alpha < \kappa\}$, where $(x_\alpha, x_{\alpha+1})$ is a maximal open interval and $(x_\alpha)_\alpha$ is an increasing continuous enumeration of $C$. Now define $f : S \rightarrow \kappa$ by $f(\xi) = x_\alpha(\xi)$ with $\alpha(\xi)$ is the unique $\nu$ so that $\xi \in (x_\nu, x_{\nu+1})$.

Now, $\text{Fix}(f) = \emptyset$ since $S \cap C = \emptyset$ and $|f^{-1}(\xi)| \leq |(x_\alpha(\xi), x_\alpha(\xi)+1)| < \kappa$ for all $\xi$’s. □

## 5 Stationary sets and real continuous functions

The following theorem shows that modulo non-stationary sets constant functions are the only real continuous functions on stationary sets.

**Theorem 5.1.** For any subset $S$ of $\kappa$, the following statements are equivalent:

(a) $S$ is stationary in $\kappa$.

(b) Every continuous function $f : S \rightarrow \mathbb{R}$ is constant on some cofinal segment of $S$.

**Proof.** (a) implies (b). Let $f : S \rightarrow \mathbb{R}$ be continuous. We claim that there is $n_0 \in \omega$ so that $f^{-1}([n_0, +\infty))$ is bounded in $\kappa$. Indeed, if $f^{-1}([n_0, +\infty))$ are clubs in $\kappa$, it follows then that $\cap f^{-1}([n_0, +\infty)) : n > 0$ is not empty and thus $f(t) \geq n_0$ for some $t < \kappa$ and all $n > 0$. This is impossible. Likewise there is $m_0 \in \omega$ so that $f^{-1}((\neg \infty, -m_0])$ is bounded in $\kappa$. Thus $\sup(f^{-1}([n_0, +\infty)) \cup f^{-1}((\neg \infty, -m_0])) = \delta < \kappa$. Now construct a sequence of closed intervals $(J_k)_k \in \omega$ so that $J_0 = [-m_0, m_0]$ and $J_{k+1} \subseteq J_k$, $d(J_k) = \frac{d(J_k)}{2^k}$. Set $A = \{k \in \omega : f^{-1}(J_k)$ is bounded in $\kappa\}$, $B = \{k \in \omega : f^{-1}(J_k)$ is unbounded in $\kappa\}$. Let $\bar{a} = \sup(f^{-1}(J_k) : k \in A)$. For any $t \in [\delta \lor \delta_1, \rightarrow) \cap S \cap (\cap f^{-1}(J_k) : k \in B)$, we have $f(t) \in \cap (J_k : k \in B) = \{a\}$. Next set $T = S \setminus (S \cap (\cap f^{-1}(J_k) : k \in B))$. Assume that $T$ is closed and write $T = \{t_\alpha : \alpha < \kappa\}$. Then for each $\alpha < \kappa$, $f(t_\alpha) \neq a$. Since $\kappa$ is regular uncountable pick a cofinal set $D \subseteq \{\alpha : \alpha < \kappa\}$ and some $b_0 \in \omega$ so that $f(D) \subseteq J_{b_0} \setminus J_{b_0+1}$. Write $J_k = [a_k, b_k]$ for all $k \in \omega$. Now let $t \in S \cap \text{lim}(D) \cap (\cap f^{-1}(J_k) : k \geq b_0)$. Pick $d(\alpha) \in D$ so that $\lim_\alpha d(\alpha) = \sup d(\alpha) = \sup_\alpha d(\alpha) = t$. It follows then that $f(t) = a$ and $f(\lim_\alpha d(\alpha) = \lim_\alpha f(\alpha))$, where $f(d(\alpha)) \in [a_k, b_k] \cup [b_{k+1}, b_k]$ and $f(t) = a$: contradiction. Therefore $\gamma = \sup T < \kappa$. Thus for $t \in [\delta, \rightarrow) \cap S$, $f(t) = a$, where $\delta_0 > \max(\gamma, \delta_1)$.

(b) implies (a). Suppose $S$ is non-stationary. So pick $C$ a club set in $\kappa$ disjoint from $S$ and write $\kappa \setminus C = \{x_\alpha, x_{\alpha+1} : \alpha < \kappa\}$, where $(x_\alpha, x_{\alpha+1})$ is a maximal open interval and $(x_\alpha)_\alpha$ is an increasing continuous enumeration of $C$. $S \subseteq \kappa \setminus C = \{x_\alpha, x_{\alpha+1} : \alpha < \kappa\}$. Define $f : S \setminus C \rightarrow \mathbb{R}$ by:

$$f(\xi) := \begin{cases} 0 & \text{if } \xi \in (x_\alpha, x_{\alpha+1}) \text{ and } \alpha = \lambda + 2n, \\ \frac{1}{n} & \text{if } \xi \in (x_\alpha, x_{\alpha+1}) \text{ and } \alpha = \lambda + 2n + 1. \end{cases}$$

$f$ is the continuous function that works. □

**References**


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