On Global Bipartite Domination Polynomials

Latheeshkumar A. R. and Anil Kumar. V.

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Abstract In this paper we introduce the concept of the global bipartite domination polynomial of a connected bipartite graph and study some of its general properties. We establish some relationships between domination polynomial and global bipartite domination polynomial of certain classes of graphs.

1 Introduction

In this paper we consider simple, connected and bipartite graphs. All notations and definitions not given here can be found in [2, 4]. A graph is an ordered pair \( G = (V(G), E(G)) \), where \( V(G) \) is a finite nonempty set and \( E(G) \) is a collection of 2-point subsets of \( V \). The sets \( V(G) \) and \( E(G) \) are the vertex set and edge set of \( G \) respectively. The degree of a vertex \( v \) in \( G \) is the number of edges incident at \( v \). The set of all neighbors of \( v \) is the open neighborhood of \( v \), denoted by \( N(v) \). Let \( P_n, C_n \) and \( K_{m,n} \) denote path, cycle and complete bipartite graph respectively. A set \( A \subseteq V(G) \) of vertices in a graph \( G = (V, E) \) is called a dominating set, if every vertex \( v \in V \) is either an element of \( A \) or adjacent to an element of \( A \). The domination number \( \gamma(G) \) of a graph \( G \) is the minimum cardinality of a dominating set in \( G \). The domination polynomial of a graph \( G \) of order \( n \) is the polynomial \( D(G, x) = \sum_{i=\gamma(G)}^n d(G, i) x^i \), where \( d(G, i) \) is the number of dominating sets of \( G \) of size \( i \) [1].

2 Main Results

In this section we introduce a new concept, namely, Global Bipartite Dominating Set of a simple bipartite graph \( G \). Then we define the Global Bipartite Domination Polynomial of \( G \).

Definition 2.1. Let \( G \) be a connected bipartite graph with bipartition \((X, Y)\), with \( |X| = m \) and \( |Y| = n \). The relative complement of \( G \) in \( K_{m,n} \) denoted by \( \bar{G} \) is the graph obtained by deleting all edges of \( G \) from \( K_{m,n} \) (i.e., \( K_{m,n} \setminus G \)). A global bipartite dominating set (GBDS) of \( G \) is a set \( S \) of vertices of \( G \) such that it dominates \( G \) and its relative complement \( \bar{G} \). The global bipartite domination number, \( \gamma_{gb}(G) \) is the minimum cardinality of a global bipartite dominating set of \( G \).

Definition 2.2. Let \( D_{gb}(G, i) \) be the family of global bipartite dominating sets of a simple connected bipartite graph \( G \) with cardinality \( i \) and let \( d_{gb}(G, i) = |D_{gb}(G, i)| \). Then the global bipartite domination polynomial \( D_{gb}(G, x) \) of \( G \) is defined as \( D_{gb}(G, x) = \sum_{i=\gamma_{gb}(G)}^n d_{gb}(G, i) x^i \).

Theorem 2.3. If \( G \) and \( \bar{G} \) are connected, then \( D_{gb}(G, x) = D_{gb}(\bar{G}, x) \).

Proof. The proof follows immediately from the definitions of GBDS and \( D_{gb}(G, x) \). \( \square \)

Theorem 2.4. For any positive integers \( m \) and \( n \),

(i) \( D_{gb}(K_{m,n}, x) = x^{m+n} \).

(ii) If \( K_{m,n} \setminus e \) is connected, then \( D_{gb}(K_{m,n} \setminus e, x) = x^{m+n-1}(x+2) \).

Proof. (i) Obviously \( \gamma_{gb}(K_{m,n}) = m+n \). Therefore \( D_{gb}(K_{m,n}, x) = x^{m+n} \).
(ii) We have \( \gamma_{gb}(K_{m,n} \smallsetminus e) = m + n - 1 \). Since \( d_{gb}(K_{m,n} \smallsetminus e, m + n - 1) = 2 \) and \( d_{gb}(K_{m,n} \smallsetminus e, m + n) = 1 \), the proof follows. \( \square \)

A bi-star graph \( B_{(m,n)} \) is a tree obtained from the graph \( K_2 \) with two vertices \( u \) and \( v \) by attaching \( m \) pendant edges in \( u \) and \( n \) pendant edges in \( v \).

**Theorem 2.5.** The global bipartite domination polynomial of bi-star graph is

\[
D_{gb}(B_{(m,n)}) = x^2 \left[ x^m + x^n + [(1 + x)^m - 1] \right] \left[ (1 + x)^n - 1 \right]
\]

**Proof.** Let \( U \) and \( V \) be the set of all pendant vertices in \( u \) and \( v \) respectively. Suppose \( S \) is a G.B.D.S of \( B_{(m,n)} \). Since the vertices \( u \) and \( v \) are isolated in \( B_{(m,n)} \), \( \{ u, v \} \subseteq S \). For \( |S| - 2 \neq m \) or \( n \), \( S \cap U \neq \emptyset \) and \( S \cap V \neq \emptyset \). If \( |S| - 2 = m \), then \( U \cup \{ u, v \} \) and if \( |S| - 2 = n \), then \( V \cup \{ u, v \} \) are G.B.D.S of \( B_{(m,n)} \). This completes the proof. \( \square \)

The next theorem follows immediately from the definition of global bipartite domination polynomial.

**Theorem 2.6.** For any spanning subgraph \( G \) of \( K_{m,n} \),

(i) \( d_{gb}(G, m + n) = 1 \).

(ii) \( d_{gb}(G, i) = 0 \) if and only if \( i < \gamma_{gb}(G) \) or \( i > m + n \).

(iii) \( D_{gb}(G, x) \) has no constant term.

(iv) \( D_{gb}(G, x) \) is a strictly increasing function in \( [0, \infty) \).

(v) If \( H \) is an induced subgraph of \( G \), then \( \deg(D_{gb}(G, x)) \geq \deg(D_{gb}(H, x)) \).

(vi) Zero is a root of \( D_{gb}(G, x) \) with multiplicity \( \gamma_{gb}(G) \).

**Theorem 2.7.** Let \( G \) be a graph with bipartition \( (X, Y) \). If \( G \) has a \( \gamma \)-set \( S = V_1 \cup V_2 \), where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \), then \( S \) is a \( \gamma_{gb} \)-set of \( G \) if and only if \( \bigcap_{x \in V_1} N(x) \subseteq V_2 \) and \( \bigcap_{y \in V_2} N(y) \subseteq V_1 \).

**Proof.** Let \( \bigcap_{x \in V_1} N(x) \subseteq V_2 \) and \( \bigcap_{y \in V_2} N(y) \subseteq V_1 \). Since \( S \) is a \( \gamma \)-set of \( G \), it suffices to show that \( S \) dominates the relative compliment of \( G \). Let \( u \in X \). If \( u \in \bigcap_{y \in V_2} N(y) \), then \( u \in V_1 \). If \( u \notin \bigcap_{y \in V_2} N(y) \) then \( u \) is adjacent to at least one vertex of \( V_2 \) in \( \hat{G} \). Similarly, we can prove that if \( v \in Y \) then \( v \in V_2 \) or \( v \) is adjacent to at least one vertex of \( V_1 \) in \( \hat{G} \). Conversely, let \( S \) dominates \( \hat{G} \). Let \( x \) be an arbitrary vertex in \( X \). If \( x \in \bigcap_{y \in V_2} N(y) \), then in \( \hat{G} \), \( x \) is not adjacent to any vertex of \( V_2 \). Since \( S \) dominates \( \hat{G} \), we can deduce that \( x \in V_1 \). If \( x \notin \bigcap_{y \in V_2} N(y) \), then \( x \) is adjacent to at least one element of \( V_2 \) in \( \hat{G} \). Hence the proof. \( \square \)

**Corollary 2.8.** For \( n \geq 10 \), \( \gamma_{gb}(P_n) = \gamma(P_n) = \left\lceil \frac{n}{2} \right\rceil \).

**Proof.** Let \( V(P_n) = \{1, 2, 3, \ldots, n\} \). Then \( X = \{ x : x \text{ is even, } x \leq n \} \) and \( Y = \{ y : y \text{ is odd, } y \leq n \} \) are the bipartition of \( P_n \). Let \( S_1 = \{ i : i \equiv 1 \pmod{3}, i \leq n \} \) and \( S_2 = \{ i : i + 1 \equiv 0 \pmod{3}, i \leq n \} \). Then either \( S_1 \) or \( S_2 \) is a \( \gamma \)-set of \( P_n \). Also for \( i = 1, 2, \bigcap_{x \in S_i \cap X} N(x) = \emptyset \) and \( \bigcap_{y \in S_i \cap Y} N(y) = \emptyset \). Thus the proof follows from Theorem 2.7. \( \square \)

**Corollary 2.9.** For an even integer \( n \geq 10 \), \( \gamma_{gb}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{2} \right\rceil \).

**Proof.** The proof is exactly similar to corollary 2.8. \( \square \)
Corollary 2.10. If \( G \) is an \( n - 1 \)-regular connected bipartite graph, then
\[
D_{gb}(G, x) = [x(x + 2)]^n - 2nx^n.
\]

Proof. Since \( G \) is \( n - 1 \)-regular, each component of \( G \) is \( P_2 \). Therefore a G.B.D.S of \( G \) contains at least one vertex from each component of \( G \). So \( \gamma_{gb}(G) = n \) and for \( 1 \leq i \leq n \), \( d_{gb}(G, n + i) = \binom{n}{i}2^{n-i} \). It follows from Theorem 2.7 that \( d_{gb}(G, n) = 2^n - 2n \). This completes the proof.

Next, we shall study the relation between domination polynomials and global domination polynomials of paths. For, we need the following:

Theorem 2.11. [1] For every \( n \geq 4 \),
\[
D(P_n, x) = x[D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)],
\]
with initial values \( D(P_1, x) = x \), \( D(P_2, x) = x^2 + 2x \), \( D(P_3, x) = x^3 + 3x^2 + x \).

Lemma 2.12. For a path \( P_n \) with bipartition \( (X, Y) \), let \( S = V_1 \cup V_2 \) where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \) be a dominating set. If \( |V_i| > 2 \), \( \forall i \) then \( S \) is a G.B.D.S of \( P_n \).

Proof. In \( P_n \) if \( |V_i| > 2 \), then \( \bigcap_{v \in V_i} N(v) = \phi \). Then by Theorem 2.7, \( S \) is a G.B.D.S of \( P_n \).

Theorem 2.13. Let \( G \) be a connected bipartite graph with partite sets \( X \) and \( Y \). Let \( S = V_1 \cup V_2 \) be a GBDS of \( G \), where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \). Then if \( V_1 = \phi \), then \( V_2 = Y \) and if \( V_2 = \phi \), then \( V_1 = X \).

Proof. Let \( S = V_1 \cup V_2 \), where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \). If \( V_1 = \phi \), then \( S \subseteq Y \). Since \( G \) is bipartite, the vertices in \( Y \) are not adjacent and so \( S \subseteq Y \). Therefore \( S = V_2 = Y \). Similarly, we can prove that if \( V_2 = \phi \) then \( V_1 = X \).

So for \( n \geq 12 \), to find \( d(P_n, i) - d_{gb}(P_n, i) \) it suffices to consider the dominating sets \( S = V_1 \cup V_2 \) of \( P_n \) with \( 1 \leq |V_1| \leq 2 \) or \( 1 \leq |V_2| \leq 2 \). To prove theorems 2.14 to 2.17, we take \( X = \{1, 3, 5, \ldots, 2n - 1\} \) and \( Y = \{2, 4, 6, \ldots, 2n\} \) be the bipartition of \( P_{2n} \) and \( S = V_1 \cup V_2 \), where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \) be a dominating set. Using the following theorems we can find the number of dominating sets which are not global bipartite dominating sets.

Theorem 2.14. For \( |V_1| = 1 \), we have

(i) \( d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2 \).

(ii) \( d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 2 \).

Proof. Since a vertex in \( X \) is adjacent to at most two vertices in \( Y \), \( n - 2 \leq |V_2| \leq n \). If \( |V_2| = n \), then \( S = V_1 \cup V_2 \) is a G.B.D.S and the proof is complete. So \( |V_2| = n - 2 \) or \( n - 1 \). We consider the following cases:

Case 1: \( V_1 = \{1\} \).
Here \( V_2 = \{4, 6, 8, \ldots, 2n\} \). Since \( N(1) = \{2\} \nsubseteq V_2, S \) is not a G.B.D.S.

Case 2: \( V_1 = \{3\} \).
Here also \( |V_2| = n - 1 \) and \( V_2 = \{2, 6, 8, \ldots, 2n\} \). Since \( N(3) = \{2, 4\} \nsubseteq V_2, S \) is not a G.B.D.S.

Case 3: \( V_1 = \{i\}, i \neq 1, 3 \).
Then for each \( i, V_1 \cup Y \setminus \{i - 1, i + 1\} \), \( V_1 \cup Y \setminus \{i - 1\} \) and \( V_1 \cup Y \setminus \{i + 1\} \) are dominating sets of \( P_{2n} \). Since \( N(i) = \{i - 1, i + 1\} \nsubseteq V_2, S \) are not G.B.D.S of \( P_{2n} \).

In cases 1 and 2 we have two dominating sets of order \( n \). In case 3 we have \( 2(n - 2) \) dominating sets of order \( n \) and \( n - 2 \) dominating sets of order \( n - 1 \). Therefore the result follows.

Theorem 2.15. For \( |V_2| = 1 \), we have
(i) \(d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2\).

(ii) \(d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 2\).

**Proof.** The proof is exactly similar to Theorem 2.14.

**Theorem 2.16.** For \(|V_1| = 2\), we have

(i) \(d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 3\).

(ii) \(d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4\).

(iii) \(d(P_{2n}, n + 1) - d_{gb}(P_{2n}, n + 1) = n - 1\).

**Proof.** Since \(|V_1| = 2\), we have \(n - 3 \leq |V_2| \leq n\). If \(|V_2| = n\), then \(S = V_1 \cup V_2\) is a G.B.D.S. So it suffices to consider the cases \(|V_2| = n - 3, n - 2\) and \(n - 1\).

**Case 1:** \(V_1 = \{1, 3\}\).

**Subcase 1:** \(|V_2| = n - 2\).

Then \(V_2 = \{6, 8, \ldots, 2n\}\). Since \(N(1) \cup N(3) = \{2\} \not\subseteq V_2\), \(S\) is not a G.B.D.S of \(P_{2n}\).

**Subcase 2:** \(|V_2| = n - 1\).

Then \(V_2 = \{4, 6, 8, \ldots, 2n\}\). Since \(N(1) \cup N(3) = \{2\} \not\subseteq V_2\), the dominating set \(S\) is not a G.B.D.S.

**Case 2:** \(V_1 = \{3, 5\}\).

As in case 1 we get two dominating sets which are not G.B.D.S of \(P_{2n}\).

**Case 3:** \(V_1 = \{i, i + 2\}, i \neq 1, 3\).

**Subcase 1:** \(|V_2| = n - 3\).

Then \(V_2 = Y \setminus \{i - 1, i + 1, i + 3\}\).

**Subcase 2:** \(|V_2| = n - 2\).

In this case we have the possibilities, \(V_2 = Y \setminus \{i - 1, i + 1\}\) and \(V_2 = Y \setminus \{i + 1, i + 3\}\).

**Subcase 3:** \(|V_2| = n - 1\).

Then \(V_2 = Y \setminus \{i + 1\}\).

In subcase 1, 2 and 3, \(S = V_1 \cup V_2\) is a dominating set but since \(N(i) \cap N(i + 1) = \{i + 1\} \not\subseteq V_2\), \(S\) is not a G.B.D.S of \(P_{2n}\).

In cases 1 and 2 we have two dominating sets of order \(n\) and \(n + 1\). In case 3 we have \(n - 3\) dominating sets of order \(n - 1, 2(n - 3)\) dominating sets of order \(n\) and \(n - 3\) dominating sets of order \(n + 1\). Hence the result follows.

**Theorem 2.17.** For \(|V_2| = 2\), we have

(i) \(d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 3\).

(ii) \(d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4\).

(iii) \(d(P_{2n}, n + 1) - d_{gb}(P_{2n}, n + 1) = n - 1\).

**Proof.** The proof is exactly similar to Theorem 2.16.

**Theorem 2.18.** For \(n \geq 6\),

\[
\mathcal{D}(P_{2n}, x) - \mathcal{D}_{gb}(P_{2n}, x) = (4n - 10)x^{n-1} + (8n - 12)x^n + (2n - 2)x^{n+1}.
\]

**Proof.** It follows from Theorems 2.14, 2.15, 2.16 and 2.17.

Next, we find the relationship between domination polynomials and global bipartite domination polynomials of \(P_{2n+1}\). To prove theorems 2.19 to 2.22, we take \(X = \{1, 3, 5, \ldots, 2n + 1\}\) and \(Y = \{2, 4, 6, \ldots, 2n\}\) be the bipartition of \(P_{2n+1}\) and \(S = V_1 \cup V_2\), where \(V_1 \subseteq X\) and \(V_2 \subseteq Y\) be a dominating set of \(P_{2n+1}\).

**Theorem 2.19.** For \(|V_1| = 1\), we have
(i) \(d(P_{2n+1}, n-1) - d_{gb}(P_{2n+1}, n-1) = n - 3\).

(ii) \(d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n - 2\).

**Proof.** Case 1: \(V_1 = \{1\}\). Let \(V_2 = Y \setminus \{2\}\). Since \(N(1) = \{2\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S.

Case 2: \(V_1 = \{3\}\). Let \(V_2 = Y \setminus \{4\}\). Since \(N(3) = \{2, 4\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S.

Case 3: \(V_1 = \{i\}, i \not\in \{1, 3, 2n-1, 2n+1\}\). In this case we have the possibilities, \(V_2 = Y \setminus \{i-1, i+1\}\) or \(V_2 = Y \setminus \{i-1\}\) and \(V_2 = Y \setminus \{i+1\}\). Since \(N(i) = \{i-1, i+1\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order \(n\) and in case 3 there are \(n-3\) dominating sets of order \(n-1\) and \(2(n-3)\) dominating sets of order \(n\). This completes the proof. \(\square\)

**Theorem 2.20.** For \(|V_2| = 1\), we have

(i) \(d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n\).

(ii) \(d(P_{2n+1}, n + 1) - d_{gb}(P_{2n+1}, n + 1) = 2n\).

**Proof.** Let \(V_2 = \{i\}, i \in Y \Rightarrow N(i) = \{i-1, i+1\}\). Then \(V_1\) can be \(X \setminus \{i-1\}\) or \(X \setminus \{i+1\}\) or \(X \setminus \{i-1, i+1\}\). Since \(i\) can be selected in \(n\) ways, we have \(2n\) dominating sets of order \(n+1\) and \(n\) dominating sets of order \(n\). Since \(N(i) = \{i-1, i+1\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S. of \(P_{2n+1}\). Hence the result follows. \(\square\)

**Theorem 2.21.** For \(|V_1| = 2\), we have

(i) \(d(P_{2n+1}, n-1) - d_{gb}(P_{2n+1}, n-1) = n - 4\).

(ii) \(d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n - 4\).

(iii) \(d(P_{2n+1}, n + 1) - d_{gb}(P_{2n+1}, n + 1) = n\).

**Proof.** Case 1: \(V_1 = \{1, 3\}\). Then \(V_2\) can be \(Y \setminus \{2\}\) or \(Y \setminus \{2, 3\}\). Since \(N(1) \cap N(3) = \{2\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S.

Case 2: \(V_1 = \{3, 5\}\). Then \(V_2\) can be \(Y \setminus \{4\}\) or \(Y \setminus \{4, 5\}\). Since \(N(3) \cap N(5) = \{4\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S.

Case 3: \(V_1 = \{i, i+2\}, i \not\in \{1, 3, 2n-3, 2n-1\}\). Then \(V_2\) can be \(Y \setminus \{i-1, i+1, i+3\}\) or \(Y \setminus \{i-1, i+1\}\) or \(Y \setminus \{i+1, i+3\}\). Since \(N(i) \cap N(i+2) = \{i+1\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order \(n\) and \(n+1\). In case 3 there are \(n-4\) dominating sets of order \(n-1\) and \(n+1\) and \(2(n-4)\) dominating sets of order \(n\). Thus the result follows. \(\square\)

**Theorem 2.22.** For \(|V_2| = 2\), we have

(i) \(d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n - 1\).

(ii) \(d(P_{2n+1}, n + 1) - d_{gb}(P_{2n+1}, n + 1) = 2n - 2\).

(iii) \(d(P_{2n+1}, n + 2) - d_{gb}(P_{2n+1}, n + 2) = n - 1\).

**Proof.** Let \(V_2 = \{i, i+2\}, i \in Y \Rightarrow N(i) \cap N(i+2) = \{i+1\}\). Then \(V_1\) can be \(X \setminus \{i-1, i+1, i+3\}\) or \(X \setminus \{i-1, i+1\}\) or \(X \setminus \{i+1, i+3\}\). Since \(V_2\) can be chosen in \(n-1\) ways, we have \(n-1\) dominating sets of order \(n\) and \(2(n-1)\) dominating sets of order \(n+1\) and \(n-1\) dominating sets of order \(n+2\). Since \(N(i) \cap N(i+2) = \{i+1\}\), \(S = V_1 \cup V_2\) is not a G.B.D.S. of \(P_{2n+1}\). This proves the result. \(\square\)
Theorem 2.23. For \( n \geq 6 \),
\[
\mathcal{D}(P_{2n+1}, x) - \mathcal{D}_{gb}(P_{2n+1}, x) = (2n-7)x^{n-1} + (6n-7)x^n + (5n-2)x^{n+1} + (n-1)x^{n+2}.
\]
Proof. It follows from Theorems 2.19, 2.20, 2.21 and 2.22. \( \square \)

Theorem 2.24. [1] For every \( n \geq 4 \),
\[
\mathcal{D}(C_n, x) = x[D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)], \text{ with initial values } D(C_1, x) = x, D(C_2, x) = x^2 + 2x, D(C_3, x) = x^3 + 3x^2 + 3x.
\]
Next, we find \( \mathcal{D}(C_{2n}, x) - \mathcal{D}_{gb}(C_{2n}, x) \).
To prove theorems 2.25 to 2.29, we take \( X = \{1, 3, 5, \ldots, 2n - 1\} \) and \( Y = \{2, 4, 6, \ldots, 2n\} \) be the bipartition of \( C_{2n} \) and \( S = V_1 \cup V_2 \) where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \) be a dominating set of \( C_{2n} \).

Theorem 2.25. For \( |V_1| = 1 \), we have
(i) \( d(C_{2n}, n) - d_{gb}(C_{2n+1}, n-1) = n \).
(ii) \( d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n \).

Proof. Let \( V_1 = \{i\}, i \in X \). Then \( N(i) = \{i - 1, i + 1\} \) (if \( i = 1 \), then we take \( i - 1 = 2n \)).
Then \( V_2 \) can be \( Y \setminus \{i - 1, i + 1\} \) or \( X \setminus \{i - 1\} \) or \( X \setminus \{i + 1\} \).
Since \( i \) can be chosen in \( n \) ways, we have \( n \) dominating sets of order \( n - 1 \) and \( 2n \) dominating sets of order \( n \).
Since \( N(i) = \{i - 1, i + 1\} \), \( S = V_1 \cup V_2 \) is not a G.B.D.S. of \( C_{2n} \). Hence the result follows. \( \square \)

Theorem 2.26. For \( |V_2| = 1 \), we have
(i) \( d(C_{2n}, n - 1) - d_{gb}(C_{2n+1}, n - 1) = n \).
(ii) \( d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n \).

Proof. The proof is exactly similar to Theorem 2.25. \( \square \)

Theorem 2.27. For \( |V_1| = 2 \), we have
(i) \( d(C_{2n}, n - 1) - d_{gb}(C_{2n}, n - 1) = n - 1 \).
(ii) \( d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n - 1) \).
(iii) \( d(C_{2n}, n + 1) - d_{gb}(C_{2n}, n + 1) = n - 1 \).

Proof. Let \( V_1 = \{i, i+2\}, i \in X \). Then \( N(i) \cap N(i+2) = \{i+1\} \) (if \( i = 2n-1 \), then we take \( i+2 = 2 \) and \( i+3 = 2 \)).
Then \( V_2 \) can be \( Y \setminus \{i-1, i+1, i+3\} \) or \( Y \setminus \{i+1\} \) or \( Y \setminus \{i+3\} \) or \( Y \setminus \{i+1\} \).
Since \( V_1 \) can be chosen in \( n - 1 \) ways, we have \( (n - 1) \) dominating sets of order \( n - 1 \) \( 2(n - 1) \) dominating sets of order \( n - 1 \) and \( 2(n - 1) \) dominating sets of order \( n - 1 \). Since \( N(i) \cap N(i+2) = \{i+1\} \), \( S = V_1 \cup V_2 \) is not a G.B.D.S. of \( C_{2n} \). Hence the result follows. \( \square \)

Theorem 2.28. For \( |V_2| = 2 \), we have
(i) \( d(C_{2n}, n - 1) - d_{gb}(C_{2n}, n - 1) = n - 1 \).
(ii) \( d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n - 1) \).
(iii) \( d(C_{2n}, n + 1) - d_{gb}(C_{2n}, n + 1) = n - 1 \).

Proof. The proof is exactly similar to Theorem 2.27. \( \square \)

Theorem 2.29. For \( n \geq 6 \),
\[
\mathcal{D}(C_{2n}, x) - \mathcal{D}_{gb}(C_{2n}, x) = (4n-2)x^{n-1} + (8n-4)x^n + (2n-2)x^{n+1}.
\]
Proof. It follows from Theorems 2.25, 2.26, 2.27 and 2.28. \( \square \)
References


Author information

Latheshkumar A. R., Department of Mathematics, St. Mary’s College, Sulthan bathery, Wayanad, Kerala, 673 592, India.
E-mail: latheshby@gmail.com

Anil Kumar V., Department of Mathematics, University of Calicut, Malappuram, Kerala, 673 635, India.
E-mail: anil@uoc.ac.in

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